Convergence theorems for generalized random variables and martingales

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Abstract— We examine generalizations of random variables and martingales. We prove a new convergence theorem for set-valued martingales. We also generalize a well known characterization of set-valued random variables to the fuzzy setting.

Keywords— Banach space, fuzzy, martingale, random variable, set-valued.

1 Set-valued random variables and martingales

The theory of conditional expectations and martingales has been established for Banach space-valued, Bochner-integrable functions. Hiai and Umegaki have then generalized random variables, conditional expectations and martingales to the set-valued (multivalued) setting in [1]. We will assume that the reader is familiar with the basic ideas of random variables and provide a brief introduction to the set-valued generalization and also to the related concept of conditional expectations which has played an important role in probability theory, ergodic theory and quantum statistical mechanics. We also contribute to the theory of set-valued martingales and since martingales are important in probability theory, we feel that there are potential applications to be developed from this work. This paper can be divided into two main results. The first result is an original convergence theorem for set-valued martingales. The second result is a generalization of a useful characterization of measurable set-valued random variables by Hiai and Umegaki in the fuzzy setting. The original theorem by Hiai and Umegaki was an essential tool used to obtain many of the subsequent results in [1] and it is clear that we can apply this new generalized theorem in an analagous way in the fuzzy setting.

We will be again considering a measure space (Ω, Σ, μ), a separable Banach space $(X, \|\cdot\|)$ with K a field of scalars (either $\mathbb{R}$ or $\mathbb{C}$). We remind the reader that functions f, g from Ω into X are said to be equal *almost everywhere*, denoted $f = g$ a.e. (μ) or $f(\omega) = g(\omega)$ $(\forall\omega \in \Omega)$ a.e. (μ) if $f(\omega) = g(\omega)$ for all $\omega \in \Omega$ except on a set of measure zero. We can omit reference to μ and Ω if there is no confusion. That is $f = g$ a.e. if $\mu(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0$. We will denote by $M[\Omega, X]$ and $L^p[\Omega, X]$ the collections of measurable and p-integrable functions $f : \Omega \to X$ respectively, for $1 \le p \le \infty$. Due to the completeness of X we have several characterizations of this class of functions (see [1] Theorem 1.0). As in [1], $K(X)$ shall denote the collection of nonempty compact subsets of X and $K_k(X)$ the collection of convex sets in $K(X)$. We denote the collection of (weakly) measurable set-valued functions $F : \Omega \to K_k(X)$ by $\mathcal{M}[\Omega, K_k(X)]$. That is $F \in \mathcal{M}[\Omega, K_k(X)]$ if and only if for all O open, $O \subset X$, $F^{-1}(O) \in \Sigma$. Let p be chosen such that $1 \le p \le \infty$ then we denote by $\mathcal{L}^p[\Omega, K_k(X)] = \mathcal{L}^p[\Omega, \Sigma, \mu, K_k(X)]$ the space of all p-integrable functions in $\mathcal{M}[\Omega, K_k(X)]$, where two functions $F_1, F_2 \in \mathcal{L}^1[\Omega, K_k(X)]$ will be considered identical if $F_1(\omega) = F_2(\omega)$ a.e.

The topology on $K_k(X)$ is induced by the Hausdorff metric defined by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\right\}, \tag{1.1}$$

for $A, B \subseteq X$. It is well known $(K_k(X), d_H)$ is a complete separable metric space. It now follows that in this topology $(F_n)_{n\in N} \subseteq \mathcal{M}[\Omega, K_k(X)]$ then $F_n \longrightarrow F$ as $n \longrightarrow \infty$ if and only if $d_H(F_n(\omega), F(\omega)) \longrightarrow 0$ for all $\omega \in \Omega$ a.e.

We can consider $(X, \|\cdot\|)$ to be a topological space since a norm defines a natural metric on X which induces the *metric topology* on X. We define the *closure* of a set $A \subseteq X$, denoted $\mathrm{cl}(A)$ to be the *norm closure* of A. That is, the smallest closed set containing A with respect to the metric topology induced by the norm $\|\cdot\|$.

A measurable function $f : \Omega \to X$ is called a *measurable selection* of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ a.e. That is f is a measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ except on a set of measure zero. We define the set $S_F^p = \{f \in L^p[\Omega, X] : f(\omega) \in F(\omega)\ a.e.\}$. It is easy to show that S_F^p is a closed subset of $L^p[\Omega, X]$ for $1 \le p \le \infty$.

Lemma 1.1 *[1] Let $F \in \mathcal{M}[\Omega, K_k(X)]$ and $1 \le p \le \infty$. If S_F^p is nonempty, then there exists a sequence $(f_i)_{i\in N}$ contained in S_F^p such that $F(\omega) = \mathrm{cl}\{f_i(\omega)\}$ for $\omega \in \Omega$.*

Corollary 1.2 *[1] Let $F_1, F_2 \in \mathcal{M}[\Omega, K_k(X)]$ and $1 \le p \le \infty$. If $S_{F_1}^p = S_{F_2}^p \neq \emptyset$ then $F_1(\omega) = F_2(\omega)$, for all $\omega \in \Omega$.*

A brief discussion of the generalization of conditional expectations is necessary before we reach our first main result. This material is covered more comprehensively in [1].

Let W be a sub-σ-algebra of Σ,

$$S_F^1(W) = \{f \in L^1[\Omega, W, \mu, X] : f(\omega) \in F(\omega)\, a.e.\}. \tag{1.2}$$

We define

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$$\int_{\Omega}^{(W)} F d\mu = \Big\{ \int_{\Omega} f d\mu : f \in S_F^1(W) \Big\}. \quad (1.3)$$

If $f \in L^1[\Omega, X]$ then the *conditional expectation* $E[f|W]$ *of* f *relative to* W is defined as a W-measurable function $E[f|W] \in L^1[\Omega, W, \mu, X]$ such that

$$\int_A E[f|W] d\mu = \int_A f d\mu, \quad (1.4)$$

for all $A \in W$.

The following theorem provides a natural extension of the notion of a conditional expectation to the set-valued setting.

Theorem 1.3 *[1] Let $F \in \mathcal{L}^1[\Omega, K_k(X)]$. Then there exists a unique $E[F|W] \in \mathcal{L}^1[\Omega, W, \mu, K_k(X)]$ such that*

$$S^1_{E[F|W]}(W) = \text{cl}\{E[f|W] : f \in S^1_F\} \quad (1.5)$$

where the closure is taken with respect to the norm in $\mathcal{L}^1[\Omega, K_k(X)]$. $E[F|W]$ is called the conditional expectation of F relative to W.

The concept of martingale in probability theory was introduced by Paul Pierre Lévy. Part of the motivation for that work was to show the impossibility of successful betting strategies. A martingale was originally devised as an indexed sequence of random variables with the index representing time. If t is a later time than s then the idea is that the conditional expected value at time t given the same observations as at time s will be equal to the expected value at time s. The notion of a martingale has proved to be a useful tool in modeling various events in probability theory. We present a precise definition of the classical notion followed by the generalization to the set-valued setting.

We remind the reader that a *filtration* on (Ω, Σ, μ) is a sequence of σ-algebras $(\Sigma_n)_{n\in N}$ such that $\Sigma_n \subseteq \Sigma$ and $\Sigma_n \subseteq \Sigma_{n+1}$ for all $n \in \mathbb{N}$.

Definition 1.4 *If $(f_n)_{n\in N}$ is a sequence of random variables and $(\Sigma_n)_{n\in N}$ a filtration then the sequence $(f_n, \Sigma_n)_{n\in N}$ is called a martingale if we have for each $n \in \mathbb{N}$:*

(a) f_n is Σ_n-measurable and

$$\int_{\Omega}^{(\Sigma_n)} \|f_n\| d\mu < \infty, \quad (1.6)$$

(b)

$$E[f_{n+1}|\Sigma_n] = f_n. \quad (1.7)$$

Alternatively, if property (b) is replaced by

(b')

$$E[f_{n+1}|\Sigma_n] \geq f_n (E[f_{n+1}|\Sigma_n] \leq f_n) \quad (1.8)$$

then $(f_n, \Sigma_n)_{n\in N}$ is called a submartingale (supermartingale), respectively.

Definition 1.5 *Let $(F_n)_{n\in N}$ be a sequence of set-valued random variables and $(\Sigma_n)_{n\in N}$ a filtration then the sequence $(F_n, \Sigma_n)_{n\in N}$ is called a set-valued martingale if we have, for each $n \geq 1$:*

(a) F_n is Σ_n-measurable and

$$\int_{\Omega}^{(\Sigma_n)} \|F_n\| d\mu < \infty, \quad (1.9)$$

(b)

$$E[F_{n+1}|\Sigma_n] = F_n. \quad (1.10)$$

Alternatively, if property (b) is replaced by

(b')

$$E[F_{n+1}|\Sigma_n] \geq F_n (E[F_{n+1}|\Sigma_n] \leq F_n) \quad (1.11)$$

then $(F_n, \Sigma_n)_{n\in N}$ is called a fuzzy submartingale (supermartingale), respectively.

Theorem 1.6 *If $(F_n, \Sigma_n)_{n\in N}$ is a set-valued martingale and $1 \leq p \leq \infty$ then $F_n \longrightarrow F$ as $n \longrightarrow \infty$ if and only if for each choice of $f_n \in S^p_{F_n}$, $n \in N$, we have that*

$$\lim_{n\longrightarrow\infty} f_n \in S^p_F. \quad (1.12)$$

Proof: Let p be such that $1 \leq p \leq \infty$. For brevity, we will henceforth denote $\lim_{n\longrightarrow\infty} f_n(\omega)$ by $f(\omega)$ for each $\omega \in \Omega$. Now $f_n \in S^p_{F_n}$ for all $n \in \mathbb{N}$ but $f \notin S^p_F \Longleftrightarrow$ for all $n \in \mathbb{N}$, there exists $A \in \Sigma$, $\mu(A) \neq 0$ such that $f(\omega) \notin F(\omega)$ for all $\omega \in A$. Now for a given $A \in \Sigma$ we have $\forall\omega \in A$, $f_n(\omega) \neq f(\omega) \Longleftrightarrow \forall\omega \in A$, $f(\omega) - f_n(\omega) \neq 0 \Longleftrightarrow \forall\omega \in A$, $\|f(\omega) - f_n(\omega)\| \geq \delta$ for some $\delta > 0$. Thus $f \notin S^p_F \Longleftrightarrow \exists A \in \Sigma, \mu(A) \neq 0, \forall\omega \in A, \forall n \in \mathbb{N}$,

$$d_H(F(\omega), F_n(\omega)) =$$

$$\max\{ \sup_{f(\omega)\in F(\omega)} \inf_{f_n(\omega)\in F_n(\omega)} \|f(\omega) - f_n(\omega)\|,$$

$$\sup_{f_n(\omega)\in F_n(\omega)} \inf_{f(\omega)\in F(\omega)} \|f(\omega) - f_n(\omega)\|\}$$

$\geq \delta \Longleftrightarrow d_H(F(\omega), F_n(\omega) \not\longrightarrow 0$ for all $\omega \in A \in \Sigma$, $\mu(A) \neq 0 \Longleftrightarrow d_H(F, F_n) \not\longrightarrow 0 \Longleftrightarrow F_n \not\longrightarrow F$ as $n \longrightarrow \infty$.

2 Fuzzy sets

We now provide a brief introduction to fuzzy sets with a particular focus on the tools that we will need in the subsequent section. These notions are all well known and standard. For further reading on the topic, the reader is referred to [2, 3, 4, 5, 8](by no means an exhaustive list). Throughout this paper, we will be considering fuzzy sets $A \in I^X$ where X is a Banach Space and I the unit interval $[0, 1]$.

For a given fuzzy set we associate collections of crisp subsets of X with it. This concept is highly useful in relating statements in the fuzzy setting to the classical setting.

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Definition 2.1 *If $A \in I^X$ and $\alpha \in I$ we define,*

$$A^\alpha = \{x \in X : A(x) > \alpha\}; \tag{2.13}$$

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}. \tag{2.14}$$

These crisp sets are referred to as α-levels (or cuts), strong and weak respectively. We call supp$A = A^0$ *the support of A.*

We shall work mainly with the weak cuts in this paper although similar results can be obtained using the strong cuts.

Throughout this paper we will denote the characteristic function of a crisp set $A \subseteq X$ by χ_A. That is $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ elsewhere. We immediately obtain a useful characterization of fuzzy sets.

Theorem 2.2 *If $A \in I^X$ and $x \in X$ then*

$$A(x) = \sup_{n \in N}\{\alpha_n \cdot \chi_{A_{\alpha_n}}(x)\} \tag{2.15}$$

where $\{\alpha_n : n \in \mathbb{N}\}$ is dense in I.

We remind the reader of the standard definitions of addition and scalar multiplication of crisp sets. $A + B$ denotes the collection $\{a + b : a \in A, b \in B\}$ and for a scalar α, we denote by $\alpha \cdot A$ the collection $\{\alpha \cdot a : a \in A\}$. There are natural extensions of these concepts in the fuzzy setting and it should be noted that the following two definitions are consequences of the well known concept of a fuzzy image.

Definition 2.3 *Let $A, B \in I^X$ we define*

$$(A + B)(x) = \sup_{a+b=x}\{A(a) \wedge B(b)\} \tag{2.16}$$

for all $x \in X$.

Definition 2.4 *Let $A \in I^X, t \in K$ and $x \in X$. Then we define $t \odot A(x) = A(\frac{x}{t})$ for $t \neq 0$ and*

$$t \odot A(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \sup A & \text{if } x = 0 \end{cases} \tag{2.17}$$

if $t = 0$.

The $\odot$ is simply used to distinguish between this type of scalar multiplication and the $\cdot$ representing ordinary pointwise multiplication of functions as used in Theorem 2.2.

Due to the lattice structure of I, we have that pointwise ordering is a natural partial ordering that can be defined on I^X. That is $\leq$ given by

$$A \leq B \Leftrightarrow A(x) \leq B(x) \tag{2.18}$$

for all $x \in X$ and $A, B \in I^X$. This leads to the following straightforward relationships between the concepts of α-levels and this ordering respectively. The proofs follow trivially.

Lemma 2.5 *If $A, B \in I^X$ then for all $\alpha \in [0, 1]$ the following statements hold*

1.

$$[A \vee B]_\alpha = A_\alpha \cup B_\alpha, \tag{2.19}$$

2.

$$[A \wedge B]_\alpha = A_\alpha \cap B_\alpha, \tag{2.20}$$

3.

$$A \leq B \Longleftrightarrow A_\alpha \subseteq B_\alpha. \tag{2.21}$$

We have the following immediate consequence of the previous lemma.

Lemma 2.6 *Let A and B be fuzzy sets. Then*

$$A = B \Longleftrightarrow A_\alpha = B_\alpha \tag{2.22}$$

for all $\alpha \in [0, 1]$.

Definition 2.7 *A fuzzy set A on X is convex if $A(kx+(1-k)y) \geq A(x) \wedge A(y)$ whenever $x, y \in X$ and $0 \leq k \leq 1$.*

The following proposition is a useful characterization of fuzzy convexity which justifies the definition and relates it to the corresponding crisp definition.

Proposition 2.8 *Let A be a fuzzy set in X then the following three assertions are equivalent:*

1. *A is convex.*
2. $\forall k \in [0, 1], kA + (1 - k)A \leq A$.
3. *$\forall \alpha \in I$, A_α is convex.*

We will later require the next theorem which illustrates the relationship between the scalar multiplication and addition of fuzzy sets and the corresponding concepts in the crisp setting.

Lastly we discuss the natural concept of fuzzy points since this will be necessary for the next section. We use $\mathcal{P}(X)$ to denote the *power set* of X - that is, $\mathcal{P}(X) = \{A : A \subseteq X\}$.

If $A \in \mathcal{P}(X)$ and $\alpha \in I$ we define

$$\alpha \cdot \chi_{\{x\}} = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.23}$$

$$\alpha \cdot \chi_{\{x\}} = \begin{cases} \alpha & \text{on } x \\ 0 & \text{elsewhere} \end{cases} \tag{2.24}$$

So

We call $\alpha \cdot \chi_{\{x\}}$ a *fuzzy point* with *support* at x and *value* α. We will denote the set of fuzzy points in I^X by $\hat{X}$.

3 Fuzzy set-valued random variables

Our next focus is on the generalization of set-valued random variables to the fuzzy setting. The following overview of fuzzy set-valued random variables is taken mainly from [5] and [9]. These concepts are also discussed in [6, 7] amongst other works.

Definition 3.1 *Let* $\mathrm{F(X)}$ *denote the collection of fuzzy sets such that*

(a) *A is uppersemicontinuous,*

(b) *$\{x \in X : A(x) = 1\} \neq \emptyset$,*

(c) supp$A \in K(X)$.

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We define $F_k(X)$ to be the collection of fuzzy sets in $F(X)$ that are fuzzy convex. We denote the collection of measurable functions $f : \Omega \to \hat{X}$ by $M[\Omega, \hat{X}]$, that is, $f \in M[\Omega, \hat{X}]$ if and only if for each $\alpha \in (0,1]$, $f_\alpha \in M[\Omega, X]$ with $f_\alpha(\omega) = [f(\omega)]_\alpha$ for all $\omega \in \Omega$. Let p be chosen such that $1 \le p \le \infty$ then $L^p[\Omega, \hat{X}]$ will denote the p-integrable functions from Ω into $\hat{X}$. A fuzzy set-valued random variable (f.r.v.) or a fuzzy random set is a function $F : \Omega \to \mathrm{F_k(X)}$ such that $F_\alpha(\omega) = \{x \in X : F(\omega)(x) \ge \alpha\}$ is a set-valued random variable for all $\alpha \in (0,1]$. A fuzzy set-valued random variable F is called measurable (integrable) if for each $\alpha \in (0,1]$, F_α is measurable (integrable). Let $\mathcal{M}[\Omega, F_k(X)]$ be the collection of all measurable fuzzy random variables and $\mathcal{L}^p[\Omega, \mathrm{F_k(X)}] = \mathcal{L}^p[\Omega, \Sigma, \mu, \mathrm{F_k(X)}]$ denote the set of all p-integrable fuzzy random variables for $1 \le p \le \infty$. Clearly, a fuzzy set-valued random variable can be considered to be a generalized set-valued random variable. F_1, F_2 fuzzy set-valued random variables then $(F_1 + F_2)(\omega) = F_1(\omega) + F_2(\omega)$ for all $\omega \in \Omega$ (the normal addition of fuzzy sets defined earlier). Similarly for a fuzzy set-valued random variable F and measurable real valued function ζ on Ω, $(\zeta F)(\omega) = \zeta(\omega)F(\omega)$ for all $\omega \in \Omega$. We will also generalize Theorem 2.8 to the fuzzy setting. There are several ways to generalize d_H to the fuzzy setting but due to certain desirable properties we will restrict our attention to d_∞ defined by

$$d_\infty(F_1, F_2) = d_H(\mathrm{supp}F_1, \mathrm{supp}F_2). \tag{3.25}$$

It follows trivially from our earlier discussion of the Hausdorff metric that $(F_k(X), d_\infty)$ is a metric space. Under the topology induced by d_∞ we have that $F_n \longrightarrow F$ if and only if $d_\infty(F_n(\omega), F(\omega)) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\omega \in \Omega$ a.e.

Based on the notation used in [1] we now generalize the definition of S_F^p to the fuzzy setting.

Definition 3.2 *A measurable function $f \in M[\Omega, \hat{X}]$ such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ a.e. is called a measurable selection of F. Let $F \in \mathcal{M}[\Omega, F_k(X)]$ and let $1 \le p \le \infty$ then $S_F^p = \{f \in L^p[\Omega, \hat{X}] : f(\omega) \in F(\omega)$ a.e. $\}$.*

We now state the second of our main results. This is a new generalization of Lemma 1.1 - notice that the collection of fuzzy points $\hat{X}$ replaces the set X in this version of the theorem. This theorem opens the way for similar results to those obtained in [1].

Theorem 3.3 *If $F \in \mathcal{M}[\Omega, F_k(X)]$ then there exists a countable collection $\{f^i\}_{i\in N} \subseteq S_F^p$, such that $F(\omega) = \mathrm{cl}_{i\in N}\{f^i(\omega)\}$ for all $\omega \in \Omega$.*

Proof: Let $F \in \mathcal{M}[\Omega, F_k(X)]$ and let $\omega \in \Omega$. Then for $n \in \mathbb{N}$ we have $F_{\alpha_n} \in \mathcal{M}[\Omega, K_k(X)]$ with $\{\alpha_n\}_{n\in N}$ a countable set dense in I as in Theorem 2.2. Therefore by Lemma 1.1, for each $n \in \mathbb{N}$, $\exists\{f^i_{\alpha_n}\}_{i\in N}$ such that

$$\mathrm{cl}_{i\in N}\{f^i_{\alpha_n}(\omega)\}(x) = F_{\alpha_n}(\omega).$$

For each $i \in \mathbb{N}$ we now define

$$f^i(\omega) = \bigvee_{n\in N} \alpha_n \cdot \chi_{f^i_{\alpha_n}(\omega)}.$$

Then $\{f^i\}_{i\in N}$ is the required collection. Let $x \in X$ and now notice that for each $i \in \mathbb{N}$, $f^i : \Omega \to \hat{X}$. We now have that $\mathrm{cl}_{i\in N}\{[f_i(\omega)]\}(x) = \mathrm{cl}_{i\in N}\{[f_i(\omega)](x)\} = \mathrm{cl}_{i\in N}\{\bigvee_{n\in N}\alpha_n \cdot \chi_{f^i_{\alpha_n}(\omega)}(x)\} = \lim_{j\longrightarrow\infty} b_j$ where for each $j \in \mathbb{N}$, $b_j \in \left\{\bigvee_{n\in N}\alpha_n \cdot \chi_{f^i_{\alpha_n}(\omega)}(x) : i \in \mathbb{N}\right\}$. Now for each $j \in \mathbb{N}$, $b_j = \lim_{k\longrightarrow\infty} b_j^k$ with $b_j^k \in \{\alpha_n \cdot \chi_{f^i_{\alpha_n}(\omega)}(x) : i, n \in \mathbb{N}\} \subseteq I$. Thus

$$\mathrm{cl}_{i\in N}\{f_i(\omega)\}(x) = \lim_{j\longrightarrow\infty}\lim_{k\longrightarrow\infty} b_j^k, \tag{3.26}$$

for $\{b_j^k\}_{j,k\in N} \subseteq \{\alpha_n \cdot \chi_{f^i_{\alpha_n}(\omega)}(x) : i, n \in \mathbb{N}\} = \left\{[f^i_{\alpha_n}(\omega)](x) : i, n \in \mathbb{N}\right\}$ and b_j^k is nondecreasing with respect to k.

We also have by Theorem 2.2 that $[F(\omega)](x) = \left(\bigvee_{n\in N}\alpha_n \cdot \chi_{F_{\alpha_n}(\omega)}\right)(x)$

$= \left(\bigvee_{n\in N}\alpha_n \cdot \chi_{\mathrm{cl}_{\{f^i_{\alpha_n}(\omega)\}}}\right)(x) = \left(\bigvee_{n\in N}\alpha_n \cdot \chi_{\mathrm{cl}_{\{f^i_{\alpha_n}(\omega)\}}}(x)\right) \in I$.

By definition $\left(\bigvee_{n\in N}\alpha_n \cdot \chi_{\mathrm{cl}_{\{f^i_{\alpha_n}(\omega)\}}}(x)\right) = \lim_{k\longrightarrow\infty} b^k$ where $(b^k)_{k\in N} \subseteq \{[f^i_{\alpha_n}(\omega)](x) : n \in \mathbb{N}\} \subseteq I$ and b^k is nondecreasing with respect to k. Also for each $k \in \mathbb{N}$ we have that $b^k = \lim_{j\longrightarrow\infty} b_j^k$ with $b_j^k \in \left\{[f^i_{\alpha_n}(\omega)](x) : i, n \in \mathbb{N}\right\}$ for each $j, k \in \mathbb{N}$. Therefore we have

$$[F(\omega)](x) = \lim_{k\longrightarrow\infty}\{\lim_{j\longrightarrow\infty} b_j^k\} \tag{3.27}$$

This concludes the proof since it is well known from real analysis that

$$\lim_{k\longrightarrow\infty}\{\lim_{j\longrightarrow\infty} b_j^k\} = \lim_{j\longrightarrow\infty}\{\lim_{k\longrightarrow\infty} b_j^k\}$$

for all $b_j^k \in I$ with $j, k \in \mathbb{N}$.

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