A link between the 2-additive Choquet Integral and Belief functions

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Abstract— In the context of decision under uncertainty, we characterize the 2-additive Choquet integral on the set of fictitious acts called binary alternatives or binary actions. This characterization is based on a fundamental property called MOPI which permits us to relate belief functions and the 2-additive Choquet integral.

Keywords— Capacity, Möbius transform, Choquet integral, k-monotone function, Belief function, Decision under uncertainty

1 Introduction

Decision under uncertainty is a part of decision making where each act has several possible consequences, depending on the state of nature whose probability of occurrence is unknown. As shown by the well-known Ellsberg's paradox [1, 2], the use of the expected utility model [3] in decision under uncertainty is limited. Therefore some non-additive models like Choquet expected utility [4] have been proposed in order to overcome the limitations of the expected utility model.

The Choquet integral is defined w.r.t. a capacity (or non-additive monotonic measure, or fuzzy measure), and can be thought of as a generalization of the expected value, the capacity playing the role of a probability measure. In this paper we focus on the 2-additive Choquet integral [5, 6], a particular Choquet integral where interaction between two states of nature can be represented, but not more complex interaction. This model is in practice already sufficiently flexible. In many situations, it is important for the Decision-Maker (DM) to construct a preference relation over the set of all acts X. Because it is not an easy task (the cardinality of X may be very large), we ask him to give, using pairwise comparisons, an ordinal information (a preferential information containing only a strict preference and an indifference relations) on a particular reference subset $\mathcal{B} \subseteq X$. The set $\mathcal{B}$ we use is the set of binary acts or binary actions. A binary action is a fictitious act which takes only two values denoted $\mathbf{1}$ and $\mathbf{0}$ belonging to the set of consequences, such that $\mathbf{1}$ is strictly preferred to $\mathbf{0}$. We present necessary and sufficient conditions on the ordinal information for the existence of a 2-additive capacity such that the Choquet integral w.r.t. this capacity represents the preference of the decision maker. We introduce the new fundamental property MOPI, a kind of monotonicity coming from the definition of a 2-additive capacity, in order to have this characterization. We found through our MOPI property the following link between the 2-additive Choquet integral and belief functions (Shafer [7]): *Any ordinal information representable by a belief function is representable by a Choquet integral w.r.t. a 2-additive capacity*.

Because a belief function is a capacity, we show another characterization of the representation of any ordinal information by a belief function. The new fundamental property defined in this case is called the 2-MOPI property. This property and the MOPI property in the previous paragraph are related by the following statement: *if the 2-MOPI property is satisfied then the MOPI property is satisfied*.

The article is organized as follows. The next section presents the basic concepts we need. Section 3 concerns a representation of ordinal information by the 2-additive Choquet integral. In the last section, after some results on the case of the k-monotone functions, we study the representation of an ordinal information by a belief function.

2 Preliminaries

Let us denote by $N = \{1, \dots, n\}$ the set of n states of nature and by 2^N the set of all subsets of N. The set of possible consequences (also called "outcomes") is denoted by C. An act x is identified to an element of $X = C^n$ with $x = (x_1, ..., x_n)$. We introduce the following convenient notation: for two acts $x, y \in X$ and a subset $A \subseteq N$, the compound act $z = (x_A, y_{N-A})$ is defined by $z_i := x_i$ if $i \in A$, and $z_i := y_i$ otherwise. For all i, j in N, the element $i \vee j$ denotes one of the elements i, j.

We want to construct a preference relation over X, but this is not easy because X may contain infinitely many acts. In practice [8] one can only ask to the DM pairwise comparisons of acts on a finite subset X' of X. Hence we get a preference relation $\succsim_{X'}$ on X'. The question is then: how to construct $\succsim_X$ from $\succsim_{X'}$? To this end, people usually suppose that $\succsim_X$ is representable by an overall utility function:

$$x \succsim_X y \Leftrightarrow F(U(x)) \geq F(U(y)) \quad (1)$$

where $U(x) = (u(x_1), \dots, u(x_n))$, $u : C \to \mathbb{R}$ is called a utility function, and $F : \mathbb{R}^n \to \mathbb{R}$ is an aggregation

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function. Usually, we consider a family of aggregation functions characterized by a parameter vector θ (e.g., a probability distribution over the states of nature). The parameter vector θ can be deduced from the knowledge of $\succsim_{X'}$, that is, we determine the possible values of θ for which (1) is fulfilled over X'. We study the case where F is the Choquet integral, the parameters are the 2-additive capacity and X' is the set of binary actions. The aim of this paper is to give necessary and sufficient conditions on $\succsim_{X'}$ to be represented by a 2-additive Choquet integral. The model obtained in X' will be automatically extended to X.

2.1 The 2-additive Choquet Integral

The 2-additive Choquet integral [6] is a particular case of the Choquet integral [9, 10]. This integral generalizes the arithmetic mean and takes into account interactions between the states of nature. A 2-additive Choquet integral is based on a 2-additive capacity defined below and its Möbius transform [11, 12]:

Definition 2.1.

1. A *capacity* on N is a set function $\mu : 2^N \to [0,1]$ such that:
   (a) $\mu(\emptyset) = 0$
   (b) $\mu(N) = 1$
   (c) $\forall A, B \in 2^N$, $[A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).
2. The *Möbius transform* of a capacity μ on N is a function $m : 2^N \to \mathbb{R}$ defined by:

$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N \quad (2)$$

   When m is given, it is possible to recover the original μ by the following expression:

$$\mu(T) := \sum_{K \subseteq T} m(K), \forall T \in 2^N \quad (3)$$

Definition 2.2. A capacity μ on N is said to be *2-additive* if

- For all subset T of N such that $|T| > 2$, $m(T) = 0$;
- There exists a subset B of N such that $|B| = 2$ and $m(B) \neq 0$.

Notations We simplify our notations by using for a capacity μ and its Möbius transform m: $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i,j\})$, $m_i := m(\{i\})$, $m_{ij} := m(\{i,j\})$, for all $i, j \in N$, $i \neq j$. Whenever we use i and j together, it always means that they are different.

The following important Lemma shows that a 2-additive capacity is entirely determined by the value of the capacity on the singletons $\{i\}$ and pairs $\{i,j\}$ of 2^N:

Lemma 1.

1. *Let μ be a 2-additive capacity on N. We have*

$$\mu(K) = \sum_{\{i,j\} \subseteq K} \mu_{ij} - (|K| - 2) \sum_{i \in K} \mu_i, \ \forall K \subseteq N, \ |K| \geq 2. \quad (4)$$

2. *If the coefficients μ_i and μ_{ij} are given for all $i, j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:*

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i = 1 \quad (5)$$

$$\mu_i \geq 0, \ \forall i \in N \quad (6)$$

$$\sum_{i \in A \setminus \{k\}} (\mu_{ik} - \mu_i) \geq (|A| - 2)\mu_k, \ \forall A \subseteq N, |A| \geq 2, \forall k \in A. \quad (7)$$

Proof. See [6] □

For an act $x := (x_1, ..., x_n) \in X$, the expression of the Choquet integral w.r.t a capacity μ is given by:

$$C_\mu((u(x_1), ..., u(x_n))) := u(x_{\tau(1)})\mu(N) + \sum_{i=2}^{n} (u(x_{\tau(i)}) - u(x_{\tau(i-1)}))\mu(\{\tau(i), ..., \tau(n)\})$$

where τ is a permutation on N such that $u(x_{\tau(1)}) \leq u(x_{\tau(2)}) \leq ... \leq u(x_{\tau(n-1)}) \leq u(x_{\tau(n)})$.

A Choquet integral with a 2-additive capacity μ is called a 2-additive Choquet integral. Given an act $x := (x_1, ..., x_n) \in X$, the 2-additive Choquet integral can be written also as follows:

$$C_\mu((u(x_1), \ldots, u(x_n))) = \sum_{i=1}^{n} v_i u(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij} |u(x_i) - u(x_j)| \quad (8)$$

where $v_i = \sum_{K \subseteq N \setminus i} \frac{(n - |K| - 1)!|K|!}{n!} (\mu(K \cup i) - \mu(K))$ represents the importance of the state of nature i and corresponds to the Shapley value [13]; $I_{ij} = \mu_{ij} - \mu_i - \mu_j$ is the interaction index between the two states of nature i and j.

The above development suggests that the Choquet integral w.r.t. a 2-additive capacity seems to be of particular interest, and offers a good compromise between flexibility of the model and complexity. Therefore, we focus in this paper on the 2-additive model.

2.2 Binary actions

We assume in this paper that the DM is able to identify in C two consequences denoted $\mathbf{1}$ and $\mathbf{0}$ such that he strictly prefers $\mathbf{1}$ to $\mathbf{0}$. In the sequel, we call $\mathbf{0}$ the "neutral level" (even if this is not the neutral level understood in bipolar model).

We call a *binary action or binary act*, an element of the set $\mathcal{B} = \{\mathbf{0}_N, \ (\mathbf{1}_i, \mathbf{0}_{N-i}), \ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}), \ i, j \in N, i \neq j\} \subseteq X$ where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is an act which has a consequence $\mathbf{0}$ on all states of nature.

- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an act which has a consequence $\mathbf{1}$ on state of nature i and a consequence $\mathbf{0}$ on the other states of nature.

- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an act which has a consequence $\mathbf{1}$ on states of nature i and j and a consequence $\mathbf{0}$ on the other states of nature.

By convention we set $u(\mathbf{0}) = 0$ and $u(\mathbf{1}) = 1$. The above convention have the following consequences:

Remark 1.

1. The Choquet integral satisfies the following property [14, 10]: if μ is a capacity then

$$C_\mu(U(\mathbf{1}_A, \mathbf{0}_{N-A})) = \mu(A),\ \forall A \subseteq N. \quad (9)$$

2. Let μ be a 2-additive capacity. We have

$$C_\mu(U(a_0)) = 0;$$

$$C_\mu(U(a_i)) = \mu_i = v_i - \tfrac{1}{2} \sum_{k \in N,\ k \neq i} I_{ik};$$

$$C_\mu(U(a_{ij})) = \mu_{ij} = v_i + v_j - \tfrac{1}{2} \sum_{k \in N,\ k \notin \{i,j\}} (I_{ik} + I_{jk})$$

Generally the DM knows how to compare some acts using his knowledge of the problem, his experience, etc. These acts form a set of reference acts and allows to determine the parameters of a model (utility functions, subjective probabilities, weights,...) in the decision process (see [8] for more details). As shown by the previous Remark 1 and Lemma 1, it should be sufficient to get some preferential information from the DM only on binary acts. To entirely determine the 2-additive capacity this information is expressed by the following relations: $P = \{(x, y) \in \mathcal{B} \times \mathcal{B} :$ DM strictly prefers x to $y\}$, $I = \{(x, y) \in \mathcal{B} \times \mathcal{B} :$ DM is indifferent between x and $y\}$.

Definition 2.3. The *ordinal information on* $\mathcal{B}$ is the structure $\{P,\ I\}$.

Now we will suppose P nonempty for any ordinal information $\{P,\ I\}$ ("non triviality axiom"). Before we end this section, let us introduce another relation M which completes the ordinal information $\{P,\ I\}$ given by the DM and models the natural relations of monotonicity between binary actions. For $(x, y) \in \{(a_i, a_0), i \in N\} \cup \{(a_{ij}, a_i), i, j \in N, i \neq j\}$,

$$x\ M\ y \text{ if not}(x\ (P \cup I)\ y).$$

The relation M models the monotonicity conditions $\mu(\{i\}) \geq 0$ and $\mu(\{i, j\}) \geq \mu(\{i\})$ for a capacity μ.

Example 1. *If we consider* $N = \{1, 2, 3\}$, $\mathcal{B} = \{a_o, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$, $P = \{(a_{13}, a_3), (a_2, a_3), (a_{23}, 0)\}$, $I = \{(a_{12}, a_1)\}$, *then the relation* M *is* $M = \{(a_{12}, a_0), (a_{13}, a_0), (a_1, a_0), (a_2, a_0), (a_3, a_0), (a_{12}, a_2), (a_{13}, a_1), (a_{23}, a_2), (a_{23}, a_3)\}$.

3 The representation of the ordinal information by the Choquet integral

An ordinal information $\{P, I\}$ is said to be *representable by a 2-additive Choquet integral* if there exists a 2-additive capacity μ such that:

1. $\forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow C_\mu(U(x)) > C_\mu(U(y))$
2. $\forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow C_\mu(U(x)) = C_\mu(U(y))$.

Given an ordinal information $\{P, I\}$, we look for the necessary and sufficient conditions on $\mathcal{B}$ for which $\{P, I\}$ is representable by a 2-additive Choquet integral. To do it, we need to define first the notion of strict cycle of the relation $(P \cup I \cup M)$.

3.1 Cycle of $(P \cup I \cup M)$

For a binary relation $\mathcal{R}$ on $\mathcal{B}$ and x, y elements of $\mathcal{B}$, $\{x_1, x_2, \cdots, x_p\} \subseteq \mathcal{B}$ is a *path* of $\mathcal{R}$ from x to y if $x = x_1\ \mathcal{R}\ x_2\ \mathcal{R} \cdots \mathcal{R}\ x_{p-1}\ \mathcal{R}\ x_p = y$. A path of $\mathcal{R}$ from x to x is called a *cycle* of $\mathcal{R}$.

- A path $\{x_1, x_2, ..., x_p\}$ of $(P \cup I \cup M)$ is said to be a *strict path* from x to y if there exists i in $\{1, ..., p-1\}$ such that $x_i\ P\ x_{i+1}$. In this case, we will write $x\ TC_P\ y$.

- A cycle $(x_1, x_2, ..., x_p)$ of $(P \cup I \cup M)$ is a *nonstrict cycle* if it is not strict.

- We note $x \sim y$ if there exists a nonstrict cycle of $(P \cup I \cup M)$ containing x and y.

Contrarily to the strict cycle which is a classic concept used in graph theory [15, 16], we need to define a new fundamental property called *MOPI*.

3.2 MOPI property and theorem of Characterization

Before defining the property *MOPI*, let us discover this new condition through a simple example:

Example 2. *Suppose that the DM says :* $a_{12}\ I\ a_3,\ a_{13}\ I\ a_2$ *and* $a_1\ P\ a_0$. *Using the relation* M*, we have* $a_{12}\ M\ a_2\ I\ a_{13}\ M\ a_3\ I\ a_{12}$. *So,* $(a_{12}, a_2, a_{13}, a_3, a_{12})$ *forms a nonstrict cycle of* $(P \cup I \cup M)$. *If* $\{P, I\}$ *is representable by a 2-additive Choquet integral* C_μ, *this implies* $\mu_{12} = \mu_{13} = \mu_2 = \mu_3$ *and* $\mu_1 > 0$. *However, we get a contradiction with the monotonicity constraint* $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$ *of a 2-additive capacity with the subset* $A = \{1, 2, 3\}$, $k = 1$ *(see Equation* (7) *in Lemma 1).*

This type of inconsistency is defined by:

Definition 3.1 (MOPI property). Let $i, j, k \in N$, i fixed.

1. We call *Monotonicity of Preferential Information in* $\{i, j, k\}$ *w.r.t.* i the following property (denoted by **(**$\{i, j, k\}$,i**)-MOPI**):

$$\left.\begin{array}{l} a_{ij} \sim a_{i\vee j} \\ a_{ik} \sim a_{i\vee k} \\ i \vee j \in \{i, j\} \\ i \vee k \in \{i, k\} \\ i \vee j \neq i \vee k \end{array}\right\} \Rightarrow \begin{array}{l} [not(a_l\ TC_P\ a_0) \\ l \in \{i, j, k\} \setminus \{i \vee k, i \vee j\}] \end{array}$$

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If the property **($\{i,j,k\}$,i)-MOPI** is satisfied then the element $a_l, l \in \{i,j,k\} \setminus \{i \vee k, i \vee j\}$ is called the *neutral binary action of* $\{i,j,k\}$ *w.r.t.* i.

2. We say that $\{i,j,k\}$ satisfies the property *Monotonicity of Preferential Information* (MOPI) if $\forall l \in \{i,j,k\}$, ($\{i,j,k\}$,l)-MOPI is satisfied.

Example 3. *Let* $N = \{1,2,3,4\}$ *and* $i = 1$ *fixed. The property* $(\{1,2,3\},1)$*- MOPI reads as follows:*

- $\begin{cases} a_{12} \sim a_2 \\ a_{13} \sim a_1 \end{cases} \Rightarrow not(a_3\ TC_P\ a_0)$
- $\begin{cases} a_{12} \sim a_1 \\ a_{13} \sim a_3 \end{cases} \Rightarrow not(a_2\ TC_P\ a_0)$
- $\begin{cases} a_{12} \sim a_2 \\ a_{13} \sim a_3 \end{cases} \Rightarrow not(a_1\ TC_P\ a_0)$

The MOPI condition given in this paper is equivalent to the MOPI property presented in [5]. We give below our theorem of characterization of consistent ordinal information $\{P, I\}$ representable by a 2-additive Choquet integral:

Theorem 1. *An ordinal information* $\{P, I\}$ *is representable by a 2-additive Choquet integral on* $\mathcal{B}$ *if and only if the following conditions are satisfied:*

1. $(P \cup I \cup M)$ *contains no strict cycle;*
2. *Any subset* K *of* N *such that* $|K| = 3$ *satisfies the MOPI property.*

Is it possible to represent an ordinal information by another operator instead of the 2-additive Choquet integral? If the answer is yes, can we give a similar characterization like in Theorem 1? In the next section, we will show that it is possible by using for instance belief functions.

4 The representation of ordinal information by belief functions

4.1 General definitions

Beliefs functions are one of the fundamental concepts used in the theory of evidence of Shafer [7]. They are defined by the belief function mass m as follows:

Definition 4.1. A function $m : 2^N \to [0,1]$ is called a *mass distribution* or a *basic belief assignment* if m satisfies the following two properties:

1. $m(\emptyset) = 0$;
2. $\sum_{A \subseteq N} m(A) = 1.$

The quantity $m(A)$ expresses the total amount of belief that supports the proposition: " the actual state of nature is in A", and does not support any more specific subset of N because of lack of information.

Based on this concept, we define the belief function Bel by:

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad \forall A \subseteq N.$$

Remark 2.

- Bel is a capacity;
- The sets A such that $m(A) > 0$ are called the focal elements;
- If all focal elements are singletons then a mass distribution can be considered as a probability distribution;
- The mass distribution m corresponds to the Möbius transform of Bel. So we have $\forall T \in 2^N$,

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B).$$

Thus, we can have a definition of the representation of ordinal information by a belief function which is similar to the same representation by a Choquet integral (see Section 3).

Definition 4.2. An ordinal information $\{P, I\}$ is said to be *representable by a belief function* if there exists a belief function Bel such that

1. $\forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow C_{Bel}(U(x)) > C_{Bel}(U(y))$
2. $\forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow C_{Bel}(U(x)) = C_{Bel}(U(y))$.

By using Definition 2.2, a 2-additive belief function has a mass distribution m characterized by:

1. $\exists i, j \in N$ tel que $m(\{i,j\}) \neq 0$;
2. $\forall K \in 2^N$ tel que $|K| \geq 3$, $m(K) = 0$.

Theorem 2 below provides a relation between a k-monotone function [6, 12] and a belief function, and a relation between k-monotone functions and the previous MOPI property.

4.2 k-monotone functions and belief functions

Given an integer $k \geq 2$, a function $\mu : 2^N \to [0,1]$ is k-monotone (shorthand for: monotone of order k) if for each family $\{A_1, A_2, \ldots, A_k\} \subseteq 2^N$, we have

$$\mu(\bigcup_{i=1}^{k} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \mu(\bigcap_{i \in I} A_i). \qquad (10)$$

A simpler characterization of k-monotone functions by their Möbius inversion is given by the following proposition:

Proposition 1. *Let* $\mu : 2^N \to [0,1]$ *and* m *its Möbius transform.* μ *is k-monotone (k integer,* $k \geq 2$*) if and only if*

$$\sum_{A \subseteq L \subseteq B} m(L) \geq 0 \quad \forall A, B \subseteq N, A \subseteq B \text{ and } 2 \leq |A| \leq k. \qquad (11)$$

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Proof. See [12] □

It is well-known that $\mu : 2^N \to [0,1]$ is a belief function if and only if μ is a k-monotone capacity for all $k \geq 2$. The following result gives another sufficient condition to obtain a belief function from a k-monotone and 2-additive capacity, and relates belief function with the MOPI condition translated in terms of capacity.

Theorem 2.
Let $\mu : 2^N \to [0,1]$ be a function and k be an integer such that $k \geq 2$.

1. *If μ is monotone, k-monotone and 2-additive then μ is a belief function (precisely a 2-additive belief function);*
2. *If μ is monotone and k-monotone then μ satisfies the following property: for all $i, j, k \in N$, i fixed*

$$\left.\begin{array}{l}\mu_{ij} = \mu_{i\vee j}\\ \mu_{ik} = \mu_{i\vee k}\\ i\vee j \in \{i,j\}\\ i\vee k \in \{i,k\}\\ i\vee j \neq i\vee k\end{array}\right\} \Rightarrow \begin{array}{l}[\mu_l = 0,\\ l \in \{i,j,k\} \setminus \{i\vee k, i\vee j\}]\end{array}$$

We end the paper by a characterization of ordinal information by belief functions.

4.3 A link between Belief functions and the 2-additive Choquet integral

In this section, we give through the MOPI property (see Section 3) a link between beliefs functions and the 2-additive Choquet integral.

Proposition 2. *Let $\{P,\ I\}$ be an ordinal information on $\mathcal{B}$.*
If there exist $i, j, k \in N$, i fixed such that the property $(\{i,j,k\}, i)$-MOPI is violated, then there is no belief function Bel which represents $\{P,\ I\}$.

Corollary 1. *Every ordinal information $\{P,\ I\}$ on $\mathcal{B}$ representable by a belief function $Bel : 2^N \to [0,1]$ is representable by a 2-additive Choquet integral.*

The inverse of Corollary 1 is false. If we suppose $P = \{(a_2, a_0)\}$, $I = \{(a_{12}, a_1)\}$ and μ a 2-additive capacity, we will have $\{P,\ I\}$ representable by a 2-additive Choquet integral and $I_{12} = m_{12} = \mu_{12} - \mu_1 - \mu_2 < 0$. So no belief function can represent $\{P,\ I\}$ in this case. Then it is interesting to look for the class of 2-additive capacities which are belief functions. In order to characterize them, we introduce a new fundamental property called 2-MOPI property:

Definition 4.3. An ordinal information $\{P,\ I\}$ satisfies the 2-*MOPI property* if

$$\forall i, j \in N, i \neq j, [a_{ij} \sim a_i \Rightarrow not(a_j\ TC_P\ a_0)]. \quad (12)$$

The relation between the 2-MOPI property and the MOPI property is given by the following proposition:

Proposition 3. *Let $\{P,\ I\}$ an ordinal information on $\mathcal{B}$.*

$\{P,\ I\}$ satisfies the 2-MOPI property

$$\Downarrow$$

$\forall i, j, k \in N,\ \{i,j,k\}$ satisfies the MOPI property

Now we have the main result of this section:

Theorem 3.
$\{P,\ I\}$ is representable by a 2-additive belief function if and only if the two following conditions are satisfied:

1. *$(P \cup I \cup M)$ contains no strict cycle;*
2. *$\{P,\ I\}$ satisfies the 2-MOPI property.*

4.4 Interpretation of 2-MOPI and MOPI properties

We try to give an interpretation in terms of decision behavior of the two main conditions introduced in this paper. We assume here for clarity that consequence **1** is a good consequence for the DM, while consequence **0** is neither bad nor good (statu quo).

Facing a situation where for two states of nature i and j the DM is indifferent between the two acts a_{ij} and a_i, the 2-MOPI property says that act a_j is equivalent to act a_0 (statu quo for every state of nature). Hence in such a situation, the DM thinks that state of nature j is unlikely to occur. This is a strong condition, since it suffices that one such state i exists to infer the "nullity" of state j. This condition can be related to the notion of null set in generalized measure theory (see, e.g., [17]): a set $A \subseteq N$ is said to be *null* for capacity μ if $\mu(B \cup A) = \mu(B)$, $\forall B \subseteq N \setminus A$. Taking $A = \{j\}$ and $B = \{i\}$ gives our condition 2-MOPI. Observe that for the nullity condition, $\{j\}$ would be null if for *all* subsets B not containing j we would have $\mu(B \cup j) = \mu(B)$, but the 2-MOPI condition asks to find only one singleton satisfying this equality.

The MOPI property is a weakening of the above one, and can be interpreted in a similar way. Let us consider now three states of nature i, j and k. The MOPI condition can be translated as follows (see Example **??**, with $i = 1$, $j = 2$, and $k = 3$). Suppose that a_{ij} and a_j are indifferent. As above, this would suggest that i is unlikely to occur for the DM, but this is *relatively to the occurence of* j, or put differently, i is much less likely than j. Suppose in addition that a_{ik} is indifferent to a_i. Again, this suggests that k much less likely to occur than i. Since i is much less likely than j, the conclusion is that k is very unlikely to occur, hence a_k is indifferent to a_0. This explains the first case in the MOPI condition. The second case (indifference between a_{ik} and a_k, and between a_{ij} and a_i) works exactly the same way. The third case says that a_{ij} and a_j are indifferent (i is much less likely than j) as well as a_{ik} and a_k (i is much less likely than k). Since i is much less likely than both j and k, the conclusion is that i is very unlikely, so that a_i is indifferent with a_0.

For $N = \{1, 2\}$, the 2-MOPI property can be also viewed as uncertainty aversion[1] (see [18]). Indeed,

[1] Uncertainty aversion, as presented in [18], is defined as follows: For three acts x, y, z, if y and z are comonotonic then:

$$x \sim y \Rightarrow x + z \succsim y + z.$$

Comonotonicity between two acts y, z means that there are no $i, j \in N$ such that $u(y_i) > u(y_j)$ and $u(z_i) < u(z_j)$.

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given the three acts $a_{12} = (\mathbf{1}, \mathbf{1})$, $a_1 = (\mathbf{1}, \mathbf{0})$, $z = (-\mathbf{1}, \mathbf{0})$ and using the property of uncertainty aversion, we have:

$$(\mathbf{1}, \mathbf{0}) \sim (\mathbf{1}, \mathbf{1}) \Rightarrow (\mathbf{0}, \mathbf{0}) \succsim (\mathbf{0}, \mathbf{1})$$

which corresponds to the 2-MOPI property in this case. However, this interpretation does not work any more for the MOPI condition.

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