Pointwise defined CRI-based aggregation distributive operators are trivial

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Abstract— In a rather general, essentially aggregation operator based discussion of the traditional fuzzy control strategies known as FATI and FITA strategies, a way to reduce these strategies to one another has been to define pairs of aggregation distributive aggregation operators. In this paper it is shown that for some often used special cases this reduction condition allows only the set theoretic union as aggregation operator.

Keywords— fuzzy control, FATI strategy, FITA strategy, aggregation operators, aggregation distributivity

1   Introduction

The standard paradigm behind standard approaches toward fuzzy control is that one supposes to have given, as an incomplete and fuzzy description of a control function Φ from an input space $\mathbb{X}$ to an output space $\mathbb{Y}$, a family

$$\mathcal{D} = (\langle A_i, B_i \rangle)_{1 \leq i \leq n} \qquad (1)$$

of (fuzzy) input-output data pairs to characterize this function Φ.

In the usual approaches such a family of input-output data pairs is provided by a finite list

$$\text{IF } x \text{ is } A_i \text{ THEN } y \text{ is } B_i, \qquad (2)$$

(with $i = 1, \ldots, n$) of linguistic control rules, also called fuzzy IF-THEN rules.

The basic examples of fuzzy control approaches are Zadeh's original approach via fuzzy relations and the compositional rule of inference (CRI), as prototypically realized by the Mamdani–Assilian approach in [7], and the Holmblad–Ostergaard approach toward fuzzy control of a cement kiln as explained in [5].

Derived from these two approaches there is the well known distinction between FATI and FITA strategies to evaluate systems of linguistic control rules w.r.t. arbitrary fuzzy inputs from $\mathcal{F}(\mathbb{X})$.

2   Preliminaries

The core idea of a FITA strategy is that it is a strategy which **F**irst **I**nfers (by reference to the single rules) and **T**hen **A**ggregates starting from the actual input information A. Contrary to that, a FATI strategy is a strategy which **F**irst **A**ggregates (the information in all the rules into one fuzzy relation) and **T**hen **I**nfers starting from the actual input information A.

Both these strategies use the set theoretic union as their aggregation operator. Furthermore, both of them refer to the CRI as their core tool of inference.

In general, however, the interpolation operators may depend more generally upon some inference operator(s) as well as upon some aggregation operator.

2.1   Aggregation operations and fuzzy control strategies

By an *inference operator* we mean here simply a mapping from the class of fuzzy subsets of the input space to the class of fuzzy subsets of the output space.[1]

And an *aggregation operator* $\mathbf{A}$, as explained e.g. in [1, 2], is a family $(f^n)_{n \in \mathbb{N}}$ of ("aggregation") operations, each f^n an n-ary one, over some partially ordered set $\mathbf{M}$, with ordering $\leqslant$, with a bottom element $\mathbf{0}$ and a top element $\mathbf{1}$, such that each operation f^n is non-decreasing, maps the bottom to the bottom: $f^n(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$, and the top to the top: $f^n(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$.

Such an aggregation operator $\mathbf{A} = (f^n)_{n \in \mathbb{N}}$ is a *commutative* one iff each operation f^n is commutative. And $\mathbf{A}$ is an *associative* aggregation operator iff

$$f^n(a_1, \ldots, a_n) = \\ f^r(f^{k_1}(a_1, \ldots, a_{k_1}), \ldots, f^{k_r}(a_{m+1}, \ldots, a_n)) \quad (3)$$

for $n = \sum_{i=1}^{r} k_i$ and $m = \sum_{i=1}^{r-1} k_i$.

Our aggregation operators further on are supposed to be commutative as well as associative ones.[2]

As in [3, 4], we now consider operators Ψ of FATI-type operators Ξ of FITA-type and which have the abstract forms

$$\Psi_{\mathcal{D}}(A) = \mathbf{A}(\theta_{\langle A_1, B_1 \rangle}(A), \ldots, \theta_{\langle A_n, B_n \rangle}(A)), \quad (4)$$

$$\Xi_{\mathcal{D}}(A) = \widehat{\mathbf{A}}(\theta_{\langle A_1, B_1 \rangle}, \ldots, \theta_{\langle A_n, B_n \rangle})(A). \quad (5)$$

Here we assume that each one of the "local" inference operators θ_i is determined by the single input-output pair $\langle A_i, B_i \rangle$. This restriction is in general sufficient. For the present purpose we assume that our inference operators are CRI-based, i.e. we assume that $\theta_{\langle A_1, B_1 \rangle}(A)$ has

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[1]This terminology has its historical roots in the fuzzy control community. There is no relationship at all with the logical notion of inference intended and supposed here; but–of course–also not ruled out.

[2]It seems that this is a rather restrictive choice from a theoretical point of view. However, in all the usual cases these restrictions are satisfied.

the form

$$\theta_{\langle A_i, B_i \rangle}(A) = R_i'' A \qquad (6)$$

for some fuzzy relation R. In this case we call the inference operator $\theta_{\langle A_1, B_1 \rangle}$ CRI-*based*.

Furthermore $\mathbf{A}$ has to be an aggregation operator for fuzzy subsets of the universe of discourse $\mathbb{X}$, and $\widehat{\mathbf{A}}$ has to be an aggregation operator for inference operators.

2.2 Stability conditions

If $\Theta_{\mathcal{D}}$ is a fuzzy inference operator of one of the types (4), (5), then the interpolation property one likes to have realized is that one has

$$\Theta_{\mathcal{D}}(A_i) = B_i \qquad (7)$$

for all the data pairs $\langle A_i, B_i \rangle$. In the particular case that the operator $\Theta_{\mathcal{D}}$ is detrmined by the CRI-methodology, this is just the usual problem to solve a system (7) of fuzzy relation equations.

In the present generalized context the property (7) has been called the $\mathcal{D}$-*stability* of the fuzzy inference operator $\Theta_{\mathcal{D}}$.

To find $\mathcal{D}$-stability conditions on this abstract level seems to be rather difficult in general. However, the restriction to fuzzy inference operators of FITA-type makes things easier.

To explain some of the known results it is necessary to have a closer look at the aggregation operator $\mathbf{A} = (f^n)_{n \in \mathbb{N}}$ involved in (4) which operates on $\mathbb{F}(\mathbb{Y})$, of course with the inclusion relation for fuzzy sets as partial ordering.

Definition 1 *Having $B, C \in \mathbb{F}(\mathbb{Y})$ we say that C is $\mathbf{A}$-negligible w.r.t. B iff $f^2(B, C) = f^1(B)$ holds true.*

The core idea here is that in any aggregation by $\mathbf{A}$ the presence of the fuzzy set B among the aggregated fuzzy sets makes any presence of C superfluous.

For examples and further interesting properties of aggregation operators the interested reader may consult [3, 4].

Now we are in a position to state one of the results from [3, 4] to give an impression of what becomes of interest in the present context.

Proposition 1 *Consider a fuzzy inference operator $\Psi_{\mathcal{D}}$ of FITA-type (4). It is sufficient for the $\mathcal{D}$-stability of $\Psi_{\mathcal{D}}$, i.e. to have*

$$\Psi_{\mathcal{D}}(A_k) = B_k \qquad \text{for all } k = 1, \ldots, n \qquad (8)$$

that one always has

$$\theta_{\langle A_k, B_k \rangle}(A_k) = B_k \qquad (9)$$

and additionally that for each $i \neq k$ the fuzzy set

$$\theta_{\langle A_k, B_k \rangle}(A_i) \quad \text{is } \mathbf{A}\text{-negligible w.r.t.} \quad \theta_{\langle A_k, B_k \rangle}(A_k). \quad (10)$$

This result has two quite interesting specializations which themselves generalize well known results about fuzzy relation equations. The interested reader may consult [3, 4].

To extend such considerations from inference operators (4) of the FITA type to those ones of the FATI type (5) let us consider the following notion.

Definition 2 *Suppose that $\widehat{\mathbf{A}}$ is an aggregation operator for inference operators, and that $\mathbf{A}$ is an aggregation operator for fuzzy sets. Then $(\widehat{\mathbf{A}}, \mathbf{A})$ is an* application distributive *pair of aggregation operators iff*

$$\widehat{\mathbf{A}}(\theta_1, \ldots, \theta_n)(X) = \mathbf{A}(\theta_1(X), \ldots, \theta_n(X)) \qquad (11)$$

holds true for arbitrary inference operators $\theta_1, \ldots, \theta_n$ and fuzzy sets X.

Using this notion it is easy to see that one has on the left hand side of (11) a FATI type inference operator, and on the right hand side an associated FITA type inference operator. So one is able to give a reduction of the FATI case to the FITA case, assuming that such application distributive pairs of aggregation operators exist.

Proposition 2 *Suppose that $(\widehat{\mathbf{A}}, \mathbf{A})$ is an application distributive pair of aggregation operators. Then a fuzzy inference operator $\Xi_{\mathcal{D}}$ of FATI-type is $\mathcal{D}$-stable iff its associated fuzzy inference operator $\Psi_{\mathcal{D}}$ of FITA-type is $\mathcal{D}$-stable.*

3 Application distributivity

Based upon the notion of application distributive pair of aggregation operators the property of $\mathcal{D}$-stability can be transferred back and forth between two inference operators of FATI-type and of FITA-type if they are based upon a pair of application distributive aggregation operators.

What has not been discussed previously was the existence and the uniqueness of such pairs. Here are some results concerning these problems.

The uniqueness problem has a simple solution.

Proposition 3 *If $(\widehat{\mathbf{A}}, \mathbf{A})$ is an application distributive pair of aggregation operators then $\widehat{\mathbf{A}}$ is uniquely determined by $\mathbf{A}$, and conversely also $\mathbf{A}$ is uniquely determined by $\widehat{\mathbf{A}}$.*

And for the existence problem we have a nice reduction to the two-argument case.

Theorem 1 *Suppose that $\mathbf{A}$ is a commutative and associative aggregation operator. For the case that there exists an aggregation operator $\widehat{\mathbf{A}}$ such that $(\widehat{\mathbf{A}}, \mathbf{A})$ form an application distributive pair of aggregation operators it is necessary and sufficient that there exists some operation G for fuzzy inference operators satisfying*

$$\mathbf{A}(\theta_1(X), \theta_2(X)) = G(\theta_1, \theta_2)(X) \qquad (12)$$

for all fuzzy inference operators θ_1, θ_2 and all fuzzy sets X.

However, there is an important restriction concerning the existence of such pairs of application distributive aggregation operators, at least for the interesting particular case that the application operation is determined by the compositional rule of inference (CRI). And this means simply that the inference operations θ_i are determined via suitable fuzzy relations R_i.

Definition 3 *An aggregation operator* $\mathbf{A} = (f^n)_{n \in \mathbb{N}}$ *for fuzzy subsets of a universe of discourse* $\mathbb{X}$ *is* pointwise defined *iff for each* $n \in \mathbb{N}$ *there exists a function* $g_n : [0,1]^n \to [0,1]$ *such that for all* $A_1, \ldots, A_n \in \mathcal{F}(\mathbb{X})$ *and all* $x \in \mathbb{X}$ *there hold*

$$f^n(A_1, \ldots, A_n)(x) = g_n(A_1(x), \ldots, A_n(x)). \quad (13)$$

And an aggregation operator $\widehat{\mathbf{A}}$ *for inference operators is* pointwise defined *iff it can be reduced to a pointwise defined aggregation operator for fuzzy relations.*

From the isotonicity behavior of the aggregation operator $\mathbf{A}$ it follows that also these characterizing functions g_n are isotonic, and similarly in the case of $\widehat{\mathbf{A}}$.

The restrictive result, first proved in [6], now reads as follows.

Theorem 2 *Suppose that all inference operators are CRI-based. Then the pair* $(\bigcup, \bigcup)$ *is the only application distributive pair among the commutative, associative, and pointwise defined aggregation operators.*

Proof: Obviously the considerations can restricted to the binary case $n = 2$. So let us start in this CRI-based case with an aggregation operator $\widehat{\mathbf{A}}$, which has to give a fuzzy relation $\widehat{\mathbf{A}}(R_1, R_2)$ for any two inference operators θ_1, θ_2 determined by the fuzzy relations R_1, R_2, respectively. Because $\widehat{\mathbf{A}}$ has to be pointwise defined, according to Definition 3 there has to be a function $\widehat{g} : [0,1] \times [0,1] \to [0,1]$ such that one has for the membership degrees of the corresponding fuzzy relations

$$\widehat{\mathbf{A}}(R_1, R_2)(x, y) = \widehat{g}(R_1(x, y), R_2(x, y)). \quad (14)$$

In a similar way, again by Definition 3, a corresponding aggregation operator $\mathbf{A}$ has to be determined by a function $g : [0,1] \times [0,1] \to [0,1]$. Assuming that these aggregation operators $(\mathbf{A}, \widehat{\mathbf{A}})$ form an aggregation distributive pair, gives for arbitrary fuzzy inputs A the condition

$$
\begin{aligned}
&\widehat{\mathbf{A}}(R_1, R_2)''(A)(y) \\
&= \bigvee_x T\Big(A(x), \widehat{\mathbf{A}}(R_1, R_2)(x, y)\Big) \\
&= \bigvee_x T\Big(A(x), \widehat{g}(R_1(x, y), R_2(x, y))\Big) \quad (15) \\
&= g\Big(\bigvee_x T(A(x), R_1(x, y)), \bigvee_x T(A(x), R_2(x, y))\Big) \\
&= \mathbf{A}(R_1'' A, R_2'' A)(y),
\end{aligned}
$$

which has to be satisfied for arbitrary fuzzy sets A and fuzzy relations R_1, R_2. Of course, T here is the t-norm involved in the CRI application process.

So let be always $A(x) = 1$ and furthermore $R_1(x, y) = a$, $R_2(x, y) = b$ for some $a, b \in [0, 1]$. Now routine calculations yield $\widehat{g}(a, b) = g(a, b)$, which means equality of the functions $\widehat{g} = g$ which determine the aggregation operators $\widehat{\mathbf{A}}, \mathbf{A}$, respectively.

So application distributivity of the pair $(\mathbf{A}, \widehat{\mathbf{A}})$ becomes a condition which has to be satisfied by the characterizing function g, and this condition reads

$$\bigvee_x T\Big(A(x), g(R_1(x, y), R_2(x, y))\Big) = \quad (16)$$

$$g\Big(\bigvee_x T(A(x), R_1(x, y)), \bigvee_x T(A(x), R_2(x, y))\Big).$$

To continue our discussion and to finish the proof of Theorem 2, we insert two lemmata.

Lemma 1 *Suppose that* $g : [0,1] \times [0,1] \to [0,1]$ *determines a commutative and associative pointwise defined* (*binary*) *aggregation operator. Then the condition*

$$\sup_{i \in I} g(a_i, b_i) = g(\sup_{i \in I} a_i, \sup_{i \in I} b_i) \quad (17)$$

is equivalent to the fact that g *is left continuous and satisfies*

$$g(a, b) = g(\max\{a, b\}, \max\{a, b\}). \quad (18)$$

It is easy to see that (17) implies the left continuity of g. So assume (17) and that (18) is not generally satisfied. Then there are $a_0, b_0 \in [0, 1]$ such that

$$g(a_0, b_0) \neq g(\max\{a_0, b_0\}, \max\{a_0, b_0\}) \quad (19)$$

and additionally, w.l.o.g., also $b_0 \leq a_0$. This last condition forces even its strengthening $b_0 < a_0$, and together with the isotonicity of g yields $g(a_0, b_0) < g(a_0, a_0)$. But this now means

$$
\begin{aligned}
\max\{g(a_0, b_0), g(b_0, a_0)\} &= g(a_0, b_0) \\
&< g(a_0, a_0) = g(\max\{a_0, b_0\}, \max\{a_0, b_0\}), \quad (20)
\end{aligned}
$$

contradicting (17). So the $(\Rightarrow)$-part of the lemma is proved.

If otherwise g is left continuous and satisfies (18), one has

$$g(\sup_{i \in I} a_i, \sup_{i \in I} b_i) \geq \sup_{i \in I} g(a_i, b_i) \quad (21)$$

by the isotonicity of g. But $g(a_j, b_j) \leq \sup_{i \in I} g(a_i, b_i)$ and (18) yield

$$g(c_j, c_j) \leq \sup_{i \in I} g(a_i, b_i) \quad (22)$$

for $c_j = \max\{a_j, b_j\}$, and by the left continuity and isotonicity of g this gives

$$g(\sup_{i \in I} c_i, c_j) \leq \sup_{i \in I} g(c_i, c_i) \leq \sup_{i \in I} g(a_i, b_i),, \quad (23)$$

and thus also

$$g(\sup_{i \in I} a_i, \sup_{i \in I} b_i) \leq g(\sup_{i \in I} c_i, \sup_{i \in I} c_i) \leq \sup_{i \in I} g(a_i, b_i). \quad (24)$$

All together gives (17) and hence the $(\Leftarrow)$-part of the lemma.

The condition (17) obviously means that the corresponding aggregation operator commutes with the supremum.

Lemma 2 *Suppose that* $g : [0, 1] \times [0, 1] \to [0, 1]$ *determines a commutative and associative pointwise defined* (*binary*) *aggregation operator. Then condition* (17) *is satisfied iff there exists a left continuous isotonic function* $h : [0, 1] \to [0, 1]$ *satisfying* $h(0) = 0$, $h(1) = 1$ *and*

$$g(a, b) = \max\{h(a), h(b)\} \, . \qquad (25)$$

For the ($\Rightarrow$)-part consider the function $h(x) = g(0, x)$. It is left continuous as well as isotonic, satisfies $h(0) = 0$ and $h(1) = 1$, and one has by Lemma 1

$$
\begin{aligned}
g(a, b) &= g(\max\{a, b\}, \max\{a, b\}) \\
&= g(\max\{0, \max\{a, b\}\}, \max\{0, \max\{a, b\}\}) \\
&= g(0, \max\{a, b\}) = \max\{h(a), h(b)\} \, . \quad (26)
\end{aligned}
$$

The ($\Leftarrow$)-part follows from routine calculations.

So we come back to the proof of Theorem 2. What we did not discuss up to now is that for having an aggregation distributive pair of pointwise defined operators the characterizing function g has to be distributive relative to the t-norm T which determines the CRI-application. So we need to have always satisfied

$$T(a, g(b, c)) = g(T(a, b), T(a, c)) \, , \qquad (27)$$

which means, via Lemmata 1 and 2, to have always satisfied

$$
\begin{aligned}
&\max\{T(a, h(b)), T(a, h(c))\} \\
&= \max\{h(T(a, b)), h(T(a, c))\} \, , \quad (28)
\end{aligned}
$$

which forces that one always has to have

$$T(a, h(b)) = h(T(a, b)) \, . \qquad (29)$$

And this yields $h = \mathsf{id}$: because otherwise there would exist some c with $h(c) \neq c$ and

$$T(c, h(1)) = T(c, 1) = c \neq h(c) = h(T(c, 1)) \, . \qquad (30)$$

So Theorem 2 is finally proved.

4   Conclusion

The type of approach explained in Section 2.2 works actually well only in the FITA case. This was the starting point for the considerations on aggregation distributive operator pairs. They give a transfer possibility to the FATI case.

However, as the main result, i.e. Theorem 2 of the present paper shows, these transfer possibilities are quite restricted under some conditions which have, up to now, been considered rather natural ones.

For the authors understanding this result points into two directions. (i) It may be appropriate to try to find other ways and to discuss the FATI case differently. And this way may become essentially different from the reduction strategy toward fuzzy relation equations which stands behind the generalization in [3, 4]. (ii) It may be suitable to move into the realm of aggregation operators which are no longer pointwise defined, and it may

also be suitable to leave the world of the CRI-based approaches.

Particularly point (ii) here seems promising, even having in mind that the actual standard cases all fall into the class of pointwise defined aggregation operators. Further investigations into this topic are necessary.

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