# The Use of Interval-Valued Probability Measures in Fuzzy Linear Programming: A Constraint Set Approach

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***Abstract***—*This paper uses a constraint set approach to linear programming problems with equality constraints whose coefficients and/or right-hand side values could be uncertain. We consider three types of uncertainty: probabilistic, fuzzy/possibilistic, and interval. The constraint set generated by the linear constraints under uncertainty is ill-defined and difficult to generate. Our approach computes an inner constraint set and an outer constraint set. Optimization is then carried out using these two sets using interval-valued probability approaches. We discuss the requisite associated semantics.*

***Keywords***—inverval-valued probability measure, random sets, inner constraint set, outer constraint set.

## I. Introduction

In order to solve linear programming problems under uncertainty, many researchers convert the problem into a sequence of deterministic problems. The approach in this paper is to work with the constraint set of the problem.

It is difficult (exponentially hard) to find the exact shape of the feasible set of uncertain constraints, as we shall see. Therefore we create two sets (boxes); one represents the set of solutions guaranteed to have a corresponding data to satisfy the constraints while the other is the set that contains all solutions. They are called the *inner set* and the *outer set*, respectively. We use these two sets in our optimization problem.

We present how to obtain these two sets in section III after giving some useful definitions. In section IV, we create an interval-valued probability measure (IVPM) by a pair of possibility and necessity measures on the space between the inner and outer sets.

In section V, we solve an uncertain optimization problem, at each $\alpha$-level of the uncertain data, on the inner box and give a guaranteed bound on the objective value. We use an IVPM to get the bound on the objective value with some $\gamma$ degree of occurrence. The final section articulates some conclusions.

## II. Background and Definitions

We consider linear programming (LP) problems with uncertain parameters of the form

$$\min_x \breve{c}^{\mathrm{T}} x \quad \text{s.t.} \quad \breve{A}x = \breve{b}. \tag{1}$$

$\breve{A}, \breve{b}$ and $\breve{c}$ may be (or are) uncertain. Without knowing the type of uncertainty, we use $\breve{b}_j$ and $\breve{c}_j$ as the $j^{\text{th}}$ component of the uncertain vectors $\breve{b}$ and $\breve{c}$, respectively. Similarly, $\breve{a}_{ij}$ is the $(i, j)$ component of the uncertain matrix $\breve{A}$.

The main types of uncertainties are probabilistic, fuzzy/possibilistic and interval. When we know the type of uncertain component $\breve{a}_{ij}$, we can specify the uncertainty by the following symbols. (The same symbols also apply for components $\breve{b}_j$ and $\breve{c}_j$.)

- $\tilde{a}_{ij} \equiv$ a fuzzy component $a_{ij}$,
- $\hat{a}_{ij} \equiv$ a possibilistic component $a_{ij}$,
- $\check{a}_{ij} \equiv$ a probabilistic component $a_{ij}$,
- $[a_{ij}] = \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \equiv$ an interval component $a_{ij}$.

*Definition 2.1:* Given an LP problem (1), the constraint set $\Omega_{\exists\exists} := \left\{x_{Ab} \mid \exists A \in \breve{A}, \exists b \in \breve{b} \ : \ Ax_{Ab} = b\right\}$ is said to be the set of all solution $x$'s for the constraints of (1).
The set $\Omega_{\exists\exists}$ may be (usually is) a non-convex set.

*Definition 2.2:* A finite *random set* on $S$ is a pair $(\mathscr{F}, m)$ where $\mathscr{F}$ is a finite family of distinct non-empty subsets of $S$ and $m$ is a mapping $\mathscr{F} \to [0, 1]$ such that $\sum_{A\in\mathscr{F}} m(A) = 1$.

We can also define a random set on $S$ when $S$ is infinite by using the measure space $(S, \mathscr{S}, m)$, where $\mathscr{S}$ is a $\sigma$-algebra of $S$. Therefore, a random set on an infinite set $S$ is a pair $(\mathscr{F}, m)$ where $\mathscr{F}$ is a subset of $\mathscr{S}$ such that $\sum_{A\in\mathscr{F}} m(A) = 1$.

In real applications, we might not know (with certainty) the probability measure for our problems. Lodwick and Jamison [5] use an IVPM, $i_{\breve{m}}(A) = \left[i^-_{\breve{m}}(A), i^+_{\breve{m}}(A)\right]$, to measure a partial representation for an unknown probability measure. The original paper for the idea of IVPM is Weichselberger [12]. We use the following notation and information throughout the paper unless stated otherwise:

- The arithmetic operations applied to intervals are those of interval arithmetic [7] and where tractable, constraint interval arithmetic [3], [6].
- The set of all intervals contained in $[0, 1]$ is denoted as

$$\mathrm{Int}_{[0,1]} \equiv \{[a, b] \mid 0 \le a \le b \le 1\}.$$

- $\mathcal{S}$ denotes the universal set and $\mathcal{A}$ is a $\sigma$-algebra of $\mathcal{S}$.

*Definition 2.3:* (Weichselberger [12]) Given measurable space $(\mathcal{S}, \mathcal{A})$, an interval valued function $i_{\breve{m}} : A \subseteq \mathcal{A} \to \mathrm{Int}_{[0,1]}$ is called an *R-probability* if:

- $i_{\breve{m}}(A) = \left[i^-_{\breve{m}}(A), i^+_{\breve{m}}(A)\right] \subseteq [0, 1]$,

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- $\exists$ a probability measure, Pr, on $\mathcal{A}$ such that $\forall A \in \mathcal{A}$, $\Pr(A) \in i_{\breve{m}}(A)$.

*Definition 2.4:* A function $p : \mathcal{S} \to [0,1]$ is called a *regular possibility distribution function* if it is a possibility such that

$$\sup\{p(x) \mid x \in \mathcal{S}\} = 1. \tag{2}$$

Possibility distribution functions (see [11]) define a possibility measure, $Pos : \mathcal{S} \to [0,1]$ where

$$Pos(A) = \sup\{p(x) \mid x \in A\} \tag{3}$$

and its dual necessity measure is

$$Nec(A) = 1 - Pos(A^c), \tag{4}$$

where $\sup\{p(x) \mid x \in \emptyset\} = 0$.
In [4], it is shown that possibility distributions can be constructed which satisfy the following consistency definition.

*Definition 2.5:* Let $p : \mathcal{S} \to [0,1]$ be a regular possibility distribution function with associated possibility measure $Pos$ and necessity measure $Nec$. Then $p$ is said to be *consistent* with random variable $X$ if for every measurable set $A$, $Nec(A) \leq \Pr(X \in A) \leq Pos(A)$. Note that this necessity measure $Nec$ may not be the dual of $Pos$ in this definition.

An R-probability from definition 2.3 is an *interval-valued probability measure* (IVPM) where $i^-_{\breve{m}}$ and $i^+_{\breve{m}}$ are constructed from a possibility (fuzzy) distribution function. To see this is an IVPM see [4].

The R-probability function $i_{\breve{m}}$ in definition 2.3 is used to define IVPMs. A possibility and necessity pair, $i_{\widehat{m}}(A) = [Nec(A), Pos(A)]$, constructed by definition 2.5 is able to bound an unknown probability of interest. Therefore it can be used to define an IVPM.

*Definition 2.6:* The Interval-Valued Probability Measure (IVPM) defined from possibility and necessity measures is $i_{\widehat{m}}(A) = [Nec(A), Pos(A)]$.
The reader could find more explanations, examples and a construction of an IVPM in [5] .

## III. Generating Two Sets from Equality Constraint Set

In this section, we consider only constraints of (1),

$$\breve{A}x = \breve{b}. \tag{5}$$

Although the set of all solution $x$'s, $\Omega_{\exists\exists}$, for (5) is difficult to construct, it is relatively easy to generate two random sets for each $\alpha$-level called *inner set*, $B^I_\alpha$, and *outer set*, $B^O_\alpha$, such that $B^I_\alpha \subseteq B^O_\alpha$. Each $x$ in $B^I_\alpha$ has a particular matrix $A_x \in A_\alpha$ and vector $b_x \in b_\alpha$, such that $A_x x = b_x$. An outer set, $B^O_\alpha$, represents a space where $x$ should be. This implies that $\Omega_{\exists\exists} \subseteq B^O$.

### A. Interval case

We drop the $\alpha$ subscript when we work with interval version of equation (5).

A system

$$[A]x = [b] \tag{6}$$

is called *solvable* if $Ax = b$ has a solution for each $A \in [A]$ and for each $b \in [b]$, where $[A] = [\underline{A}, \overline{A}]$ and $[b] = [\underline{b}, \overline{b}]$.

Gay [2] gives a method for solving interval linear equations when $[A]$ is a regular square interval uncertain matrix, i.e., $A$ is invertible, $\forall A \in [A]$. Rohn [8] provides a powerful theorem for any $m \times n$ interval matrix $[A]$ that guarantees solvability by testing nonnegative solutions of a finite number of linear systems.

The following notations are useful for theorem 3.1 below.

- $A_c = \frac{(\underline{A}+\overline{A})}{2}$, the center matrix.
- $\Delta = \frac{(\overline{A}-\underline{A})}{2}$, the radius matrix.
- $b_c = \frac{(\underline{b}+\overline{b})}{2}$ and $\delta = \frac{(\overline{b}-\underline{b})}{2}$.
- $Y_m = \{y \in \mathrm{R}^m \mid y_j \in \{-1,1\}, \forall j = 1,2,\ldots m\}$.
- $T_y = \mathrm{diag}(y_1, y_2, \ldots, y_m)$, for each $y \in Y_m$.
- $\mathrm{Conv}(X) = \{\gamma p_1 + (1-\gamma)p_2 \mid p_1, p_2 \in X,\ \gamma \geq 0\}$.

*Theorem 3.1 (see Rohn [8]):* A system of linear equations $[A]x = [b]$ is solvable if and only if for each $y \in Y_m$ the system

$$\left.\begin{array}{c}(A_c - T_y\Delta)\,x^1 - (A_c + T_y\Delta)\,x^2 = b_c + T_y\delta, \\ x^1 \geq 0,\ \ x^2 \geq 0\end{array}\right\} \tag{7}$$

has a solution $x^1_y, x^2_y$. Moreover, if this is the case, then for each $A \in [A], b \in [b]$ the system $Ax = b$ has a solution in the set $\mathbf{C} := \mathrm{Conv}\left\{x^1_y - x^2_y \mid y \in Y_m\right\}$.

Theorem 3.1 tells us that there is a solution for the system $Ax = b$ in the set $\mathbf{C}$. However, it does not mean that all the solutions of the system $Ax = b$ are in $\mathbf{C}$, especially when $A$ is not a square matrix.

*Corollary 3.1 (see [1]):* A system of linear equations $[A]x = [b],\ x \geq 0$ is solvable if and only if for each $y \in Y_m$ the system

$$(A_c - T_y\Delta)\,x = b_c + T_y\delta, \tag{8}$$

has a nonnegative solution $x_y$. Moreover, if this is the case, then for each $A \in [A], b \in [b]$ the system $Ax = b,\ x \geq 0$ has a solution in the set $\mathrm{Conv}\{x_y \mid y \in Y_m\}$.

Now we are going to generate outer and inner sets given the assumption that our constraint set is solvable.

*1) An outer set, $B^O$:* We will consider two cases when $m = n$ and when $m < n$.

- When $m = n$. Suppose that a system of linear equation (6) is solvable by theorem 3.1 and $[A]$ is a regular matrix, we can create a rectangular box that contains all the solution points $x^1_y - x^2_y$ of the system (7). To see this simply take $\max/\min$ of components. We prefer to use the rectangular box as an outer set over the set $\mathrm{Conv}\left\{x^1_y - x^2_y;\ y \in Y_m\right\}$ because it is easier to construct. We can use $2^{m^2}2^m$ endpoint equations to generate $\mathbf{C}$, hence creating a rectangular box, $B^O$.
  We will not consider the case when $[A]$ is not a regular matrix. If there exists a singular matrix $A \in [A]$, then $B^O$ is an unbounded set.
- When $m < n$. Each system $Ax = b$ in the constraint set (6) provides infinitely many solutions in the form of $n - r$ parameters where $\mathrm{rank}(A) = r$. For

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example, $2x_1 + x_2 = 3$ has solutions as set $S = \{(t, 3-2t) \mid t \in \mathrm{R}\}$. This leads unbounded objective value for the linear programming (1). However, if the constraint set is in the form

$$[A]x = [b];\ x \geq 0, \tag{9}$$

the LP might be bounded (even if it has unbounded feasible set).

Suppose that a system of linear equation (9) is solvable by corollary 3.1. Each system (8) has infinitely many solutions of dimension $n - r$ which are bounded below by $x = 0$. For each $y \in Y_m$, let $S_y = \{x_y \mid (A_c - T_y\Delta)\, x_y = b_c + T_y\delta,\ x_y \geq 0\}$. Then an outer set $B^O = \mathrm{Conv}\,\{S_y \mid y \in Y_m\}$.

*2) An inner set, $B^I$:* To build an inner set, $B^I$, we also consider two cases when $m = n$ and when $m < n$.

- When $m = n$, we work through the following steps.
  - Step 1 : Choose $x_*$ as one of the point $x_y^1 - x_y^2$ in theorem 3.1 that creates $B^O$.
  - Step 2 : Find $A_*$ and $b_*$ such that $A_* x_* = b_*$ The matrix $A_*$ and vector $b_*$ is one of the endpoint matrices and vectors created by endpoints of $[A]_{m\times n}$ and $[b]_{n\times 1}$, respectively.
  - Step 3 : An inner set, $B^I$, is the set $\mathrm{Conv}\,\{x^* \mid A_* x^* = b^*\}$. There are $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{m}$ different $b^*$'s created by changing any component(s) of $b_*$ to the other endpoint(s). For example, if $[b] = \begin{bmatrix}[1,3]\\ [0,2]\end{bmatrix}$ and $b_* = \begin{bmatrix}1\\2\end{bmatrix}$, then the 4 choices of $b^*$ are $b_*$, $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$.
- When $m < n$, we work through the following steps.
  - Step 1 : Choose two elements $y^1$ and $y^2$ in $Y_m$.
  - Step 2 : $B^I = \mathrm{Conv}\left\{S_{y^1}, S_{y^2}\right\}$.

To see that these $B^O$ and $B^I$ are random sets, we define $m(B^I) = 1$ and $m(K) = 0$ for other $\sigma$-algebra $K$'s of $B^O$.

### *B. General case*

When we have mixed uncertain information for (5); for instance, when $\breve{A}$ is an uncertain matrix of $1 \times 2$ dimension such that $\breve{a}_{11} := \hat{a}_{11}$ with some possibility distribution and $\breve{a}_{12} := [\underline{a}_{12}, \overline{a}_{12}]$, we need to consider each component separately by using $\alpha$-levels. The semantics for each $\alpha$-level of different types of uncertainty is as follow. (Similar idea applies for $\breve{b}_i$).

*1) Fuzzy uncertainty:* An $\alpha$-level set, $\left[\underline{a}_{ij_\alpha}, \overline{a}_{ij_\alpha}\right]$, of a fuzzy number $\tilde{a}_{ij}$ represents an interval that every element in it has degree of satisfaction equal to or greater than $\alpha$.

*2) Possibilistic uncertainty:* An $\alpha$-level set, $\left[\underline{a}_{ij_\alpha}, \overline{a}_{ij_\alpha}\right]$, of a possibility distribution $\hat{a}_{ij}$ expresses that every element in the set has degree of occurrence equal to or greater than $\alpha$.

*3) Probabilistic uncertainty:* A probabilistic component $\breve{a}_{ij}$ with the corresponding density function $f$. Define $h := \max\{f(a) \mid a \in \breve{a}_{ij}\}$, an interval $\left[\underline{a}_{ij_{\alpha h}}, \overline{a}_{ij_{\alpha h}}\right]$ coincides with the $\alpha$-level set of two previous cases and it means that $\forall a \in \left[\underline{a}_{ij_{\alpha h}}, \overline{a}_{ij_{\alpha h}}\right],\ f(a) \geq \alpha h$.

At each $\alpha$-level, the uncertainty acts as an interval. However, each interval has a history of the original type of uncertainty, (III-B.1, III-B.2 or III-B.3). So the semantics must be different from the interval case.

We present $B_\alpha^I$ and $B_\alpha^O$ when $\alpha = 0, \frac{1}{k}, \frac{2}{k}, \ldots, 1,\ k \in \mathrm{N}$. We define $[A]_\alpha$ as a matrix uncertainty whose each component is restricted to that particual $\alpha$-level. (Similar definition applies to $[b]_\alpha$). Moreover, if system $[A]_\alpha x = [b]_\alpha$ is solvable, we write $\mathbf{C}$ in the result of the theorem 3.1 as $\mathbf{C}_\alpha$.

To generate $B_\alpha^O$, we start with $\alpha = 0$. That means we consider a system $[A]_0 x = [b]_0$. Suppose that this system is solvable, we can create $B_0^O$ as shown in the interval case III-A.1. Therefore it is not hard to see that $B^O_{\frac{i}{k}} \supseteq B^O_{\frac{j}{k}}, \forall\, i \leq j \leq k,\ i, j, k \in \mathrm{N}$.

In the case of $B_\alpha^I$, we adopt the construction in III-A.2 as follows.

- When $m = n$:
  - Step 1 : When $\alpha = 0, (i = 0)$, we deal with $[A]_0 x = [b]_0$. We use the method in III-A.2 for finding $B_0^I$. We also get $x_*, A_*$ and $b_*$ s.t. $A_* x_* = b_*$.
  - Step 2 : Set $c := x_*$ as a reference point.
  - Step 3 : Set $i := i + 1$. Now we deal with $[A]_{\frac{i}{k}} x = [b]_{\frac{i}{k}}$. Choose the point $x_*$ that creates $B^O_{\frac{i}{k}}$ so that $\|x_* - c\| = \min_x \{\|x - c\|\}$, where $x$ is the other point $x_y^1 - x_y^2$ of $B^O_{\frac{i}{k}}$. Follow Step 2 in III-A.2.
  - Step 4 : An inner set $B^I_{\frac{i}{k}} = B^I_{\frac{i-1}{k}} \cap \mathrm{Conv}\,\{x^* \mid A_* x^* = b^*\}$ where $b^*$'s are the different endpoints of $[b]_{\frac{i}{k}}$.
  - Step 5 : Go back to Step 2 , or quit when $i = k$.
- When $m < n$: At each $i^{\text{th}}$ iteration, we need to choose $y^1, y^2 \in Y_m$ so that $B^I_{\frac{i-1}{k}} \cap \mathrm{Conv}\left\{S_{y^1}, S_{y^2}\right\} \neq \emptyset$. Then $B^I_{\frac{i}{k}} = B^I_{\frac{i-1}{k}} \cap \mathrm{Conv}\left\{S_{y^1}, S_{y^2}\right\}$.

Using this construction, we have $B^I_{\frac{i}{k}} \supseteq B^I_{\frac{j}{k}};\ \forall i \leq j \leq k,\ i, j, k \in \mathrm{N}$.

Therefore the outer and inner sets maintain the following properties.

*Theorem 3.2:* For all $i \leq j \leq k$ and $i, j, k \in \mathrm{N}$,

$$B^O_{\frac{j}{k}} \subseteq B^O_{\frac{i}{k}} \text{ and } B^I_{\frac{j}{k}} \subseteq B^I_{\frac{i}{k}}.$$

*Proof:* Follow directly from the construction of the sets. ⋄

Using $m(B^I_{\frac{i}{k}}) = \frac{1}{k}, \forall\ i = 1, 2, \ldots, k$ and $m(K) = 0$ for other $\sigma$-algebra $K$'s of $B_0^O$, we create the random set $\left\{B^0_{\frac{i}{k}}, i = 0, 1, \ldots, k\right\}$ and the random set $\left\{B^I_{\frac{i}{k}}, i = 0, 1, \ldots, k\right\}$.

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## IV. Interval-valued probability measure on the set $B^O_\alpha \backslash B^I_\alpha$

In this section we consider each $\alpha$-level separately. For convenience, we drop the subscript $\alpha$ of $B^O_\alpha$ and $B^I_\alpha$.

After we have an inner set $B^I$ and an outer set $B^O$, we consider the space between these two sets, $B^O \backslash B^I$, as our universal space of consideration. Let $X$ be a random variable with an unknown distribution. For any space $B \subseteq B^O \backslash B^I$, there is a probability that $X = x \in \Omega_{\exists\exists}$ is in $B$. We write this probability as $\Pr(x \in B)$. However, we lack information to justify for sure what the value is.

Each of the figures 1(a) and 1(b) illustrates a geometric interpretation of $B^I$ and $B^O$ in $\mathrm{R}^2$ of a system $[A]x = [b]$ where $[A]$ is regular with $m = n = 2$. Inner sets are the black areas while outer sets are the areas of rectangular boxes (that look like squares in the figures below). It can be difficult to give a geometric interpretation of $B^I$ and $B^O$, especially in higher dimension.

The points $p, q, r$ and $s$ that create $B^O$ in each figure are $x^1_y - x^2_y$ where $(x^1_y, x^2_y)$ is a solution of each system (7) in theorem 3.1. As we said in the construction of an outer set that it is easier to get a rectangular box in practice. Therefore, we use the rectangular box in 1(b) to represent $B^O$ instead of $\mathrm{Conv}\,\{p, q, r, s\}$.

When we choose $x_* = p$ in the construction of an inner set, we obtain an inner set as the parallelogram black set in both 1(a) and 1(b) as can be seen.

A point $z$ is more likely to be in the set $\Omega_{\exists\exists}$ than a point $y$ that is further away from $B^I$. We can create an possibility distribution of a particular nested sets by using this fact.

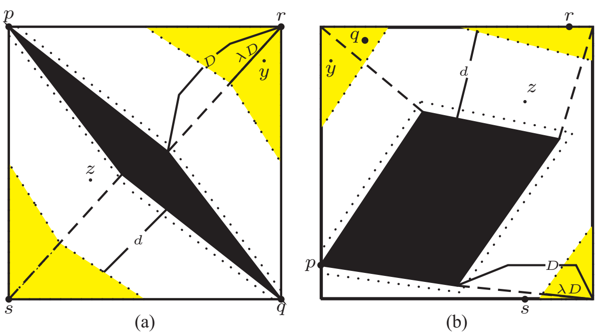


Fig. 1. Inner and outer boxes: inner boxes are the black areas and outer boxes are the rectangular sets.

We choose two corresponding corner points of $B^I$ and $B^O$ that make the longest distance, i.e., the length of one of dashed lines in figure 1(a) (or (b)). We use $D$ to represent this longest distance. Let $0 \le \lambda \le 1$ be a scalar. An easy possibility distribution of the nested sets $V(\lambda D)$ could be a function in the term of $\lambda$, for instance,

$$\mathrm{Poss}\ (V(\lambda D)) = \lambda, \quad \text{when} \quad 0 \le \lambda \le 1. \tag{10}$$

$V(\lambda D)$ in the domain of the function (10) represents the yellow areas (volume for higher dimensions) in the Figure 1. For $\lambda_1 \le \lambda_2$, we have $V(\lambda_1 D) \subseteq V(\lambda_2 D)$. When $\lambda$ is increasing, the possibility that $V(\lambda D)$ contains solutions $x \in \Omega_{\exists\exists}$ is also increasing.

By restricting to the nested domain, we would be able to work in the white space between $B^I$ and $B^O$. The white space has the width from the edge of the black and yellow areas to be $d$, as shown in figure 1. The distance $d$ is less than or equal to $(1 - \lambda)D$. We also can use an appropriate necessity distribution, Nec $(V(\lambda D))$.

However, in order to satisfy the definition of an IVPM, we need to make sure that the unknown probability measure is in the bound of our necessity and possibility measures. Therefore, the equation (10) may not be appropriate for most of the systems. We leave this part to the decision maker to come up with a reasonable nestedness necessity and possibility distributions that (s)he thinks they are an lower and upper bounds on the unknown probability.

Thus an IVPM with the restricted nested domain $V(\lambda D)$ is

$$i_{\mathrm{m}}(V(\lambda D)) = [\mathrm{Nec}\ (V(\lambda D)), \mathrm{Poss}\ (V(\lambda D))] \tag{11}$$

where Poss $(V(\lambda D))$ and Nec $(V(\lambda D))$ are defined to cover the unknown probability by the expert's opinion. In the next section, we show the use this IVPM in LP problems.

## V. Optimization on boxes using IVPMs

To deal with an optimization problem (1), it would make sense to consider on each $\alpha$-level of each uncertainty.

As we know for sure that the solutions in $B^I_\alpha$ have their correspondent matrix $A$ and vector $b$, one approach for this problem (1), restricted on the constraint set $B^I_\alpha$, can be

$$\min_{x \in B^I_\alpha} \check{c}_\alpha x, \tag{12}$$

where $\check{c}_\alpha = [\underline{c}_\alpha, \overline{c}_\alpha]$. By solving $\min \underline{c}_\alpha x$ and $\min \overline{c}_\alpha x$ on the convex polyhedral $B^I_\alpha$, we obtain the guaranteed bound on the objective value.

Moreover, since $\Omega_{\exists\exists_\alpha} \supseteq B^I_\alpha$, we use the idea in IV to deal with the solution between inner and outer sets. We use a nestedness possibility distribution in a restricted IVPM obtained form IV to determine the space of consideration. We illustrate through the following examples.

*Example 5.1:*

$$\left.\begin{array}{ll} \min & [2,4]x_1 + [1,3]x_2 \\ \text{s.t.} & \widehat{3}x_1 + [-2,1]x_2 = [-2,2] \\ & \widetilde{0.5}x_1 + \check{3}x_2 = \widehat{0}, \end{array}\right\} \tag{13}$$

where the uncertain data are presented in the figure 2 above. For this example, we work at the $\alpha$-level when $\alpha = 0.5$. Therefore, the problem (13) with $\alpha = 0.5$ becomes

$$\left.\begin{array}{ll} \min & [2,4]x_1 + [1,3]x_2 \\ \text{s.t.} & [2,4]x_1 + [-2,1]x_2 = [-2,2] \\ & [-1,2]x_1 + [2,4]x_2 = [-2,2]. \end{array}\right\} \tag{14}$$

Figure 3 shows the inner box and outer box for (14). So the guaranteed bound on the objective value of

$$\min_{x \in B^I_{0.5}} [2,4]x_1 + [1,3]x_2,$$

is $[-25, 25]$. From figure 3, we get $D = 5$ units. By using possibility distribution as in (10) for simplicity, if we want

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degree of possibility of an IVPM (as constructed in IV) to be 0.6, then

$$\min_{x\in\left(B^O_{0.5}\backslash B^I_{0.5}\right)_{0.6}} \quad [2,4]x_1+[1,3]x_2$$

would have the guaranteed bound, $[-28, 28]$, on the

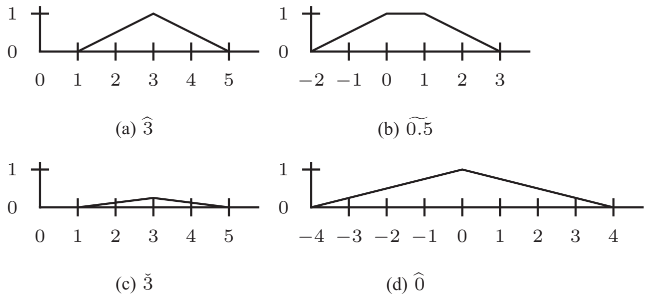


Fig. 2. Uncertain data for example 5.1.

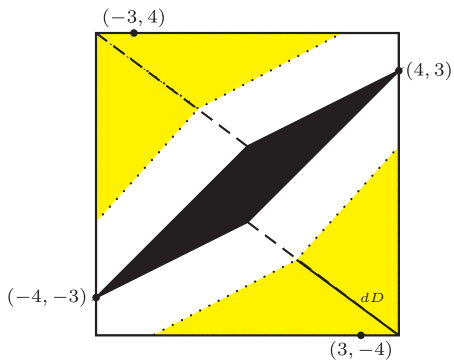


Fig. 3. Inner and outer boxes for the problem (14). Inner box is the black area. Outer box is the square generated by points (-3,4), (4,3), (3,-4) and (-4,-3).

objective value with 0.6 degree of occurrence. Similarly, we can have the guaranteed bound with a certain degree of belief by using an appropriate necessity measure of the IVPM.

*Example 5.2:* A drug company wants to create two supplement products X and Y that extract mainly from a super food F. This super food F has a fuzzy color 'orange-red-pink'. Some of fully grown F may have the color red to pink or could be more in the orange shade. The company will be totally satisfied if each of the plant F has red color. So the fuzzy color (from the company's point of view) has the membership function as in figure 4(a). There is only one place P that grows F. So P can charge the price that he wants. Although P can not predict his harvest's color, he also prefers F to be red to please his customer. P sets the price of each F relating to its fuzzy color as in figure 4(b).

The company found out that each 1g of F (depend on the color) contains an amount of Omega III and Zinc (in mg) with possibility distribution in figure 4(c) and 4(d), respectively. The company has a secret formula for X and Y. To produce X, the company need to make sure that the basic ingredient has Omega III in the range of [1280,1300]mg. The product Y need to have $\widehat{586}$mg of Zinc, (Fig. 5(a)).

Therefore the company needs to add pure Omega III and Zinc from a substance T if F does not provide enough as required. One gram of T obtains 5mg of Omega III and $\check{1.8}$mg of Zinc , (Fig. 5(b)), which means Zinc in T is not certain. The price of T depends on the market which is given by \$$\check{3}$ per 1g of T, (Fig. 5(c)). We can formulate a

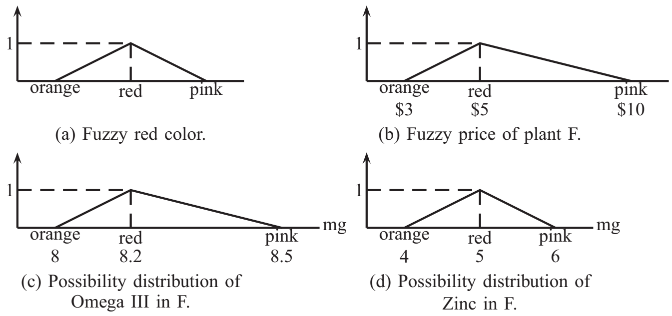


Fig. 4. Membership functions and possibility distributions.

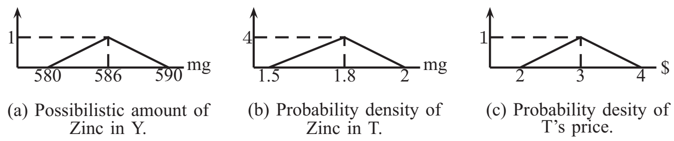


Fig. 5. Possibility distribution and probability density functions.

linear program to minimize the cost to produce these two supplements.

$$\begin{aligned} \min \quad & \widetilde{5}\text{f} + \check{3}\text{t} \\ \text{s.t.} \quad & \widehat{8.2}\text{f} + 5\text{t} = [1280, 1300] \\ & \widehat{5}\text{f} + \check{1.8}\text{t} = \widehat{586} \\ & \text{f}, \text{t} \geq 0, \end{aligned}$$

where f is the amount of grams of plant F and t is the amount of grams of substance T. When $\alpha = 0.5$, the problem becomes

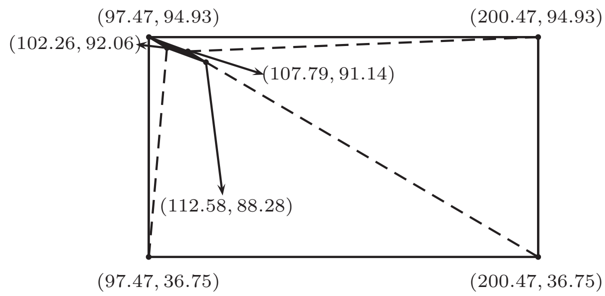


Fig. 6. Inner and outer boxes for the problem (15). Each point represents (t,f).

$$\left.\begin{aligned} \min \quad & [4, 7.5]\text{f} + [2.5, 3.5]\text{t} \\ \text{s.t.} \quad & [8.1, 8.35]\text{f} + 5\text{t} = [1280, 1300] \\ & [4.5, 5.5]\text{f} + [1.65, 1.9]\text{t} = [583, 588]. \end{aligned}\right\} \quad (15)$$

The inner and outer sets of the system (15) is shown in figure 6. The inner set is the black space generated by points (97.47,94.93), (102.26,92.06), (112.58,88.28) and (107.79,91.14). The outer set is the rectangular box. However, points (200.47,94.93) and (97.47,36.75) are not feasible for any $A \in [A]$ and $b \in [B]$.

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The semantics explanation for this example is as follows. The company sets the degree of satisfaction of F's price to be at least 0.5 together with 75% confidence interval of probabilistic price of T. It also wants 0.5 or more degree of occurrences of Omega III and Zinc in the food F and 93.75% confidence interval of probabilistic amount of Zinc in T. By its secret formula, the company sets at least 0.5 degree of occurrence of Zinc in product Y. Then the guaranteed bound on the objective value with $\alpha = 0.5$ is $\$[623.40, 1060.82]$ by using the corner point (97.47,94.93) and (107.79,91.14) of the black space. We also can find the guaranteed bound for some curtain degree of occurrence in the similar way as presented in the example 5.1.

## VI. Conclusion and Further Research

From the construction of $B^I_\alpha$ and $B^O_\alpha$, we can use $B^I_\alpha$ to find a guaranteed set of solutions and bound on an objective value of an optimization problem. We use the set $B^O_\alpha \setminus B^I_\alpha$ to create an IVPM, then use a possibility to get a set of solutions with some level $\gamma$ of occurrence. This paper considered merely equality constraints. The next step is to consider the inequality case.

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