Fuzzy Integrals over Complete Residuated Lattices

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Abstract— *The aim of this paper is to introduce two new types of fuzzy integrals, namely, $\otimes$-fuzzy integral and $\rightarrow$-fuzzy integral, where $\otimes$ and $\rightarrow$ are the multiplication and residuum of a complete residuated lattice, respectively. The first integral is based on a fuzzy measure of $\mathbf{L}$-fuzzy sets and the second one on a complementary fuzzy measure of $\mathbf{L}$-fuzzy sets, where $\mathbf{L}$ is a complete residuated lattice. Some of their properties and a relation to the fuzzy (Sugeno) integral are investigated.*

Keywords— fuzzy measure, fuzzy integral, fuzzy quantifier.

1 Introduction

Let us consider two time series $t_1 = (t_{1k})_{k\in T}$, $t_2 = (t_{2k})_{k\in T}$ displayed on Fig. 1 and suppose that our goal is to compare them and to find the "better" time series, where a greater value of time series at some time point k means a better value. To solve this task it is reasonable, firstly, to determine degrees of truth saying how much formulas

$$\varphi(i,j,k) := \text{“}\textit{the value } t_{ik} \textit{ is better than the value } t_{jk}\text{”}$$

are true, where, first, $i = 1$ and $j = 2$, and then $i = 2$ and $j = 1$. Obviously, the degrees of truth of formula $\varphi(i,j,k)$ may be modeled by a fuzzy relation $R : [0,1]^2 \rightarrow [0,1]$, e.g.

$$R(a,b) = \max(a - b, 0), \tag{1}$$

where $R(a,b)$ determines the degree that a is better than b. Note that $R(a,b) = \neg(a \rightarrow b)$, where $a \rightarrow b$ is the operation of residuum and $\neg$ is the operation of negation in the Łukasiewicz algebra (see Example 2.1 and consider $\neg a = 1 - a$). If we know how one value of time series is better than the other one for each time k, then we have to solve a task how to aggregate the obtained values to find a degree in which one times series is better than the second one. One of the natural approaches could be to evaluate the following formula of the second order logic

$$\varphi(i,j) := (\exists Y \in \mathcal{P}(T) \setminus \{\emptyset\})(\forall k \in Y)(\varphi(i,j,k) \& \psi(Y)),$$

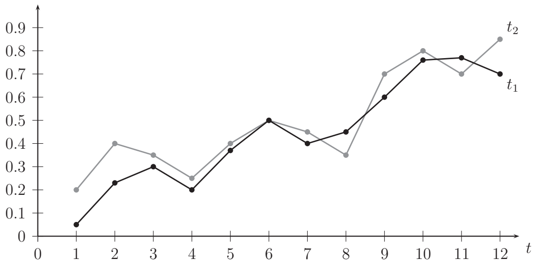

Figure 1: Time series.

where $\mathcal{P}(T)$ is the power set of T and

$$\psi(Y) := \text{“}\textit{the set } Y \textit{ is a big subset of } T\text{”}.$$

Note that formula $\varphi(i,j)$ may be defined using the formula $(Qk \in T)\varphi(i,j,k)$ with a fuzzy quantifier Q like *for nearly all* or *many* etc. Some types of fuzzy quantifiers could be determined by the interpretations of the formula ψ (cf. [1] and also see [2]). For example, if we consider the Łukasiewicz algebra as the structure of truth values for our logic, the truth value of the formula $\varphi(i,j,k)$ is defined as $R(t_{ik}, t_{jk})$ from (1), i.e., as the degree that t_{ik} is better than t_{jk}, and the truth value of the formula $\psi(Y)$ is interpreted by the value $\mu(Y)$, where $\mu : \mathcal{P}(T) \rightarrow [0,1]$ is a fuzzy measure (see Definition 3.2), then the evaluation of the formula $\varphi(i,j)$ is given by

$$||\varphi(i,j)|| = \bigvee_{Y\in\mathcal{P}(T)\setminus\{\emptyset\}} \bigwedge_{k\in Y} (R(t_{ik}, t_{jk}) \otimes \mu(Y)), \tag{2}$$

where $\otimes$ is the operation of multiplication in the Łukasiewicz algebra (see Example 2.1) interpreting the logical connective $\&$. Finally, we can conclude that, for example, the time series t_1 is better than t_2, if $||\varphi(1,2)|| > ||\varphi(2,1)||$.

Let us define a mapping $I_\mu : \mathcal{F}(T) \rightarrow [0,1]$ by

$$I_\mu(A) = \bigvee_{Y\in\mathcal{P}(T)\setminus\{\emptyset\}} \bigwedge_{k\in Y} (A(k) \otimes \mu(Y)), \tag{3}$$

where $\mathcal{F}(T)$ denotes the set of all fuzzy sets over T and $\mu : \mathcal{P}(T) \rightarrow [0,1]$ is a fuzzy measure. One could simply verify that if $c \in [0,1]$ is a constant and $A(k) = c$ for any $k \in T$, then $I_\mu(A) = c$, and if $A(k) \leq B(k)$, then $I_\mu(A) \leq I_\mu(B)$. Hence, I_μ is a fuzzy measure (in the sense of Definition 3.2) which could be understood, according to Mesiar [3], as an example of (fuzzy) integral. Putting $A(k) = R(t_{ik}, t_{jk})$ for any $k \in T$, we can write $||\varphi(i,j)|| = I_\mu(A)$ or also $||(Qk \in T)\varphi(i,j,k)|| = I_\mu(A)$ and, hence, we can see that fuzzy integrals may be used to model some types of fuzzy quantifiers. This idea is not new (see e.g. [4] or [5]), but some disadvantage of proposed approaches is that fuzzy quantifiers are defined as mappings from the set of all measurable (fuzzy) sets over a set M to $[0,1]$, although, the classical definition introduces fuzzy quantifiers as mappings from the set of all (fuzzy) sets over a set M to $[0,1]$ (see e.g. [6] or [7]). It will be clear that this drawback vanishes when fuzzy quantifiers are modeled by fuzzy integrals in a form similar to (3).

The aim of this contribution is to generalize the fuzzy integral defined in (3), namely, to introduce a $\otimes$-fuzzy integral that could be used for modeling fuzzy quantifiers like *all*, *some*, *for nearly all*, or *many*, etc. and then to introduce a $\rightarrow$-fuzzy integral that could be used for modeling quantifiers like *no*, *not all*, etc. Both types of fuzzy integrals are defined over a complete residuated lattice and it could be shown than, in general, one type of fuzzy integral cannot be expressed by another one. Nevertheless, if the structure of truth values is complete MV-algebra, then it is possible to define the $\rightarrow$-fuzzy integral from the $\otimes$-fuzzy integral using negation (see Theorem 4.14). Moreover, it is surprising that we are able to show that the well-known Sugeno integral [8] is, under certain conditions, a special case of our fuzzy integral (see Theorem 4.7).

2 Preliminaries

2.1 Structures of truth values

Let us suppose that the structure of truth values is a *complete residuated lattice* (see e.g. [9]), i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \perp, \top \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \perp, \top \rangle$ is a complete lattice, where $\perp$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \otimes, \top \rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{4}$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. The operations $\otimes$ and $\rightarrow$ are usually called the multiplication and residuum, respectively. A residuated lattice is *divisible*, if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$, and satisfies the *law of double negation*, if $(a \rightarrow \perp) \rightarrow \perp = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For other information about residuated lattices we refer to [9].

Example 2.1. It is easy to prove (see e.g. [10]) that the algebra

$$\mathbf{L}_{\mathrm{T}} = \langle [0,1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where T is a left continuous t-norm and $a \rightarrow_T b = \bigvee \{c \in [0,1] \mid T(a,c) \leq b\}$ defines the residuum, is a complete residuated lattice. Moreover, if T is the Łukasiewicz t-norm, i.e., $T(a,b) = \max(a+b-1, 0)$ for all $a, b \in [0,1]$, then $\mathbf{L}_{\mathrm{T}}$ is a complete MV-algebra called a *Łukasiewicz algebra* (on $[0,1]$). One checks easily that $a \rightarrow_T b = \max(1-a+b, 0)$ is the residuum in the Łukasiewicz algebra.

Example 2.2. One checks easily that

$$\mathbf{L}_{[0,\infty]} = \langle [0,\infty], \min, \max, \rightarrow, 0, \infty \rangle,$$

where $\otimes = \min$ and

$$a \rightarrow b = \begin{cases} b, & \text{if } b < a, \\ \infty, & \text{otherwise,} \end{cases} \tag{5}$$

is a complete residuated lattice. Note that $\mathbf{L}_{[0,\infty]}$ is a special example of more general residuated lattice called a *Heyting algebra*.[1]

Let us define the following additional operations for all $a, b \in L$:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) \qquad (\textit{biresiduum})$$

$$\neg a = a \rightarrow \perp. \qquad (\textit{negation})$$

[1] A Heyting algebra is a residuated lattice with $\otimes = \wedge$.

2.2 L-fuzzy sets

Let $\mathbf{L}$ be a complete residuated lattice and M be a universe of discourse. A mapping $A : M \rightarrow L$ is called an **L**-*fuzzy set on* M. A value $A(m)$ is called a *membership degree of* m *in the* **L**-*fuzzy set* A. The set of all **L**-fuzzy sets on M is denoted by $\mathcal{F}_{\mathbf{L}}(M)$. An **L**-fuzzy set A on M is called *crisp*, if there is a subset X of M such that $A = 1_X$, where 1_X denotes the characteristic function of X. Particularly, $1_\emptyset$ denotes the empty **L**-fuzzy set on M, i.e. $1_\emptyset(m) = \perp$ for any $m \in M$. The set of all crisp **L**-fuzzy sets on M is denoted by $\mathcal{P}_{\mathbf{L}}(M)$. An **L**-fuzzy set A is *constant*, if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant **L**-fuzzy set is denoted by the corresponding element of L, e.g., a, b, c.[2] Let us denote $\mathrm{Supp}(A) = \{m \mid m \in M \ \& \ A(m) > \perp\}$ and $\mathrm{core}(A) = \{m \mid m \in M \ \& \ A(m) = \top\}$ the *support* and *core* of an **L**-fuzzy set A, respectively. Obviously, $\mathrm{Supp}(1_X) = \mathrm{core}(1_X) = X$ for any crisp **L**-fuzzy set. An **L**-fuzzy set A is called *normal*, if $\mathrm{core}(A) \neq \emptyset$.

Let $\{A_i \mid i \in I\}$ be a non-empty family of **L**-fuzzy sets on M. Then the *union of* A_i is defined by

$$\left(\bigcup_{i \in I} A_i\right)(m) = \bigvee_{i \in I} A_i(m) \tag{6}$$

for any $m \in M$ and the *intersection of* A_i is defined by

$$\left(\bigcap_{i \in I} A_i\right)(m) = \bigwedge_{i \in I} A_i(m) \tag{7}$$

for any $m \in M$. Let A be an **L**-fuzzy set on M. The *complement* of A is an **L**-fuzzy set $\overline{A}$ on M defined by $\overline{A}(m) = \neg A(m)$ for any $m \in M$. Finally, an extension of the operations $\otimes$ and $\rightarrow$ on L to the operations on $\mathcal{F}_{\mathbf{L}}(M)$ is given by

$$(A \otimes B)(m) = A(m) \otimes B(m) \tag{8}$$

$$(A \rightarrow B)(m) = A(m) \rightarrow B(m) \tag{9}$$

for any $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $m \in M$, respectively. The following theorem shows the well-known relation between the operations of the union and intersection of sets which also holds for **L**-fuzzy sets, if we restrict ourselves to a special class of complete residuated lattices.

Theorem 2.1. *Let* $\mathbf{L}$ *be a complete residuated lattice satisfying the law of double negation and* $\{A_i \mid i \in I\}$ *be a non-empty family of* **L**-*fuzzy sets on* M. *Then*

$$\bigcup_{i \in I} A_i = \overline{\bigcap_{i \in I} \overline{A_i}} \quad \textit{and} \quad \bigcap_{i \in I} A_i = \overline{\bigcup_{i \in I} \overline{A_i}}. \tag{10}$$

We say that an **L**-fuzzy set A is *less than or equal to* an **L**-fuzzy set B and denote by $A \subseteq B$, if, for any $m \in M$, we have $A(m) \leq B(m)$. Let $f : M \rightarrow M'$ be a mapping. Then $f^{\rightarrow}(A)(m) = \bigvee_{m' \in f^{-1}(m)} A(m')$ defines a mapping $f^{\rightarrow} : \mathcal{F}_{\mathbf{L}}(M) \rightarrow \mathcal{F}_{\mathbf{L}}(M')$. Obviously, if f is a bijective mapping, then $f^{\rightarrow}(A)(f(m)) = A(m)$ for any $m \in M$.

[2] We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant **L**-fuzzy set is assumed.

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3 Fuzzy measures

In this section, we will introduce a notion of fuzzy measure and complementary fuzzy measure of **L**-fuzzy sets. More information about fuzzy measures could be found in [11, 12].

For our considerations we will consider algebras of **L**-fuzzy sets as a base for defining fuzzy measures of **L**-fuzzy sets.

Definition 3.1 ([11])**.** Let M be a non-empty universe of discourse. A subset $\mathcal{M}$ of $\mathcal{F}_{\mathbf{L}}(M)$ is an *algebra of* **L**-*fuzzy sets on* M, if the following conditions are satisfied

(i) $1_\emptyset, 1_M \in \mathcal{M}$,

(ii) if $A \in \mathcal{M}$, then $\overline{A} \in \mathcal{M}$,

(iii) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

A couple $(M, \mathcal{M})$ is called a *fuzzy measurable space*, if $\mathcal{M}$ is an algebra of **L**-fuzzy sets on M.

Example 3.1. The sets $\{1_\emptyset, 1_M\}$, $\mathcal{P}_{\mathbf{L}}(M)$, σ-algebras on M, or $\mathcal{F}_{\mathbf{L}}(M)$ are algebras of **L**-fuzzy sets on M.

Example 3.2. Let us say that an **L**-fuzzy set A on M is a *simple* **L**-*fuzzy set* on M, if there exists a family of sets $\{M_i \mid i = 1, \dots, n\}$ such that $\bigcup_{i=1}^{n} M_i = M$, $M_i \neq M_j$ for any $i \neq j$ and $A(m) = A(m')$ holds for each $m, m' \in M_i$, where $i = 1, \dots, n$. Obviously, the set of all simple **L**-fuzzy sets on M is an algebra of **L**-fuzzy sets on M.

Example 3.3. Let **L** be the Łukasiewicz algebra on $[0,1]$ (see Example 2.1) and $M = [0,1]$. Then the set of all continuous mappings $A : [0,1] \to [0,1]$ is an algebra of **L**-fuzzy sets in M.[3]

Let us introduce the concepts of fuzzy measure and complementary fuzzy measure as follows. The first definition is a modification of the definition of a normed fuzzy measure with respect to truth values (see e.g. [12, 13]).

Definition 3.2. Let $(M, \mathcal{M})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{M} \to L$ is called a *fuzzy measure* on $(M, \mathcal{M})$, if

(i) $\mu(1_\emptyset) = \bot$ and $\mu(1_M) = \top$,

(ii) if $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet $(M, \mathcal{M}, \mu)$ is called a *fuzzy measure space*, if $(M, \mathcal{M})$ is a fuzzy measurable space and μ is a fuzzy measure on $(M, \mathcal{M})$.

Definition 3.3. Let $(M, \mathcal{M})$ be a fuzzy measurable space. A mapping $\nu : \mathcal{M} \to L$ is called a *complementary fuzzy measure* on $(M, \mathcal{M})$, if

(i) $\nu(1_\emptyset) = \top$ and $\nu(1_M) = \bot$,

(ii) if $A, B \in \mathcal{M}$ such that $A \leq B$, then $\nu(A) \geq \nu(B)$.

A triplet $(M, \mathcal{M}, \nu)$ is called a *complementary fuzzy measure space*, if $(M, \mathcal{M})$ is a fuzzy measurable space and ν is a complementary fuzzy measure on $(M, \mathcal{M})$.

[3] Note that the set of all continuous mappings need not be an algebra of **L**-fuzzy sets for other residuated lattices determined by left continuous T-norms, because the negation is not a continuous mapping in general.

Example 3.4. Let $(M, \mathcal{M})$ be the fuzzy measurable space of all continuous mappings from Example 3.3. It is easy to see that

$$\mu(A) = \int_0^1 A(m)\, dm,$$

where $\int_0^1 A(m)\, dm$ denotes the Riemann integral, defines a fuzzy measure on $(M, \mathcal{M})$.

Example 3.5. Let **L** be a complete residuated lattice with the support $[0,1]$ and $\mathbb{N}$ be the set of natural numbers with 0. For any non-empty countable (i.e., finite or denumerable) universe M, injective mapping $f : M \to \mathbb{N}$, $n \in \mathbb{N}$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, denote

$$A_{f,n}(m) = \begin{cases} A(m), & \text{if } f(m) \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Further, for any injective mapping $f : M \to \mathbb{N}$ and $n \in \mathbb{N}$, define $\mu_{f,n} : \mathcal{F}_{\mathbf{L}}(M) \to [0,1]$ as follows

$$\mu_{f,n}(A) = \frac{\sum_{m \in \mathrm{Supp}(A_{f,n})} A_{f,n}(m)}{|\mathrm{Supp}(1_{M_{f,n}})|} \tag{12}$$

and, finally, define $\underline{\mu}_f, \overline{\mu}_f : \mathcal{F}_{\mathbf{L}}(M) \to [0,1]$ as follows

$$\underline{\mu}_f = \liminf_{n \to \infty} \mu_{f,n}(A), \tag{13}$$

$$\overline{\mu}_f = \limsup_{n \to \infty} \mu_{f,n}(A). \tag{14}$$

It is easy to see that $\mu_{f,n}$, $\underline{\mu}_f$ and $\overline{\mu}_f$ are fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by an injective mapping f.[4] If, for example, $M = \mathbb{N}$ and $f = \mathrm{id}$, then $\underline{\mu}_f(A) = \overline{\mu}_f(A) = \bot$ for any **L**-fuzzy set with finite universe. For the set of all even or odd numbers, both fuzzy measures give $\frac{1}{2}$ and, for the set of all prime numbers, we obtain 0.

If M is finite, then $\underline{\mu}_f = \underline{\mu}_g = \overline{\mu}_f = \overline{\mu}_g$ for any injective mappings $f, g : M \to \mathbb{N}$ and

$$\underline{\mu}_f(A) = \overline{\mu}_f(A) = \frac{\sum_{m \in M} A(m)}{|M|}. \tag{15}$$

Hence, it is easy to see that $\mu_f(A) = \mu_f(h^{\to}(A))$ holds for any non-empty finite universe M, $A \in \mathcal{F}_{\mathbf{L}}(M)$, injective mapping $f : M \to \mathbb{N}$ and bijective mapping $h : M \to M$. Unfortunately, this equality fails for denumerable universes in general. In fact, consider $M = \mathbb{N}$, $f = \mathrm{id}$ and a bijective mapping $h : \mathbb{N} \to \mathbb{N}$ such that the image of all even numbers is the set of prime numbers. Then both fuzzy measures give $\frac{1}{2}$ for the set of all even numbers, however, 0 for the set of all prime numbers.

Example 3.6. Let μ_f be one of the fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by f defined in (13) and (14). If $h : [0,1] \to [0,1]$ is a non-decreasing mapping with $h(0) = 0$ and $h(1) = 1$, then $h \circ \mu_f$ is a fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ_f and h. If $h : [0,1] \to [0,1]$ is a non-increasing mapping with $h(0) = 1$ and $h(1) = 0$, then $h \circ \mu_f$ is a complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ_f and h.

[4] Note that $\underline{\mu}_f$ and $\overline{\mu}_f$ could be understood as a generalization of lower and upper weighted densities well known in the number theory which are examples of so-called lower and upper asymptotic fuzzy measures (see [14]).

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Theorem 3.1. *Let $(M, \mathcal{M})$ be a fuzzy measurable space. If μ (ν) is a fuzzy measure (a complementary fuzzy measure) on $(M, \mathcal{M})$, then $\nu'(A) = \neg\mu(A)$ ($\mu'(A) = \neg\nu(A)$) defines a complementary fuzzy measure (a fuzzy measure) on $(M, \mathcal{M})$.*

Definition 3.4. Let $(M, \mathcal{M})$ be a fuzzy measurable space and $X \in \mathcal{F}_{\mathbf{L}}(M)$. We say that X is *$\mathcal{M}$-fuzzy measurable*, if $X \in \mathcal{M}$.

Let $(M, \mathcal{M})$ be a fuzzy measurable space and $X \in \mathcal{F}_{\mathbf{L}}(M)$. Denote $\mathcal{M}_X$ the set of all $\mathcal{M}$-fuzzy measurable sets which are contained in X, i.e.,

$$\mathcal{M}_X = \{A \mid A \in \mathcal{M} \text{ and } A \subseteq X\}. \tag{16}$$

Note that $1_\emptyset \in \mathcal{M}_X$ for each $X \in \mathcal{F}_{\mathbf{L}}(M)$ and if X is $\mathcal{M}$-fuzzy measurable set, then also $X \in \mathcal{M}_X$. If $X = M$, then we will write only $\mathcal{M}$ instead of $\mathcal{M}_M$.

Theorem 3.2. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. A mapping $\mu^* : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\mu^*(X) = \bigvee_{A \in \mathcal{M}_X} \mu(A) \tag{17}$$

is a fuzzy measure on the fuzzy measurable space $(M, \mathcal{F}_{\mathbf{L}}(M))$. We say that μ^* is the inner fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by μ.

Example 3.7. Let $(M, \mathcal{P}_{\mathbf{L}}(M), \mu)$ be an arbitrary fuzzy measurable space (recall that $\mathcal{P}_{\mathbf{L}}(M)$ is the power set of M). Then the inner fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ is defined by

$$\mu^*(A) = \begin{cases} \mu(A'), & \text{if } 1_{\text{core}(A)} = A', \\ \bot, & \text{otherwise.} \end{cases} \tag{18}$$

Thus all $\mathbf{L}$-fuzzy sets that are not normal have the inner fuzzy measure equal to $\bot$.

Example 3.8. Let $\mathbf{L}$ be the Łukasiewicz algebra on $[0, 1]$, $(M, \mathcal{M}, \mu)$ be the fuzzy measure space of continuous $\mathbf{L}$-fuzzy sets from Example 3.4. Then, for example, we have $\mu^*(1_{[a,b]}) = b - a$, however, $1_{[a,b]} \notin \mathcal{M}$.

Theorem 3.3. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. A mapping $\nu^* : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\nu^*(X) = \bigwedge_{A \in \mathcal{M}_X} \nu(A) \tag{19}$$

is a complementary fuzzy measure on the fuzzy measurable space $(M, \mathcal{F}_{\mathbf{L}}(M))$. We say that ν is the inner complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by ν.

In the following part we will define an isomorphism between fuzzy measure spaces and then between complementary fuzzy measure spaces.

Definition 3.5. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$*, if

(i) g is a bijective mapping with $g(1_\emptyset) = 1_\emptyset$,

(ii) $g(A \cup B) = g(A) \cup g(B)$ and $g(\overline{A}) = \overline{g(A)}$ hold for any $A, B \in \mathcal{M}$,

(iii) there exists a bijective mapping $f : M \to M'$ with $A(m) = g(A)(f(m))$ for any $A \in \mathcal{M}$ and $m \in M$.

Theorem 3.4. *Let $(M, \mathcal{M})$, $(M', \mathcal{M}')$ be fuzzy measurable spaces and $g : \mathcal{M} \to \mathcal{M}'$ be a surjective mapping. Then g is an isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ if and only if there exists a bijective mapping $f : M \to M'$ such that $g = f^{\to}$.*

Definition 3.6. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$* (or *between $(M, \mathcal{M}, \nu)$ and $(M', \mathcal{M}', \nu')$*), if

(i) g is an isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$,

(ii) $\mu(A) = \mu'(g(A))$ (or $\nu(A) = \nu'(g(A))$) for any $A \in \mathcal{M}$.

If g is an isomorphism between fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ or between complementary fuzzy measure spaces $(M, \mathcal{M}, \nu)$ and $(M', \mathcal{M}', \nu')$, then we write $g(M, \mathcal{M}, \mu) = (M', \mathcal{M}', \mu')$ or $g(M, \mathcal{M}, \nu) = (M', \mathcal{M}', \nu')$, respectively.

Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. If $f : M \to M'$ is a bijective mapping, then $(M', f^{\to}(\mathcal{M}), \mu_{f^{\to}})$, where

$$\mu_{f^{\to}}(f^{\to}(A)) = \mu(A) \tag{20}$$

holds for any $A \in \mathcal{M}$, is a fuzzy measure space isomorphic to $(M, \mathcal{M}, \mu)$. A simple consequence of Theorem 3.4 is the fact that each fuzzy measure space $(M', \mathcal{M}', \mu')$ isomorphic to $(M, \mathcal{M}, \mu)$ has the form $(M', f^{\to}(\mathcal{M}), \mu_{f^{\to}})$ for a suitable bijective mapping $f : M \to M'$. Analogously, to every couple of isomorphic complementary fuzzy measure spaces $(M, \mathcal{M}, \nu)$ and $(M', \mathcal{M}', \nu')$ there is a bijective mapping $f : M \to M'$ such that $(M', f^{\to}(\mathcal{M}), \nu_{f^{\to}}) = (M', \mathcal{M}', \nu')$.

4 Fuzzy integrals

In this section, we will introduce two types of fuzzy integrals. The first one is a generalization of the formula (3) derived in Introduction. For more information about fuzzy integrals we refer to [11, 12].

4.1 $\otimes$-fuzzy integral

In this part, we will introduce a type of fuzzy integral that can be defined on an arbitrary fuzzy measure space $(M, \mathcal{M}, \mu)$. The form of this integral is motivated by our need to describe a class of models of $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1 \rangle$. In [2], we show that this class of models is bounded by the models of determiners *all* and *some*. Note that models of *all* and *some* are the same as the interpretations of quantifiers $\forall$ and $\exists$, respectively, in fuzzy logic (see e.g. [6, 7, 15, 4]).

Definition 4.1. Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and X be a $\mathcal{M}$-fuzzy measurable $\mathbf{L}$-fuzzy set. The *$\otimes$-fuzzy integral of A on X* is given by

$$\int_X^{\otimes} A \, d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)). \tag{21}$$

If $X = 1_M$, then we write $\int^{\otimes} A \, d\mu$.

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Remark 4.1. It is easy to see that $\int_{1_\emptyset}^{\otimes} A\,d\mu = \bigvee\emptyset = \bot$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $\int_X^{\otimes} A\,d\mu \le \int_Y^{\otimes} A\,d\mu$, whenever $X \subseteq Y$. Since $\int_{1_M}^{\otimes} A\,d\mu \ne \top$ in general, $\mu_A(X) = \int_X^{\otimes} A\,d\mu$ does not define a fuzzy measure on $(M,\mathcal{M})$ in the sense of Definition 3.2.

Remark 4.2. One can also define a $\wedge$-fuzzy integral of A on X in such way that $\otimes$ is replaced by $\wedge$ in (21). Since $\otimes$ and $\wedge$ have many common properties, both types of fuzzy integral will have similar properties. Nevertheless, we prefer the $\otimes$-fuzzy integral in this paper, because it is closely related (due to the adjointness property) to $\to$-fuzzy integral that will be introduced in the following subsection.

Theorem 4.1. *Let $(M,\mathcal{M},\mu)$ be a fuzzy measure space. Then $\mu' : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\mu'(A) = \int^{\otimes} A\,d\mu \tag{22}$$

is a fuzzy measure on $(M,\mathcal{F}_{\mathbf{L}}(M))$.

Theorem 4.2. *Let $(M,\mathcal{M},\mu)$ be a fuzzy measure space. Then*

(i) $\int_X^{\otimes}(A\cap B)\,d\mu \le \int_X^{\otimes} A\,d\mu \wedge \int_X^{\otimes} B\,d\mu$,

(ii) $\int_X^{\otimes}(A\cup B)\,d\mu \ge \int_X^{\otimes} A\,d\mu \vee \int_X^{\otimes} B\,d\mu$,

(iii) $\int_X^{\otimes}(c\otimes A)\,d\mu \ge c\otimes\int_X^{\otimes} A\,d\mu$,

(iv) $\int_X^{\otimes}(c\to A)\,d\mu \le c\to\int_X^{\otimes} A\,d\mu$,

hold for any $X\in\mathcal{M}$, $A,B\in\mathcal{F}_{\mathbf{L}}(M)$ and $c\in L$.

Theorem 4.3. *Let $(M,\mathcal{M},\mu)$ be a fuzzy measure space and $c\in L$. Then we have*

(i) $\int^{\otimes}(c\otimes 1_X)\,d\mu = c\otimes\mu^*(1_X)$ *for any $X\subseteq M$,*

(ii) $\int^{\otimes}(c\otimes 1_X)\,d\mu = c\otimes\mu(1_X)$ *for any $X\subseteq M$ such that $1_X\in\mathcal{M}$,*

(iii) $\int^{\otimes} 1_X\,d\mu = \mu(1_X)$ *for any $X\subseteq M$ such that $1_X\in\mathcal{M}$,*

(iv) $\int^{\otimes} c\,d\mu = c$.

Theorem 4.4. *Let $(M,\mathcal{M},\mu)$ be a fuzzy measure space. If $X\in\mathcal{M}$ is such that $1_{\mathrm{Supp}(Y)}\in\mathcal{M}_X$ for any $Y\in\mathcal{M}_X$, then, for any $A\in\mathcal{F}_{\mathbf{L}}(M)$, we have*

$$\int_X^{\otimes} A\,d\mu = \bigvee_{1_Y\in\mathcal{P}_X\setminus\{1_\emptyset\}}\ \bigwedge_{m\in Y}(A(m)\otimes\mu(1_Y)), \tag{23}$$

where $\mathcal{P}_X = \{1_{\mathrm{Supp}(Z)} \mid Z\in\mathcal{M}_X\}$.

Theorem 4.5. *Let $\mathbf{L}$ be a complete MV-algebra, $(M,\mathcal{M},\mu)$ be a fuzzy measure space, $A\in\mathcal{F}_{\mathbf{L}}(M)$ and $X\in\mathcal{M}$. Then*

$$\int_X^{\otimes} A\,d\mu = \bigvee_{Y\in\mathcal{M}_X\setminus\{1_\emptyset\}}\big(\mu(Y)\otimes\bigwedge_{m\in\mathrm{Supp}(Y)} A(m)\big). \tag{24}$$

Moreover,

$$\int_X^{\otimes}(c\otimes A)\,d\mu = c\otimes\int_X^{\otimes} A\,d\mu \tag{25}$$

for any $c\in L$.

Theorem 4.6. *Let g be an isomorphism between fuzzy measure spaces $(M,\mathcal{M},\mu)$ and $(M',\mathcal{M}',\mu')$ and $X\in\mathcal{M}$. Then we have*

$$\int_X^{\otimes} A\,d\mu = \int_{g(X)}^{\otimes} g(A)\,d\mu' \tag{26}$$

for any $A\in\mathcal{F}_{\mathbf{L}}(M)$.

In the end of this part, we will show that the Sugeno integral is a special case of our proposed integral. For this purpose we will use a slight modification of the usual Sugeno integral definition with respect to the fuzzy measurable spaces over complete residuated lattices.

Let $\mathbf{L}$ be a complete residuated lattice and $(M,\mathcal{M})$ be a fuzzy measurable space such that $A\cap B\in\mathcal{M}$ for any $A,B\in\mathcal{M}$.[5] Denote $A_a = \{m \mid m\in M \,\&\, A(m)\ge a\}$. We say that an $\mathbf{L}$-fuzzy set A is $\mathcal{M}$-*Sugeno measurable*, if $1_{A_a}\in\mathcal{M}$ for any $a\in L$. The *Sugeno integral* is given, for any fuzzy measure space $(M,\mathcal{M},\mu)$ with $B\cap C\in\mathcal{M}$ for any $B,C\in\mathcal{M}$, for any $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set A and for any $X\in\mathcal{M}$, by

$$\int_X A\,d\mu = \bigvee_{a\in L}(a\wedge\mu(1_{A_a}\cap X)). \tag{27}$$

Theorem 4.7. *Let $\mathbf{L}$ be a complete Heyting algebra, $(M,\mathcal{M},\mu)$ be a fuzzy measure space with $B\cap C\in\mathcal{M}$ for any $B,C\in\mathcal{M}$, A be a $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set and $X\in\mathcal{M}$. Then $\int_X A\,d\mu = \int_X^{\otimes} A\,d\mu$.*

4.2 $\to$-fuzzy integral

In this part, we will introduce another type of fuzzy integral that can be defined on an arbitrary complementary fuzzy measure space $(M,\mathcal{M},\nu)$. The form of this integral is motivated by our need to describe another class of models of $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1\rangle$ which are kind of negations of the previous ones.

Definition 4.2. Let $(M,\mathcal{M},\nu)$ be a complementary fuzzy measure space, $A\in\mathcal{F}_{\mathbf{L}}(M)$ and X be a $\mathcal{M}$-fuzzy measurable $\mathbf{L}$-fuzzy set. The $\to$*-fuzzy integral of A on X* is given by

$$\int_X^{\to} A\,d\nu = \bigwedge_{Y\in\mathcal{M}_X\setminus\{1_\emptyset\}}\ \bigvee_{m\in\mathrm{Supp}(Y)}(A(m)\to\nu(Y)). \tag{28}$$

If $X = 1_M$, then we write $\int^{\to} A\,d\nu$.

Remark 4.3. It is easy to see that $\int_{1_\emptyset}^{\to} A\,d\nu = \bigwedge\emptyset = \top$ for any $A\in\mathcal{F}_{\mathbf{L}}(M)$ and $\int_X^{\to} A\,d\nu \le \int_Y^{\to} A\,d\nu$, whenever $Y\subseteq X$. Since $\int_{1_M}^{\to} A\,d\nu \ne \bot$ in general, $\nu_A(X) = \int_X^{\to} A\,d\nu$ does not define a complementary fuzzy measure on $(M,\mathcal{M})$ in the sense of Definition 3.3.

[5]Note that, according to Theorem 2.1, p. 2, each complete residuated lattice satisfying the law of double negation has this property. Nevertheless, there are fuzzy measurable spaces which keep this property, but $\mathbf{L}$ does not satisfy the law of double negation. A simple example is a fuzzy measurable space $(M,\mathcal{M})$ such that $\mathcal{M}\subseteq\mathcal{P}_{\mathbf{L}}(M)$ and $\mathbf{L}$ is an arbitrary complete residuated lattice (e.g. $L_{[0,\infty]}$ from Example 2.2).

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Theorem 4.8. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. Then $\nu' : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by*

$$\nu'(A) = \int^{\rightarrow} A \, d\nu \tag{29}$$

is a complementary fuzzy measure on $(M, \mathcal{F}_{\mathbf{L}}(M))$.

Theorem 4.9. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. Then*

(i) $\int_X^{\rightarrow} (A \cap B) \, d\nu \geq \int_X^{\rightarrow} A \, d\nu \vee \int_X^{\rightarrow} B \, d\nu$,

(ii) $\int_X^{\rightarrow} (A \cup B) \, d\nu \leq \int_X^{\rightarrow} A \, d\nu \wedge \int_X^{\rightarrow} B \, d\nu$,

(iii) $\int_X^{\rightarrow} (c \otimes A) \, d\nu \leq c \to \int_X^{\rightarrow} A \, d\nu$,

(iv) $\int_X^{\rightarrow} (c \to A) \, d\nu \geq c \otimes \int_X^{\rightarrow} A \, d\nu$

hold for any $X \in \mathcal{M}$, $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $c \in L$.

Theorem 4.10. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space and $c \in L$. Then we have*

(i) $\int^{\rightarrow} (c \otimes 1_X) \, d\nu = c \to \nu^*(1_X)$ *for any* $X \subseteq M$,

(ii) $\int^{\rightarrow} (c \otimes 1_X) \, d\nu = c \to \nu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$,

(iii) $\int^{\rightarrow} 1_X \, d\nu = \nu(1_X)$ *for any* $X \subseteq M$ *such that* $1_X \in \mathcal{M}$,

(iv) $\int^{\rightarrow} c \, d\nu = \neg c$.

Theorem 4.11. *Let $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space. If $X \in \mathcal{M}$ is such that $1_{\mathrm{Supp}(A)} \in \mathcal{M}_X$ for any $A \in \mathcal{M}_X$, then, for any $A \in \mathcal{F}_{\mathbf{L}}(M)$, we have*

$$\int_X^{\rightarrow} A \, d\nu = \bigvee_{1_Y \in \mathcal{P}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in Y} (A(m) \to \nu(1_Y)), \tag{30}$$

where $\mathcal{P}_X = \{1_{\mathrm{Supp}(A)} \mid A \in \mathcal{M}_X\}$.

Theorem 4.12. *Let $\mathbf{L}$ be a complete MV-algebra, $(M, \mathcal{M}, \nu)$ be a complementary fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $X \in \mathcal{M}$. Then*

$$\int_X^{\rightarrow} A \, d\nu = \bigwedge_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \big((\bigwedge_{m \in \mathrm{Supp}(Y)} A(m)) \to \nu(Y) \big). \tag{31}$$

Moreover,

$$\int_X^{\rightarrow} (c \otimes A) \, d\nu = c \to \int_X^{\rightarrow} A \, d\nu \tag{32}$$

for any $c \in L$.

Theorem 4.13. *Let g be an isomorphism between complementary fuzzy measure spaces $(M, \mathcal{M}, \nu)$ and $(M', \mathcal{M}', \nu')$ and $X \in \mathcal{M}$. Then we have*

$$\int_X^{\rightarrow} A \, d\nu = \int_{g(X)}^{\rightarrow} g(A) \, d\nu' \tag{33}$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

The following statement shows that if we consider a complete MV-algebra, then we can restrict ourselves, for example, to $\otimes$-fuzzy integrals, since each $\rightarrow$-fuzzy integral is uniquely determined by the negation of a suitable $\otimes$-fuzzy integral.

Theorem 4.14. *Let $\mathbf{L}$ be a complete MV-algebra and $(M, \mathcal{M})$ be a fuzzy measurable space. Then*

$$\int_X^{\rightarrow} A \, d\nu' = \neg \int_X^{\otimes} A \, d\mu, \tag{34}$$

$$\int_X^{\otimes} A \, d\mu' = \neg \int_X^{\rightarrow} A \, d\nu \tag{35}$$

hold for any fuzzy measure μ and complementary fuzzy measure ν, where $\nu' = \neg\mu$ and $\mu' = \neg\nu$.

5 Conclusions

In this contribution, new types of fuzzy integrals, that are quite useful for modeling fuzzy quantifiers (of the type $\langle 1 \rangle$), are introduced and some of their properties are studied. The definitions of fuzzy quantifiers using these types of fuzzy integrals and some of their semantical properties could be found in [2].

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