On 'family resemblances' with fuzzy sets*

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Abstract— *This paper takes into account the Wittgenstein's idea on family resemblances as a particular crisp relation between some fuzzy sets, that is, between some predicates representable from its use. It is shown that all uses of the same predicate actually do have family resemblance, that some pairs of predicates cannot, and a numerical degree of family resemblance is introduced.*

Keywords— Degree of family resemblance, Family resemblance, Fuzzy sets.

1 Introduction

Ludwig Wittgenstein headed two influential traditions in the so-called philosophy of language, that were originated by his famous books, *Tractatus logico-philosophicus* (1922, [1]), and *Philosophical Investigations* (1953, [2]), respectively.

The *Tractatus* does not properly deal with ordinary language, but with the logical analysis of propositions built up from atomic propositions, considered as 'pictures' of facts and keeping a strict correspondence with the world, understood through the totality of facts. Instead, the *Philosophical Investigations* in which Wittgenstein abandoned logical analysis, meant a shift in conferring a main role to the ways of designating facts, as an activity-oriented perspective on language. What is central at this respect, is that language does not primarily consist on describing the facts, but on playing 'language's games', or ways of dynamically using words to define their meaning, and that are to be described. In order to fix the meaning of a word, to show how a word works, it should be placed in the context and environment it is used. Language is not yet defined through propositions that, now, in Wittgenstein's view, come from their function in a language's game, and to note the absence of boundaries for describing such use of words, Wittgenstein introduced the term 'family resemblances'. Of course, Wittgenstein's idea on 'family resemblances', is broader than the relation of *family resemblance* in next section.

Fuzzy logic manages the extensional meaning of predicates through its use, once captured by the corresponding membership functions. This paper is nothing else than a first approach, in the path towards Zadeh's *Computing With Words*, to introduce 'family resemblances' between full-normalized fuzzy sets, and a numerical degree for measuring such relation.

Usually, full-normalized fuzzy sets, $\mu_P \in [0,1]^X$, those for which there are $x, y \in X$ such that $\mu_P(x) = 1$, $\mu_P(y) = 0$, appear as 'data' in the modeling of fuzzy systems, and they are neither self-contradictory ($\mu_P \leq \mu'_P$), nor negatively self contradictory ($\mu'_P \leq \mu_P$). Of course, μ'_P denotes the fuzzy set corresponding to 'not p'. Points x verifying $\mu_P(x) = 1$ can be taken as *the prototypes of P in X*, and points y verifying $\mu_P(y) = 0$ as *the anti-prototypes of P in X*.

In its own nature, this paper is not a conclusive one, but only a tentative to reflect the potentially interesting subject of the family resemblances shown by 'data' fuzzy sets. That is, by fuzzy sets with prototypes and anti-prototypes.

2 Family resemblance of fuzzy sets

Let $P, Q, ...$ be predicates on a universe of discourse X, such that their use, or meaning, is described by fuzzy sets $\mathbb{P}, \mathbb{Q}, ...$, given by membership functions $\mu_P, \mu_Q, ...$ in $[0,1]^X$.
For each membership function μ in $[0,1]^X$, define the sets of

- its 0-points, $Z(\mu) = \{x \in X; \mu(x) = 0\}$
- its 1-points, $S(\mu) = \{x \in X; \mu(x) = 1\}$

Definition 2.1 *With $X \subset \mathbb{R}$, the relation of family resemblance,* **fr** $\subset [0,1]^X \times [0,1]^X$*, is defined by $(\mu, \sigma) \in$* **fr** *if and only if,*

1. $Z(\mu) \cap Z(\sigma) \neq \emptyset$, $S(\mu) \cap S(\sigma) \neq \emptyset$
2. *μ is non-decreasing in $A \subset X$ iff σ is non-decreasing in A.*
3. *μ is decreasing in $A \subset X$ iff σ is decreasing in A.*

In this definition, both 'decreasing' and 'non-decreasing', are not in strict sense, but allowing some constant pieces that can be taken as the first, or the second, by following what happens before or after.

Notice that the binary relation **fr** is only predicable between full-normalized fuzzy sets, that is, such that $Z(\mu) \neq \emptyset$, and $S(\mu) \neq \emptyset$. Denote

$$\mathfrak{F}^*(X) = \{\mu \in [0,1]^X - \{0,1\}^X; Z(\mu) \neq \emptyset, S(\mu) \neq \emptyset\}.$$

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It should be pointed out that fuzzy sets μ in $\{0,1\}^X$ (crisp sets) are excluded, since such functions μ are neither decreasing, nor non-decreasing, but only piecewise constant with the values 0 or 1. In addition, in many cases if $(\mu,\sigma) \in \mathbf{fr}$, with $\mu, \sigma \in \{0,1\}^X$, it should be $\mu = \sigma$. Obviously, also constant fuzzy sets μ_r ($\mu_r(x) = r$, with $r \in [0,1], x \in X$) are not in $\mathfrak{F}^*(X)$, since $Z(\mu_r) = \emptyset$, or $S(\mu_r) = \emptyset$.
Of course, if σ results from a translation of μ keeping $Z(\mu) \cap Z(\sigma) \neq \emptyset$, and $S(\mu) \cap S(\sigma) \neq \emptyset$, it is obvious that $(\mu,\sigma) \in \mathbf{fr}$.

Remark 2.2 *Any non full-normalized fuzzy set, represented by a non-constant but continuous membership function $\mu \in [0,1]^{\mathbb{R}}$, can be re-scaled to a full-normalized one μ^* by*

$$\mu^* = \frac{\mu - \min(\mu)}{\max(\mu) - \min(\mu)}$$

Obviously, it is $Z(\mu^*) \neq \emptyset, S(\mu^*) \neq 0$, that is $\mu^* \in \mathfrak{F}^*(\mathbb{R})$, and μ^* is non-decreasing (decreasing) if and only if μ is non-decreasing (decreasing), but μ^* and μ can differ in the respective slopes (provided they have derivatives, it is $\mu^{*\prime} = \mu' / \max(\mu) - \min(\mu)$.

Example 2.3

1. *Fuzzy sets μ, σ in figure 1 verify $(\mu,\sigma) \in \mathbf{fr}$, since $Z(\mu) \cap Z(\sigma) = [0,2] \neq \emptyset$, and $S(\mu) \cap S(\sigma) = \{10\} \neq \emptyset$, and both are non-decreasing in $X = [0,10]$.*

2. *Fuzzy set μ in figure 1, and fuzzy set λ in figure 2, verify $(\mu,\lambda) \notin \mathbf{fr}$, since $S(\mu) \cap S(\lambda) = \emptyset$, and λ is decreasing in $[0,2]$, but μ is not.*

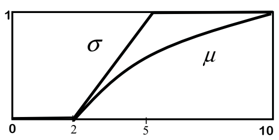

Figure 1:

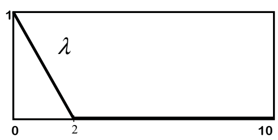

Figure 2:

Theorem 2.4 $\mathbf{fr}$ *is a reflexive and symmetric relation in $\mathfrak{F}^*(X)$.*

Proof. It is immediate to check that $\mathbf{fr}$ does verify the reflexive and symmetric properties. $\square$

As it is intuitive, crisp relation $\mathbf{fr}$ is not transitive. A counterexample is given by membership functions μ, σ, λ in the figures

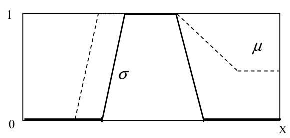

Figure 3:

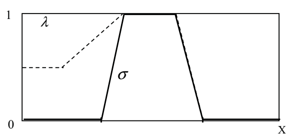

Figure 4:

that verify $(\mu,\sigma) \in \mathbf{fr}$, and $(\sigma,\lambda) \in \mathbf{fr}$, but $(\mu,\lambda) \notin \mathbf{fr}$, since $Z(\mu) \cap Z(\lambda) = \emptyset$.

Hence, if sets $[\mu] = \{\sigma \in \mathfrak{F}^*(X); (\mu,\sigma) \in \mathbf{fr}\}$ are not empty and cover $\mathfrak{F}^*(X)$, because it is $\bigcup_{\mu \in \mathfrak{F}^*(X)} [\mu] = \mathfrak{F}^*(X)$, they do not give a partition of $\mathfrak{F}^*(X)$, and the quotient $\mathfrak{F}^*(X)/\mathbf{fr}$ does not exist. For example, in the above figures, it is $\sigma \in [\mu] \cap [\lambda']$. Anyway, $[\mu]$ can be called the *family of* $\mu \in \mathfrak{F}^*(X)$, and $\sigma \in [\mu]$ a *relative* of μ, although μ could have relatives in other families, like it happens in people's families.

Theorem 2.5 *For no complement, μ' of $\mu \in \mathfrak{F}^*(X)$, is $(\mu,\mu') \in \mathbf{fr}$.*

Proof. If for $x, y \in A$, $x \leq y$, and $\mu(x) \leq \mu(y)$, it is $\mu'(y) \leq \mu'(x)$. $\square$

Theorem 2.6 *For no opposite, or antonym, $'\mu$ of $\mu \in \mathfrak{F}^*(X)$, given by a symmetry α in X, $'\mu = \mu \circ \alpha$, is $(\mu, {}'\mu) \in \mathbf{fr}$.*

Proof. If $x \leq y$, and $\mu(x) \leq \mu(y)$, it follows $\alpha(y) \leq \alpha(x)$, and $\mu(\alpha(y)) \leq \mu(\alpha(x))$, or $\mu \circ \alpha(y) \leq \mu \circ \alpha(x)$, that is $'\mu(y) \leq' \mu(x)$. (see [3]) $\square$

Theorem 2.7 *Let $u : X \to X$ be a bijective mapping such that*

- *If $x \leq y$, then $u(x) \leq u(y)$*
- *$u^{-1}(Z(\mu)) \subset Z(\mu)$*
- *$u^{-1}(S(\mu)) \subset S(\mu)$*

for $\mu \in \mathfrak{F}^*(X)$, $X \subset \mathbb{R}$. It is $(\mu, \mu \circ u^{-1}) \in \mathbf{fr}$.

Proof. It is $Z(\mu) \cap Z(\mu \circ u^{-1}) \neq \emptyset$, since $x \in Z(\mu)$, or $\mu(x) = 0$, implies $u^{-1}(x) \in Z(\mu)$, or $\mu(u^{-1}(x)) = \mu \circ u^{-1}(x) = 0$, it is, $x \in Z(\mu \circ u^{-1})$. Analogously, $x \in S(\mu)$, or $\mu(x) = 1$, that is, $x \in S(\mu \circ u^{-1})$. Hence, $Z(\mu) \cap Z(\mu \circ u^{-1}) \neq \emptyset$, and $S(\mu) \cap S(\mu \circ u^{-1}) \neq \emptyset$. If μ is non-decreasing, '$x \leq y \Rightarrow \mu(x) \leq \mu(y)$, from $u^{-1}(x) \leq u^{-1}(y)$, follows $\mu(u^{-1}(x)) \leq \mu(u^{-1}(y))$. Analogously, if μ is decreasing, so it is $\mu \circ u^{-1}$. $\square$

In particular, if Q represented by $\mu_Q = \mu_P \circ u^{-1}$ is a synonym of P (see[4]), then $(\mu_P, \mu_Q) \in \mathbf{fr}$.

Example 2.8

1. *Fuzzy sets μ, σ in figure 5 verify $\sigma = \mu \circ \alpha$, with the symmetry $\alpha(x) = 10 - x$, and represent two antonyms. Obviously, μ and $\mu \circ \alpha$ do not show family resemblance.*

Figure 5:

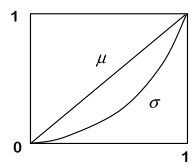

Figure 6:

2. *Fuzzy sets μ, σ in figure 6 verify $\sigma = \mu \circ u^{-1}$, with $u(x) = \sqrt{x}$, and show family resemblance.*

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3 Degree of family resemblance

Obviously, if σ_1, σ_2 are in the family $[\mu]$, they do verify $(\mu, \sigma_1) \in \mathbf{fr}$, and $(\mu, \sigma_2) \in \mathbf{fr}$, but this does not mean that σ_1, σ_2 show the same extent of family resemblance with μ. A way of measuring such extent is by means of a convenient T-indistinguishability (see([5])) $DR : \mathbf{fr} \to [0, 1]$, defining,

If $(\mu, \sigma) \in \mathbf{fr}$, the degree up to which μ resembles σ is $DR(\mu, \sigma) \in [0, 1]$.

3.1

Any prod-indistinguishability E in $\mathfrak{F}^*(X)$, can be represented by Ovchinnikov's theorem (see [6])

$$E(\mu, \sigma) = \underset{f \in F}{Inf}\left(\frac{f(\mu)}{f(\sigma)}, \frac{f(\sigma)}{f(\mu)}\right),$$

with F a family of functions $f : \mathfrak{F}^*(X) \to \mathbb{R}^+ - \{0\}$.

Provided X is a closed interval in $\mathbb{R}$, consider only the functions $\mu \in \mathfrak{F}^*(X)$ that are Riemann-integrable in X, and such that both $Z(\mu)$ and $S(\mu)$ are intervals in X. Denote by $\mathfrak{F}^{**}(X)$ this subset of functions in $\mathfrak{F}^*(X)$, and take

$$f_1(\mu) = \int_X \mu(x)dx, \quad f_2(\mu) = \text{length of the interval } X - Z(\mu).$$

Then,

$$Ovch(\mu, \sigma) = \min\left(\frac{f_1(\mu)}{f_1(\sigma)}, \frac{f_1(\sigma)}{f_1(\mu)}, \frac{f_2(\mu)}{f_2(\sigma)}, \frac{f_2(\sigma)}{f_2(\mu)}\right),$$

for all $\mu, \sigma \in \mathfrak{F}^{**}(X)$, is a prod-indistinguishability in the part of $\mathbf{fr}$ in $\mathfrak{F}^{**}(X) \times \mathfrak{F}^{**}(X)$.

3.2

If $X = [a, b] \subset \mathbb{R}$, define

$$I(\mu, \sigma) = 1 - \frac{1}{b-a}\left|\int_X \mu(x)dx - \int_X \sigma(x)dx\right|,$$

for all $\mu, \sigma \in \mathfrak{F}^{**}(X)$.

Function I is a W-indistinguishability (W is the Łukasiewicz t-norm), since:

- $I(\mu, \mu) = 1$, and $I(\mu, \sigma) = I(\sigma, \mu)$, for all $\mu, \sigma \in \mathfrak{F}^{**}(X)$.
- $W(I(\mu, \sigma), I(\sigma, \lambda)) = \max(0, 1 - \frac{1}{b-a}[|\int_X \mu(x)dx - \int_X \sigma(x)dx| + |\int_X \sigma(x)dx - \int_X \lambda(x)dx|]) \leq \max(0, 1 - \frac{1}{b-a}|\int_X \mu(x)dx - \int_X \lambda(x)dx|) = 1 - \frac{1}{b-a}|\int_X \mu(x)dx - \int_X \lambda(x)dx| = I(\mu, \lambda)$, for all $\mu, \sigma, \lambda \in \mathfrak{F}^{**}(X)$, because of the triangular inequality, $|\int_X \mu(x)dx - \int_X \lambda(x)dx| \leq |\int_X \mu(x)dx - \int_X \sigma(x)dx| + |\int_X \sigma(x)dx - \int_X \lambda(x)dx|$.

3.3

Since it is $Ovch(\mu, \sigma) \neq I(\mu, \sigma)$ and, in general, $Ovch(\mu, \sigma) < I(\mu, \sigma)$, a better degree could be obtained with a mean like,

$$DR(\mu, \sigma) = \frac{r_1 Ovch(\mu, \sigma) + r_2 I(\mu, \sigma)}{r_1 + r_2},$$

$$\text{with } r_1, r_2 \in \mathbb{R}^+ - \{0\},$$

of which the only DR that is symmetric is the one with $r_1 = r_2 = \frac{1}{2}$ (arithmetic mean). In addition, and since $W \leq prod$, $Ovch$ is also W-transitive: $W(Ovch(\mu, \sigma), Ovch(\sigma, \lambda)) \leq Ovch(\mu, \sigma) \cdot Ovch(\sigma, \lambda) \leq Ovch(\mu, \lambda)$. Thus, also DR with $r_1 = r_2 = \frac{1}{2}$ is W-transitive, since:

$$W(DR(\mu, \sigma), DR(\sigma, \lambda)) =$$

$$\max(0, \frac{Ovch(\mu, \sigma) + Ovch(\sigma, \lambda) - 1}{2} + \frac{I(\mu, \sigma) + I(\sigma, \lambda) - 1}{2}) \leq$$

$$\max(0, \frac{Ovch(\mu + \lambda)}{2} + \frac{I(\mu + \lambda)}{2}) \leq$$

$$\max(0, DR(\mu, \lambda)) = DR(\mu, \lambda),$$

because $Ovch(\mu, \sigma) + Ovch(\sigma, \lambda) - 1 \leq \max(0, Ovch(\mu, \sigma) + Ovch(\sigma, \lambda) - 1)$, and $I(\mu, \sigma) + I(\sigma, \lambda) - 1 \leq \max(0, I(\mu, \sigma) + I(\sigma, \lambda) - 1)$. Hence, we will take

$$DR(\mu, \sigma) = \frac{Ovch(\mu, \sigma) + I(\mu, \sigma)}{2},$$

as the *index of family resemblance*.

Remark 3.1 *Obviously, functions DR are not only applicable to pairs in* **fr**, *but also to all pairs in $\mathfrak{F}^{**}(X) \times \mathfrak{F}^{**}(X)$*

3.4

Let E be a T-indistinguishability relation. The crisp relation

$$\mu \equiv \sigma \Leftrightarrow E(\mu, \sigma) > 0,$$

is reflexive, and symmetric. Obviously, and provided $T = \min$, or $T = prod$, it is also transitive. Nevertheless, if $T = W$, since "$E(\mu, \sigma) > 0$ and $E(\sigma, \lambda) > 0$" is equivalent to the existence of $\varepsilon > 0$ such that "$E(\mu, \sigma) \geq \varepsilon$ and $E(\sigma, \lambda) \geq \varepsilon$", it is

$$W(E(\mu, \sigma), E(\sigma, \lambda)) \geq W(\varepsilon, \varepsilon) = \max(0, 2\varepsilon - 1), \text{ or}$$

$$E(\mu, \lambda) \geq \max(0, 2\varepsilon - 1).$$

To have $E(\mu, \lambda) > 0$, it is sufficient that $2\varepsilon - 1 > 0$, or $\varepsilon > 0.5$. In this case, $\mu \equiv \sigma$ and $\sigma \equiv \lambda$ imply $\mu \equiv \lambda$ (transitivity). Hence $E(\mu, \sigma) > 0.5$ allows to take μ and σ as 'equivalent'.

If $0.5 < Ovch(\mu, \sigma)$, and $0.5 < I(\mu, \sigma)$, it is $\mu \equiv \sigma$ for both T-indistinguishabilities $E = Ovch$ and $E = I$. Thus, if in addition to $(\mu, \sigma) \in \mathbf{fr}$, is $0.5 < \varepsilon \leq Ovch(\mu, \sigma)$, and $0.5 < \delta \leq I(\mu, \sigma)$, it is also $0.5 < \frac{\varepsilon + \delta}{2} \leq DR(\mu, \sigma)$, implying $\mu \equiv \sigma$ for $E = DR$. For example if $0.7 \leq Ovch(\mu, \sigma)$, and $0.7 \leq I(\mu, \sigma)$, it is $0.7 \leq DR(\mu, \sigma)$. In these cases, *it can be said that μ and σ show high family resemblance*.

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When $(\mu, \sigma) \in \mathbf{fr}$, if e.gr., $Ovch(\mu, \sigma) \geq 0.8$ and $I(\mu, \sigma) \geq 0.8$ implying $DR(\mu, \sigma) \geq 0.8$, there is a so high family resemblance between μ and σ that μ and σ can be taken as strongly representing similar uses of the same predicate (linguistic label).

Example 3.2

1. *Functions μ, σ in figure 7, do obviously verify $(\mu, \sigma) \in$ $\mathbf{fr}$. For them, $f_1(\mu) = 5$, $f_2(\mu) = 8$, $f_1(\sigma) = 4$ and $f_2(\sigma) = 7$,*

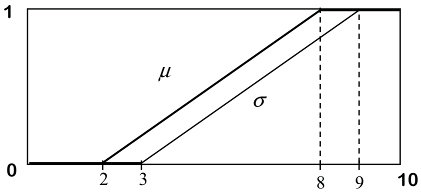

Figure 7:

hence, $Ovch(\mu, \sigma) = \min(\frac{5}{4}, \frac{4}{5}, \frac{8}{7}, \frac{7}{8}) = 0.8$.
Since, $I(\mu, \sigma) = 1 - \frac{1}{10}|5 - 4| = 0.9$, *it results*

$$DR(\mu, \sigma) = \frac{0.8 + 0.9}{2} = 0.85,$$

μ and σ show *high family resemblance.*

2. *Functions μ, σ in figure 8, verify $(\mu, \sigma) \in \mathbf{fr}$. For then, $f_1(\mu) = 4 + 2 \cdot 0.5 = 5$, $f_2(\mu) = 6$, $f_1(\sigma) = 2$, $f_2(\sigma) = 4$,*

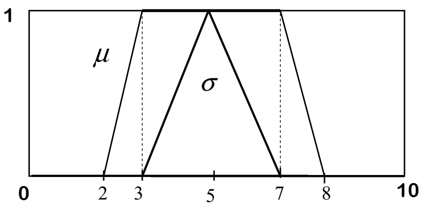

Figure 8:

hence, $Ovch(\mu, \sigma) = \min(\frac{5}{2}, \frac{2}{5}, \frac{6}{4}, \frac{4}{6}) = \frac{2}{5} = 0.4$.
Since, $I(\mu, \sigma) = 1 - \frac{1}{10}(|5 - 2|) = 0.7$, *it results*

$$DR(\mu, \sigma) = \frac{0.4 + 0.7}{2} = 0.55,$$

μ and σ *do not show* high family resemblance.

3. *Functions μ, σ in figure 9, verify $(\mu, \sigma) \in \mathbf{fr}$. For them $f_1(\mu) = 5 + 1 = 6$, $f_2(\mu) = 7$, $f_1(\sigma) = 4 + 2 = 6$ and $f_2(\sigma) = 8$, hence, $Ovch(\mu, \sigma) = \min(\frac{6}{6}, \frac{7}{8}, \frac{8}{7}) = \frac{7}{8} =$* 0.88.

   Since $I(\mu, \sigma) = 1 - \frac{1}{10}(|6 - 6|) = 1$, *it results*

$$DR(\mu, \sigma) = \frac{0.88 + 1}{2} = 0.94,$$

μ and σ *show* high family resemblance.

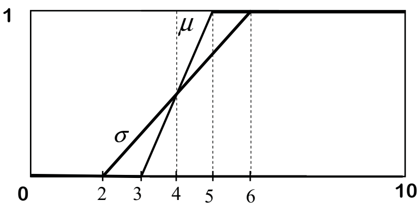

Figure 9:

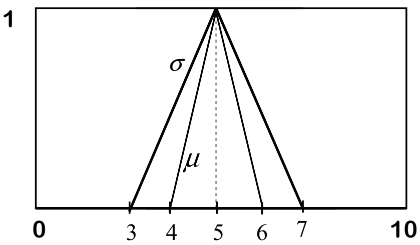

Figure 10:

4. *Functions μ, σ in figure 10, verify $(\mu, \sigma) \in \mathbf{fr}$. For them $f_1(\mu) = 1$, $f_2(\mu) = 2$, $f_1(\sigma) = 2$ and $f_2(\sigma) = 4$,*

   hence, $Ovch(\mu, \sigma) = \min(\frac{1}{2}, \frac{2}{1}, \frac{2}{4}, \frac{4}{2}) = 0.5$.

   Since $I(\mu, \sigma) = 1 - \frac{1}{10}(2 - 1) = 0.9$, *it results*

$$DR(\mu, \sigma) = \frac{0.5 + 0.9}{2} = 0.7,$$

μ and σ *show* high family resemblance.

5. *Functions μ, σ in figure 11, verify $(\mu, \sigma) \in \mathbf{fr}$. For them $f_1(\mu) = 3 + 2 \cdot 0.5 + 1 = 5$, $f_2(\mu) = f_2(\sigma) = 10$ and $f_1(\sigma) = 0.5 + 3 \cdot 0.5 + \frac{3 \cdot 0.5}{2} + 5 = 7.75$, hence, $Ovch(\mu, \sigma) = \frac{5}{7.75} = 0.64$.*

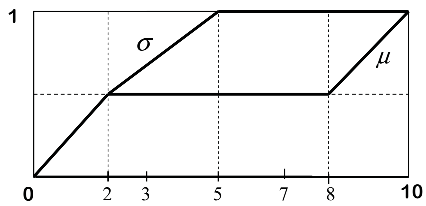

Figure 11:

Since $I(\mu, \sigma) = 1 - \frac{1}{10}(7.75 - 5) = 0.725$, *it results*

$$DR(\mu, \sigma) = \frac{0.64 + 0.725}{2} = 0.6825,$$

μ and σ *do not show* high family resemblance.

6. *Functions μ, σ in figure 12, verify $(\mu, \sigma) \in \mathbf{fr}$. It is:*

   $f_1(\mu) = 3$, $f_2(\mu) = 4$, $f_1(\sigma) = 3$ *and* $f_2(\sigma) = 5$.*Then,*

$$Ovch(\mu, \sigma) = \frac{4}{5}, \quad I(\mu, \sigma) = 1$$

That is,

$$DR(\mu, \sigma) = 0.9,$$

and μ, σ show high family resemblance.

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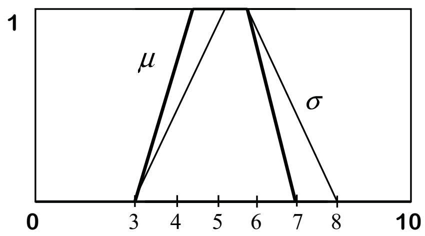

Figure 12:

4 The case of membership functions with the same predicate.

Given a predicate, or linguistic label, P on X, there is not a single fuzzy set generated by P, but each use of P in X, once reflected by its membership function μ_P (see [4]), defines a fuzzy set $\mathbb{P}$ by

$$x \in_r \mathbb{P} \Leftrightarrow \mu_P(x) = r, \text{ for all } x \in X, r \in [0,1].$$

Nevertheless, all functions μ_P do have some 0-points, and some 1-points, in common, as well as they should either decrease or non-decrease simultaneously in the same parts of X. Hence, *all μ_P should show some degree of family resemblance.*

This is the case for example, with P =*small* in $X = [0, 10]$, if two of its uses are represented by the membership functions μ_P^1 and μ_P^2 in figure 13.

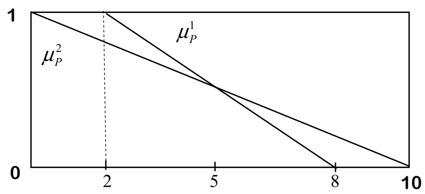

Figure 13:

It is clear that $(\mu_P^1, \mu_P^2) \in$ **fr**, and:

- $Ovch(\mu_P^1, \mu_P^2) = \min(\frac{5}{5}, \frac{10}{8}, \frac{8}{10}) = 0.8$
- $I(\mu_P^1, \mu_P^2) = 1 - \frac{1}{10}(|5 - 5|) = 1.$

  Hence,

$$DR(\mu_P^1, \mu_P^2) = \frac{0.8 + 1}{2} = 0.9,$$

and μ, σ *show high family resemblance*.

Examples 1 and 4 in section 3, do correspond to uses of P =*big* and P =*around five*, respectively.

Remark 4.1 *In natural language, families $[\mu]$ must be 'open', but not 'closed' like they were defined in section 2. These families are here static (sets), but in natural language they should be dynamical. With time, an element σ that was not in $[\mu]$, but possibly with a not too low degree $DR(\sigma, \mu)$, could be included in $[\mu]$ and thus generating a new family, indeed, changing the relation* **fr**. *Actually,* **fr** *is not a permanent relation, in natural language it is a changing one. Only with families of resemblance taken as classical sets, human thought seems to be impossible (see [7]).*

5 Last remarks

5.1

In his *Philosophical Investigations*, Wittgenstein conceived language in a way close to how people manages it. Thus, the meaning of an imprecise predicate is not given by necessary and sufficient conditions, but is built up by similarity with its prototypes in the universe of discourse, like in the case of *big* in [0,10] with, at least, the prototype 10. Notwithstanding, for more complex predicates like P =*beautiful* in a set of art's objects, it could be not clear the existence of prototypes and, since in such cases it could be $S(\mu_P) = \emptyset$, the study of the family resemblance for these predicates remains an open problem.

5.2

Certainly, to consider $X \subset \mathbb{R}$ is a restriction for this paper's results. Nevertheless, most of the predicates fuzzy logic considers are those exhibiting a numerical characteristic, allowing to translate them into an interval in $\mathbb{R}$. This is the case, for example, of P = *tall* in a big population X, that is translated into the interval $[0, 2]$, in meters, by the numerical characteristic 'Height', and the predicate $Q = big$ in such interval. That is, by the identification $\mu_{tall} = \mu_{big} \circ Height$, or $\mu_{tall}(x) = \mu_{big}(Height(x))$, for all x in X, once the use of *big* in $[0, 2]$ is chosen accordingly with that of *tall* interpreted, as it turn, by *big height*.

There are not so obvious cases that, also, can be translated into an interval through a more complex process, consisting in identifying each x in X with an n-tuple of significative parts in x, $x := (x_1, \ldots, x_n)$, with numerical characteristics $Ch_i(x_i) \in [a_i, b_i]$, and taking $Ch(x) = A(Ch_1(x_1), \ldots, Ch_n(x_n)) \in [a, b]$, with some n-place aggregation function A. An example is given by P= *beautiful* in a set X of paintings, if each painting x can be partitioned in $x = x_1 \cup \ldots \cup x_n (x_i \cap x_j = \emptyset, i \neq j)$, allowing to interpret the statement '*x* is *P*' as the composite one (x_1 is $P, \ldots, x_n$ is P). Provided each component 'x_i is P' is numerically evaluable by a clearly explicitable characteristic $Ch_i(x_i) \in \mathbb{R}^+$, the values of μ_P could be obtained by

$$\mu_P(x) = \frac{r_1 Ch_1(x_1) + \ldots + r_n Ch_n(x_n)}{r_1 + \ldots + r_n} (r_i \geq 0),$$

that are in the interval $[\min \mu_P, \max \mu_P]$.

Of course, $A(a_1, \ldots, a_n) = \frac{r_1 a_1 + \ldots + r_n a_n}{r_1 + \ldots + r_n}$, is not the only way of reasonably aggregating the n-tuples $(Ch_1(x_1), \ldots, Ch_n(x_n)) \in \mathbb{R}^n$. For example, if '*x* is *P*' is interpretable by 'x_1 is P' *and* ... *and* 'x_n is P', A can be taken as a t-norm, provided $Ch_i(x_i) \in [0, 1], 1 \leq i \leq n$.

6 Conclusions

6.1

The concept of family resemblances is here introduced as a crisp-binary relation for only a particular type of fuzzy sets in the real line, with 0-points and 1-points. Even more restrictive is the class of Riemann-integrable membership functions, to which pairs a degree of family resemblance is assigned. Anyway, this paper should be viewed as only a first trial to consider Wittgenstein's idea on family resemblances with fuzzy

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sets . For example, the definition of **fr** could be, perhaps, extended to other fuzzy sets without 1-points (non-normalized fuzzy sets), or without 0-points, as well as to pairs (μ, σ) with either μ, or σ in $\{0,1\}^X$. That is, to non-data fuzzy sets, to fuzzy sets resulting from computations with data ones. In addition, the selected degree DR cannot be considered as the definitive definition to measure to what extent there is family resemblance. More study on the subject is deserved since, for example, DR can be applied to pairs $(\mu, \sigma) \notin$ **fr**, and more accurate values can be obtained in some cases with non symmetric functions $DR(\mu, \sigma) = \frac{r_1 \cdot Ovch(\mu,\sigma) + r_2 \cdot I(\mu,\sigma)}{r_1 + r_2}$.

6.2

Of course, definition 2.1 could also be applied to crisp sets by simply allowing either μ or σ to only have 'constant pieces'. At this respect, examples like the one in figure 14 are of some interest. In such figure, μ is the trapezoidal fuzzy set $(3.8, 4, 6, 6.2)$, and σ does represent the crisp subset $[4, 6]$.

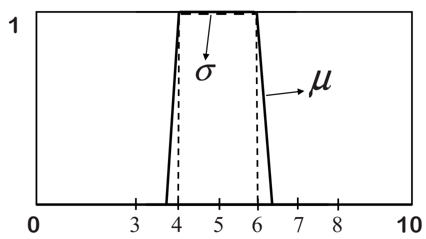

Figure 14:

Since $Ovch(\mu, \sigma) = 0.83$, $I(\mu, \sigma) = 0.98$, it follows $DR(\mu, \sigma) = 0.90$, a big value suggesting a so high resemblance between μ and σ that, perhaps, could allow to identify μ with μ_P, $P =$ *Almost between 4 and 6*. That is, μ could be 'linguistically approached' by the imprecise predicate P.

6.3

Maybe what is here introduced could be useful to approach the (pending) problem of 'linguistic approximation'. Namely, given the output $f : X \to [0, 1]$ of a system of fuzzy rules, how to find a predicate P in X such that f could be identified with μ_P? A possible way could come once identified a membership function μ_P (representing a use of the known predicate P), such that, $DR(f, \mu_P) > \varepsilon$, and $Sup|f - \mu_P| < \delta$ (for fixed $\varepsilon > 0, \delta > 0$), even if $(f, \mu_P) \notin$ **fr**. Then 'P' can be called an (ε, δ)-linguistic approximation of f. For example, in 6.2, *Almost between 4 and 6* is an $(0.89, 0.2)$-linguistic approximation of μ.

6.4

In this paper, the threshold 0.7 is taken for the sake of illustrating the idea of 'high family resemblance'. Nevertheless, and although 0.7 seems to be a good enough value for the examples shown in figures 7 to 14, such threshold's value still remains to be studied.

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