A characterization of residual implications derived from uninorms

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Abstract— *In this paper, a set of axioms is given that characterizes those functions* $I : [0,1]^2 \to [0,1]$ *for which a left-continuous uninorm* U *exists in such a way that* I *is the residual implication derived from* U*. A characterization for the particular case when* U *is representable is also given.*

Keywords— Implication function, left-continuity, residual implication, uninorm.

1 Introduction

Implication functions are probably the most important operations in fuzzy logic, approximate reasoning and fuzzy control, because they are used not only to model fuzzy conditionals, but also to make inferences in any fuzzy rule based system (see for instance [1] or [2]). Moreover, they are useful not only in approximate reasoning and fuzzy control, but also in many other fields like fuzzy relational equations and fuzzy mathematical morphology ([3]), fuzzy DI-subsethood measures and image processing ([4] and [5]), and data mining ([6]). An excellent and very recent book on fuzzy implications is [7].

Among other models (see [3]), residual implications and strong implications are the most usual ones. They are usually derived from t-norms and t-conorms respectively as follows:

- Given a t-norm T, the residual implication or R-implication derived from T is given by

  $$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\} \qquad (1)$$

  for all $x, y \in [0,1]$.

  R-implications come from residuated lattices based on the so-called residuation property that can be written as

  $$T(x,y) \le z \qquad \Longleftrightarrow \qquad I_T(x,z) \ge y.$$

  Note that this property is satisfied if and only if T is left-continuous and then the supremum in (1) can be substituted by maximum.

- Given a t-conorm S and a strong negation N, the strong implication or (S,N)-implication derived from S, N is given by

  $$I_{S,N}(x,y) = S(N(x),y) \qquad \text{for all} \quad x,y \in [0,1].$$

  In this case, (S,N)-implications appear as a generalization of the classical boolean implication $p \to q \equiv \neg p \vee q$.

Axiomatic characterizations for R-implications derived from left-continuous t-norms, as well as for (S,N)-implications (even when N is only a continuous fuzzy negation, not necessarily strong), have appeared along the time (see for instance [8], [9], [10], [11], [12], [13], and the recent surveys [3] and [14]).

On the other hand, conjunctive and disjunctive uninorms (see [15]) are a class of associative binary aggregation functions that generalize both t-norms and t-conorms, and that has been successfully used in many application fields where t-norms and t-conorms apply. In particular, R and (S,N)-implications have also been derived from uninorms, obtaining in this way new implication functions with nice and interesting properties (see [16], [17] and [18]). However, an axiomatic characterization for these new classes of implications has been done for the case of (S,N)-implications derived from disjunctive uninorms in [19] (again in the general case when N is not necessarily strong), but not for R-implications.

In this paper, we deal with this problem and we give characterizations for the class of R-implications derived from left-continuous conjunctive uninorms in a similar way as it is done for left-continuous t-norms. Moreover, a characterization for the special case of left-continuous representable uninorms is given. From this case we derive a characterization of a kind of quasi-continuous implications (continuous except at two points) that can be viewed as a generalization of the well known Smets-Magrez Theorem (see the Preliminaries).

2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms, t-conorms and fuzzy negations (all necessary results and notations can be found in [20]). We recall here only some facts on implications.

Definition 1 *A binary operator* $I : [0,1] \times [0,1] \to [0,1]$ *is said to be an* implication function*, or an* implication*, if it satisfies:*

(I1) $I(x,z) \ge I(y,z)$ *when* $x \le y$*, for all* $z \in [0,1]$.

(I2) $I(x,y) \le I(x,z)$ *when* $y \le z$*, for all* $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Special interesting properties for implication functions are:

- The *ordering property*,

  $$x \le y \Longleftrightarrow I(x,y) = 1, \quad \text{for all} \quad x,y \in [0,1]. \qquad (\mathbf{OP})$$

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- The *exchange principle*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad \text{for all} \quad x, y, z \in [0, 1]. \quad (\mathbf{EP})$$

With respect to R-implications derived from left-continuous t-norms (see for instance [9] and the survey [14] with the references therein), they satisfy both properties (OP) and (EP). Moreover, we have the following characterization.

Theorem 1 *Let $I : [0, 1]^2 \to [0, 1]$ be a function. Then I is an R-implication derived from a left-continuous t-norm, if and only if, I satisfies $(I2), (OP), (EP)$ and I is right-continuous with respect to the second variable.*

With the assumption of continuity we have a characterization of the following subclass of R-implications, that are also (S, N)-implications, known as the Smets-Magrez Theorem, see [21] and see also [12] for the current version.

Theorem 2 *Let $I : [0, 1]^2 \to [0, 1]$ be a function. Then I is a continuous function satisfying $(OP), (EP)$, if and only if, I is conjugate with the Łukasiewicz implication, that is, there exists an increasing bijection $\varphi : [0, 1] \to [0, 1]$ such that*

$$I(x, y) = \varphi^{-1}(\min(1, 1 - \varphi(x) + \varphi(y)))$$

for all $x, y \in [0, 1]$.

We also suppose that some basic facts on uninorms are known as well as the class of representable ones (see for instance [15]). Let us only recall here the definition.

Definition 2 *A* uninorm *is a two-place function $U : [0, 1]^2 \longrightarrow [0, 1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0, 1]$, called* neutral element*, such that $U(e, x) = x$ for all $x \in [0, 1]$.*

It is clear that uninorms generalize both t-norms and t-conorms, since they are retrieved from uninorms just taking $e = 1$ and $e = 0$, respectively. Moreover, for a uninorm U, it is always $U(1, 0) \in \{0, 1\}$ and U is said to be *conjunctive* when $U(1, 0) = 0$, and *disjunctive* when $U(1, 0) = 1$.

On the other hand, residual implications derived from uninorms have been also studied (see [16]).

Definition 3 *Let U be a uninorm. The* residual operation *derived from U is the binary operation given by*

$$I_U(x, y) = \sup\{z \in [0, 1] \mid U(x, z) \leq y\} \tag{2}$$

for all $x, y \in [0, 1]$.

Proposition 1 *([16]) Let U be a uninorm and I_U its residual operation. Then I_U is an R-implication if and only if the following condition holds*

$$U(x, 0) = 0 \quad \textit{for all} \quad x < 1. \tag{3}$$

This includes all conjunctive uninorms but also many disjunctive ones, for instance in the classes of representable and idempotent uninorms (see [16] and [17]). However, when we deal with left-continuous uninorms U we clearly have that U satisfies condition (3) if and only if it is conjunctive.

3 R-implications derived from left-continuous uninorms

There are some properties of R-implications derived from uninorms, that can be deduced directly from the definition, or can be proved in the same way as for those derived from t-norms (see [16]).

Proposition 2 *Let U be a uninorm with neutral element e satisfying condition(3) and I_U its residual implication. Then*

i) *$I_U(e, y) = y$ for all $y \in [0, 1]$ counterpart for uninorms of the* neutrality principle*, that will be denoted by* $(\mathbf{NP_U})$.

ii) *$I_U(x, y) \geq e$ for all $x, y \in [0, 1]$ such that $x \leq y$.*

iii) *$y \leq I_U(x, U(x, y))$ for all $x, y \in [0, 1]$.*

In this section we will deal specially with R-implications derived from left-continuous uninorms. Recall that in this case condition (3) is equivalent to the uninorm U be conjunctive and so, we will refer only to left-continuous conjunctive uninorms. As for t-norms, this case of left-continuity is specially important because then (and only then) I_U satisfies the residuation property. The proof of this fact is similar to the case of t-norms (see [2]) and thus we do not include it.

Proposition 3 *Let U be a conjunctive uninorm and I_U its residual implication. Then U is left-continuous if and only if I_U satisfies the residuation property:*

$$U(x, y) \leq z \iff I_U(x, z) \geq y \qquad (\mathbf{RP})$$

for all $x, y, z \in [0, 1]$.

Moreover, when U is left-continuous additional properties are satisfied.

Proposition 4 *Let U be a left-continuous conjunctive uninorm with neutral element $e \in (0, 1]$. Then I_U satisfies*

i) *Counterpart for uninorms of the ordering property:*

$$x \leq y \iff I_U(x, y) \geq e \quad \textit{for all } x, y \in [0, 1]. \quad (\mathbf{OP_U})$$

ii) *Exchange principle,*

$$I_U(x, I_U(y, z)) = I_U(y, I_U(x, z)) \qquad (\mathbf{EP})$$

for all $x, y, z \in [0, 1]$.

iii) *The* modus ponens property*:*

$$U(x, I_U(x, y)) \leq y \qquad (\mathbf{MP})$$

for all $x, y \in [0, 1]$.

iv) *$I_U(x, -)$ is right-continuous for all $x \in [0, 1]$.*

Proof: Part $i)$ is straightforward from the definition, part $iv)$ follows as for the case of t-norms and the other properties were already proved in [16]. ■

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In order to characterize all functions $I : [0,1]^2 \to [0,1]$ that are R-implications derived from left-continuous uninorms, we firstly study those functions with properties (OP_U) and (EP).

Proposition 5 *Let $I : [0,1]^2 \to [0,1]$ be a function satisfying properties (OP_U) and (EP). Then*

i) $I(x,x) \geq e$ for all $x \in [0,1]$.

ii) I satisfies property (I1): $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

iii) I satisfies (NP_U). In particular, $I(e,e) = e$.

iv) $I(0,y) = 1$ for all $y \in [0,1]$.

v) If $I(1,e) = 0$ then $N(x) = I(x,e)$ is a fuzzy negation with $N(e) = e$.

Proof:

i) It is obvious from condition (OP_U).

ii) Let us consider $x, y, z \in [0,1]$ with $x \leq y$. Then we have

$$I(y, I(I(y,z),z)) = I(I(y,z), I(y,z)) \geq e$$

$$\Longrightarrow y \leq I(I(y,z),z)$$

by i) and conditions (OP_U) and (EP). Then,

$$x \leq y \leq I(I(y,z),z)$$

and this implies similarly that

$$e \leq I(x, I(I(y,z),z) = I(I(y,z), I(x,z))$$

and thus $I(y,z) \leq I(x,z)$.

iii) First of all,

$$I(x, I(e,x)) = I(e, I(x,x)) \geq e$$

where the last inequality holds because $e \leq I(x,x)$ by i). But the equation above ensures that $x \leq I(e,x)$. To prove the other inequality, it is sufficient to apply i) and conditions (OP_U), (EP) to obtain

$$I(e, I(I(e,x),x)) = I(I(e,x), I(e,x)) \geq e$$

$$\Longrightarrow I(I(e,x),x) \geq e$$

and thus $I(e,x) \leq x$.

iv) We have for all $y \in [0,1]$

$$\begin{aligned} 0 \leq I(1,y) &\Longrightarrow I(0, I(1,y)) \geq e \\ &\Longrightarrow I(1, I(0,y)) \geq e \\ &\Longrightarrow 1 \leq I(0,y) \end{aligned}$$

and thus $I(0,y) = 1 \ \forall y \in [0,1]$.

v) From ii) we have that N is decreasing and iv) implies that $N(0) = I(0,e) = 1$. Thus, since $N(1) = I(1,e) = 0$, N is a fuzzy negation. Finally, by iii) $N(e) = e$. ∎

Now, we can already give the axiomatic characterization of residual implications derived from left-continuous uninorms. Recall that in this case, since the uninorm must satisfy $U(x,0) = 0$ for all $x < 1$, it must be conjunctive.

Theorem 3 *Let $I : [0,1]^2 \to [0,1]$ be a function. The following statements are equivalent:*

i) *I is an R-implication derived from a left-continuous uninorm U with neutral element $e \in (0,1]$.*

ii) *I satisfies $I2$), (OP_U), (EP) and $I(x,-)$ is right-continuous for all $x \in [0,1]$.*

Moreover, in this case the uninorm U must be conjunctive and it is given by:

$$U(x,y) = \inf\{z \in [0,1] \mid I(x,z) \geq y\}.$$

Proof: Suppose first that

$$I(x,y) = I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$$

for all $x, y \in [0,1]$ where U is a left-continuous conjunctive uninorm with neutral element $e \in (0,1]$. Then we already know that I_U is an implication and consequently satisfies $(I2)$. The other properties follow from Proposition 4 and Proposition 2-iv).

Suppose now that I satisfies the required conditions. We have to prove that the function $U : [0,1]^2 \to [0,1]$ defined by

$$U(x,y) = \inf\{z \in [0,1] \mid I(x,z) \geq y\}$$

for all $x, y \in [0,1]$, is a left-continuous conjunctive uninorm U with neutral element e and also that $I = I_U$.

- $U(x,e) = x \ \forall x \in [0,1]$:

  We have by condition (OP_U) that

$$\begin{aligned} U(x,e) &= \inf\{z \in [0,1] \mid I(x,z) \geq e\} \\ &= \inf\{z \in [0,1] \mid z \geq x\} = x. \end{aligned}$$

- $U(x,y) = U(y,x) \ \forall x, y \in [0,1]$:

  Observe that

$$\begin{aligned} I(y,z) \geq x &\iff I(x, I(y,z)) \geq e \\ &\iff I(y, I(x,z)) \geq e \\ &\iff I(x,z) \geq y \end{aligned}$$

  by conditions (OP_U) and (EP). Thus

$$\begin{aligned} U(y,x) &= \inf\{z \in [0,1] \mid I(y,z) \geq x\} \\ &= \inf\{z \in [0,1] \mid I(x,z) \geq y\} \\ &= U(x,y) \end{aligned}$$

- $x_1 \leq x_2 \Longrightarrow U(x_1,y) \leq U(x_2,y)$:

  Let $x_1 \leq x_2$. Now if $I(x_2,z) \geq y$, then $I(x_1,z) \geq I(x_2,z) \geq y$ by property ii) of Proposition 5, and thus $U(x_1,y) \leq U(x_2,y)$.

- $y_1 \leq y_2 \Longrightarrow U(x,y_1) \leq U(x,y_2)$:

  It follows immediately by commutativity and the previous step.

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- $U(U(x,y),z) = U(x,U(y,z))\ \forall x,y,z \in [0,1]$:

  We have that $U(U(x,y),z) = U(z,U(x,y))$ by commutativity, and thus $U(U(x,y),z)$ is given by

  $$U(U(x,y),z) = \inf\{t \in [0,1] \mid I(z,t) \geq U(x,y)\}.$$

  On the other hand we have that

  $$U(x,U(y,z)) = \inf\{t \in [0,1] \mid I(x,t) \geq U(y,z)\},$$

  and consequently it is sufficient to prove that for all $t \in [0,1]$,

  $$I(z,t) \geq U(x,y) \Longleftrightarrow I(x,t) \geq U(y,z).$$

  Let us prove the left-to-right implication. Since $I(x,-)$ is right-continuous, we have from the expression of $U(x,y)$ that $I(x,U(x,y)) \geq y$. Then

  $$I(z,t) \geq U(x,y) \Longrightarrow I(x,I(z,t)) \geq I(x,U(x,y)) \geq y.$$

  Now, the increasingness with respect to the second component gives that:

  $$\begin{array}{rcl} U(y,z) = U(z,y) & \leq & U(z,I(x,I(z,t))) \\ & = & U(z,I(z,I(x,t))) \\ & \leq & I(x,t) \end{array}$$

  where the last inequality follows from the definition of U. The right-to-left implication follows similarly.

- U is conjunctive and left-continuous:

  Obviously since

  $$U(1,0) = \inf\{z \in [0,1] \mid I(1,z) \geq 0\} = 0,$$

  and left-continuity also follows easily from the definition of U.

- $I(x,y) = I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$:

  First of all, observe that the residuation property for I_U allows to write

  $$\begin{array}{rl} I_U(z,I_U(x,y)) = & \\ = & \sup\{t \in [0,1] \mid U(z,t) \leq I_U(x,y)\} \\ = & \sup\{t \in [0,1] \mid U(x,U(z,t)) \leq y\} \\ = & \sup\{t \in [0,1] \mid U(U(x,z),t) \leq y\} \\ = & I_U(U(x,z),y) \end{array}$$

  Now, for all $z \in [0,1]$ we have

  $$U(x,z) \leq y \Longleftrightarrow I_U(U(x,z),y) = I_U(z,I_U(x,y)) \geq e$$

  which implies $I_U(x,y) \geq z$. Thus, taking $z = I(x,y)$ we know, by definition of U, that $U(x,I(x,y)) \leq y$ and consequently we will have $I_U(x,y) \geq I(x,y)$.

  On the other hand, right-continuity of $I(x,-)$ implies that $I(x,U(x,z)) \geq z$. Now if we take $z = I_U(x,y)$,

  $$I_U(x,y) \leq I(x,U(x,I_U(x,y))) \leq I(x,y)$$

  again by the residuation property of I_U and the increasingness of I in the second variable. ■

Remark 1 *Note that when $e = 1$, condition (OP_U) becomes the ordering property (OP) and so, the axiomatic characterization of R-implications derived from left-continuous t-norms (Theorem 1) follows as a particular case.*

The mutual-independence of the properties in the theorem above is an open problem as it already is for the case of t-norms in Theorem 1 (see for instance [7], Remark 2.5.18). However, there are some of these properties that are independent to each other. The same examples for the case of t-norms work here as follows.

Example 1 *i) The Rescher implication given by*

$$I(x,y) = \begin{cases} 1 & if\ x \leq y \\ 0 & if\ x > y \end{cases}$$

satisfies (I2), (OP_U) with respect to any $e \in (0,1]$, and $I(x,-)$ is right-continuous for all $x \in [0,1]$, but it does not satisfy (EP).

ii) The Kleene-Dienes implication given by $I(x,y) = \max(1-x,y)$ satisfies (I2), (EP) and $I(x,-)$ is right-continuous for all $x \in [0,1]$, but there is no $e \in (0,1)$ for which I satisfies (OP_U).

On the other hand, the previous theorem gives the axiomatic characterization of R-implications derived from left-continuous conjunctive uninorms in general. Some particular characterizations can also be given for special classes of left-continuous conjunctive uninorms. Let us study the case of representable uninorms in the next section.

3.1 Residual implications derived from representable uninorms

Recall that a uninorm U with neutral element $e \in (0,1)$ is said to be representable (see [15]) if there exists a strictly increasing and continuous function $h : [0,1] \to [-\infty,+\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by

$$U(x,y) = \begin{cases} h^{-1}(h(x)+h(y)) & \text{if } (x,y) \notin \{(0,1),(1,0)\} \\ 0 \text{ (or 1)} & \text{otherwise.} \end{cases}$$

Function h is called an additive generator of U. Thus it is clear that there are two different representable uninorms with the same generator h, a conjunctive one (which is then left-continuous) and a disjunctive one (which is right-continuous). Representable uninorms are clearly continuous in $[0,1]^2 \setminus \{(1,0),(0,1)\}$ and this is in fact a characterization of such class of uninorms.

Proposition 6 *([22]) A uninorm U with neutral element $e \in (0,1)$ is representable if and only if it is continuous in $[0,1]^2 \setminus \{(1,0),(0,1)\}$*

Note that in both cases, conjunctive and disjunctive, it is $U(0,y) = 0$ for all $y < 1$ and so the residual implication from U can be derived. However, when we consider left-continuous representable uninorms they will be necessarily conjunctive as in the general case.

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Proposition 7 *([16]) Let U be a representable uninorm with additive generator h. Then its residual implication I_U is given by*

$$I_U(x,y) = \begin{cases} h^{-1}(h(y)-h(x)) & \text{if } (x,y) \notin \{(0,0),(1,1)\} \\ 1 & \text{otherwise.} \end{cases}$$

Let us now prove the characterization theorem for residual implications derived from this kind of uninorms.

Theorem 4 *Let $I : [0,1]^2 \to [0,1]$ be a function. The following statements are equivalent:*

i) *I is an R-implication derived from a left-continuous representable uninorm with neutral element $e \in (0,1)$.*

ii) *I satisfies (OP_U), (EP), $I(1,e)=0$ and I is continuous except at points $(0,0)$ and $(1,1)$.*

Proof: Suppose first that I is an R-implication derived from a left-continuous representable uninorm with neutral element $e \in (0,1)$. In this case U is conjunctive and by Theorem 3, I satisfies (OP_U) and (EP). Moreover, since U is representable, I must be continuous except at points $(0,0)$ and $(1,1)$ from Proposition 7.

Now suppose that the required conditions for I are satisfied. First consider $N(x) = I(x,e)$ and let us prove that N is a strong negation with fixed point e. Since $I(1,e) = 0$, we already know from Proposition 5 that N is a fuzzy negation with $N(e) = e$. Moreover, since $e \neq 0,1$, $N(x) = I(x,e)$ is continuous.

On the other hand, again from Proposition 5 we have for all $x \in [0,1]$

$$I(x, I(I(x,e),e)) = I(I(x,e), I(x,e)) \geq e$$

and consequently

$$x \leq I(I(x,e),e) = N(N(x)) = N^2(x) \quad \text{for all} \quad x \in [0,1].$$

Thus, by decreasingness of N we have $N(x) \geq N(N(N(x))) = N^3(x)$. On the other hand,

$$I(I(x,e), I(I(I(x,e),e),e)) =$$

$$= I(I(I(x,e),e), I(I(x,e),e)) \geq e$$

and, thus $N(x) = I(x,e) \leq I(I(I(x,e),e),e) = N^3(x)$. From both inequalities we deduce $N(x) = N^3(x)$ for all $x \in [0,1]$. Now, since $N(x) = I(x,e)$ is continuous in [0,1] by hypothesis, given $x \in [0,1]$, there exists $y \in [0,1]$ such that $N(y) = x$. Then $N(N(x)) = N(N(N(y))) = N(y) = x$, that is, N is involutive and so is a strong negation.

Now, define

$$U(x,y) = N(I(x, N(y))) \quad \text{for all} \quad x,y \in [0,1]$$

and let us prove that U is a uninorm with neutral element e.

- Commutativity comes from condition (EP),

$$\begin{aligned} U(y,x) &= N(I(y,N(x))) \\ &= I(I(y,N(x)),e) \\ &= I(I(y,I(x,e)),e) \\ &= I(I(x,I(y,e)),e) \\ &= N(I(x,N(y))) \\ &= U(x,y) \end{aligned}$$

- From condition (EP) we have

$$\begin{aligned} I(N(y),N(x)) &= I(I(y,e),I(x,e)) \\ &= I(x, I(I(y,e),e)) = I(x,y) \end{aligned}$$

since $I(I(y,e),e) = N^2(y) = y$.

Now, associativity can be proved as folows

$$\begin{aligned} U(x,U(y,z)) &= N(I(x,N(U(y,z)))) \\ &= N(I(x,N(N(I(y,N(z)))))) \\ &= N(I(x,I(y,N(z)))) \\ &= N(I(y,I(x,N(z)))) \\ &= N(I(y,I(z,N(x)))) \\ &= N(I(z,I(y,N(x)))) \\ &= N(I(N(I(y,N(x))),N(z))) \\ &= N(I(N(I(x,N(y))),N(z))) \\ &= N(I(U(x,y),N(z))) \\ &= U(U(x,y),z) \end{aligned}$$

- The increasingness of U comes from Proposition 5 and the decreasingness of N. Given $x_1 \leq x_2$ and $y \in [0,1]$, we have

$$I(x_1, N(y)) \geq I(x_2, N(y)).$$

Thus,

$$N(I(x_1,N(y))) \leq N(I(x_2,N(y)))$$

$$\Longrightarrow U(x_1,y) \leq U(x_2,y)$$

- Finally,

$$\begin{aligned} U(x,e) &= N(I(x,N(e)) = N(I(x,e)) \\ &= N^2(x) = x. \end{aligned}$$

Now, since I is continuous except at points $(0,0)$ and $(1,1)$, U will be continuous except at points $(0,1)$ and $(1,0)$. Moreover, since $I(1,1) \geq e$ we have

$$U(1,0) = N(I(1,1)) \leq e$$

and consequently $U(1,0) = 0$. This proves that U is a conjunctive representable uninorm (in particular it is left-continuous).

Finally, to see that $I = I_U$, it is enough to prove the residuation property:

$$U(x,y) \leq z \Longleftrightarrow I(x,z) \geq y$$

since then $I(x,z)$ will be given by the supremum of $y \in [0,1]$ such that $U(x,y) \leq z$. We have

$$\begin{aligned} U(x,y) \leq z &\Longleftrightarrow N(I(x,N(y))) \leq z \\ &\Longleftrightarrow I(x,N(y)) \geq N(z) \\ &\Longleftrightarrow I(N(z), I(x,N(y))) \geq e \\ &\Longleftrightarrow I(x, I(N(z),N(y))) \geq e \\ &\Longleftrightarrow I(x, I(y,z)) \geq e \\ &\Longleftrightarrow I(y, I(x,z)) \geq e \\ &\Longleftrightarrow I(x,z) \geq y \end{aligned}$$

■

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With respect to the mutual independence of the properties in the theorem above, we already know that (OP_U), and (EP) can not be derived from the others (see the counterexamples below). Now, we are currently working in the independence of the other two conditions.

Example 2 *Consider* $e \in (0,1)$ *and let* $h : [0,1] \to [-\infty,+\infty]$ *be a strictly increasing continuous function with* $h(0) = -\infty, h(e) = 0$ *and* $h(1) = +\infty$. *Then*

- *The function*

$$I(x,y) = \begin{cases} e & if\ x \leq y \\ h^{-1}(h(y) - h(x)) & if\ x > y \end{cases}$$

satisfies all conditions in Theorem 4 except (EP).

- *The function*

$$I(x,y) = \begin{cases} 0 & if\ x = y = 0 \\ & or\ x = y = 1 \\ h^{-1}(h(y) - h(x)) & otherwise \end{cases}$$

satisfies all conditions in Theorem 4 except (OP_U).

Corollary 1 *Let* $I : [0,1]^2 \to [0,1]$ *be a function and* $e \in (0,1)$. *Then there exists a strictly increasing continuous function* $h : [0,1] \to [-\infty,+\infty]$ *with* $h(0) = -\infty, h(e) = 0$ *and* $h(1) = +\infty$ *such that*

$$I(x,y) = \begin{cases} h^{-1}(h(y) - h(x)) & \text{if } (x,y) \notin \{(0,0),(1,1)\} \\ 1 & \text{otherwise.} \end{cases}$$

if and only if I *satisfies* (OP_U), (EP), $I(1,e) = 0$ *and it is continuous except at points* $(0,0)$ *and* $(1,1)$.

This corollary can be viewed as a generalization of the well known Smets-Magrez Theorem (Theorem 2) since it characterizes a kind of implications that are continuous except at two points.

Finally, we want to claim that other characterization theorems can be given for residual implications derived from other classes of left-continuous conjunctive uninorms, and we are currently working in this direction. For instance, from left-continuous conjunctive idempotent uninorms or from left-continuous conjunctive uninorms continuous at the open unit square.

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