Multiplicative and implicative importance weighted averaging aggregation operators with accurate andness direction

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Abstract— *Importance weighted averaging aggregation operators play a key role in utilizations of electronic data and information resources for retrieving, fusing, and extracting information and knowledge, as needed for decision making. Two central issues in the choice of such operators are the kind of importance weighting and the andness (or conjunction degree) of the operator. We present and discuss two main kinds of importance weighting, namely multiplicative and implicative, each yielding a particular kind of operator for a particular kind of aggregation problems. Importance weighting generalizations of each kind are proposed for each of two classes of averaging operators, namely the Power Means and the Ordered Weighted Averaging operators, each in a De Morgan dual form for increased efficacy. For each class is proposed a function for a rather accurate direct control of the andness. Operators of the same kind appear to behave rather similarly at the same andness, independent of the class of averaging operators generalized.*

Keywords— Aggregation operators, averaging operators, andness, orness, importance weighting, implicative importance.

1 Introduction

With the increasing amount of electronically accessible data in databases, document bases, and data streams, and the need of efficient utilization of such data, aggregation operators have attained new interest. This is due to the central role of such operators in the key reasoning tasks, such as information fusion and pattern recognition, applied in utilizations of many kinds, *e.g.*, information retrieval, information extraction, object recognition, and knowledge discovery. Of particular interest are the aggregations operators between AND (conjunction) and OR (disjunction), *i.e.*, the averaging operators, or generalized conjunction/disjunction functions (GCD) [1], especially the importance weighted generalizations of such operators.

An importance weighted averaging aggregation operator aggregates a number of arguments, each qualified by an importance weight, into a single score. We distinguish between two kinds of importance weighting, namely multiplicative and implicative. Each kind provides an importance weighting generalization of the unweighted averaging operator, in the sense that the latter is retained by the case where all arguments have the same importance weight. We propose such generalizations of two classes of averaging operators, namely the Power Means (PM) and the Ordered Weighted Averaging (OWA) operators [2], each in a De Morgan dual form for increased efficacy, and each with an accurate direct control of the andness, *i.e.*, the degree the operator aggregates like an AND rather than an OR. The generalizations of the PM include the AIWA operators [3].

A few words on the notation applied in the following. $\bigodot_i a_i$, where $\bigodot$ is an operator, is an abbreviation for $\bigodot_{i=1}^{n}(a_i)$; a bold letter like $\mathbf{x}$ denotes the vector $(x_1,\ldots,x_n)$; $\overline{x}$ denotes the standard negation $\overline{x} = 1 - x$; $\overline{\mathbf{x}}$ denotes $(\overline{x_1},\ldots,\overline{x_n})$. Some specific letters are used all over with the same meaning: I denotes the real unit interval $[0,1]$; n is the dimension, *i.e.*, number of arguments aggregated by the averaging operator considered; ρ denotes the targeted andness of an averaging operator; $\mathbf{a}$ denotes a vector of arguments $(a_1,\ldots,a_n) \in I^n$; $\mathbf{v}$ denotes an importance weighting vector $(v_1,\ldots,v_n) \in I^n$ that is max-normalized, *i.e.*, $\max_i v_i = 1$; $\mathbf{w}$ denotes an importance weighting vector $(w_1,\ldots,w_n) \in I^n$ that is sum-normalized, *i.e.*, $\sum_i w_i = 1$; $h : I^n \times I^n \to I : (\mathbf{v},\mathbf{a}) \mapsto h(\mathbf{v},\mathbf{a})$ is an importance weighted averaging operator, such that v_i is the importance of a_i; for a particular operator h, its name acronym and parameters are attached on the form $h^{\text{acronym}}_{\text{parameters}}$.

Section 2 introduces some basic concepts applied in the following. Sections 3 and 4 introduce, for each class of averaging operators, the PM based and the OWA based, the basis of the class, its multiplicative and implicative importance weighting generalizations, and a function for accurate direct control of the andness of its operators. Sections 5 and 6 discuss, for each kind of importance weighting, the multiplicative and the implicative, the kind of applications they apply to and some key issues of relevance for their application. Section 7 compares and discusses the effect of the two kinds of importance weighting on the common problem of ordering a set of options by their score. Section 8 concludes.

2 Basic concepts

2.1 *The andness of an averaging operator*

The andness may be viewed as the degree of universal quantification over the arguments, with AND (andness $= 1$) representing *all* and OR (andness $= 0$) representing *at least one*. In this view, the andness is the degree to which all arguments, rather than at least one argument, must support the result of the aggregation. In importance weighted averaging operators, each argument is considered to the degree it is important. The dual interpretation applies to the orness, defined by: orness $= 1 -$ andness.

Since, AND and OR are evaluated by the operators min and max, respectively, the andness of an averaging operator h may be defined as distance between h and the max, relative to the distance between the min and the max, with operators h, max, and min evaluated as the mean of these operators over the ar-

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gument space, as proposed by Dujmović [4]:

$$\text{andness}(h) = \frac{E(\max_i(\mathbf{x})) - E(h(\mathbf{x}))}{E(\max_i(\mathbf{x})) - E(\min_i(\mathbf{x}))} \quad (1)$$

where $E(f(\mathbf{x}))$ is the mean of $f(\mathbf{x})$ over the argument space I^n, as defined by $E(f(\mathbf{x})) = \int_{I^n} f(\mathbf{x})d\mathbf{x}$. The terms *andness* and *orness* were coined by Yager for a measure of the OWA operators [2] that is fully consistent with the measure defined by (1).

The andness is, by OWA operators, obtained by positional weights (OWA weights), such that the OWA aggregate is the sum of the products of the i'th OWA weight and the i'th largest argument, whereas it by PM operators is obtained by a power function, yielding emphasis on the smaller or larger valued arguments to the degree the PM represents, respectively, AND and OR (see, *e.g.*, [1]).

2.2 *The two kinds of importance weighting*

A well-known example of an importance weighted averaging operator is the Weighted Arithmetic Mean (WAM) defined by $h^{\text{WAM}}(\mathbf{v}, \mathbf{a}) = \sum_i (w_i, a_i)$ where, for all i, $w_i = \frac{v_i}{\sum_i v_i}$, for which andness$(h^{\text{WAM}}) = $ orness$(h^{\text{WAM}}) = \frac{1}{2}$. The unweighted case, as obtained by $v_i = 1$ for all i, is the Arithmetic Mean (AM): $h^{\text{WAM}}((1, 1, \ldots, 1), \mathbf{a}) = \frac{1}{n}\sum_i a_i = h^{\text{AM}}(\mathbf{a})$. The WAM appears to be the only importance weighted averaging operator that represents both kinds of importance weighting, namely the multiplicative and the implicative. For other degrees and andness than $\frac{1}{2}$, the two kinds of importance weighting provide, as we shall see, different behaviors that apply to different kinds of aggregation problems.

While Multiplicative importance Weighted Averaging (for short, MWA) operators are weighted means, Implicative importance Weighted Averaging (for short, IWA) operators are logic operators for pattern matching inference. In general, the andness of an IWA operator "penalizes" (decreases) the score (the aggregate) to the degree that there are criteria that are important, but not satisfied, while the orness "rewards" (increases) the score to the degree that there are criteria that are important and satisfied.

By the importance weighting, the arguments are, essentially, transformed by the importance weights in a way giving the desired effect, depending on the kind of importance weighting and the andness, in the aggregation. In principle, an argument a with the importance weight w is, by the two kinds of weighting, transformed as follows: by MWA, to the product wa, and by IWA at andness ρ to $\rho(w \Rightarrow a) + \overline{\rho}(\overline{w \Rightarrow \overline{a}})$ where $\Rightarrow$ is a fuzzy implication operator. It is the multiplication in the MWA case and the implication in the IWA case that give name to the two kinds of importance weighting.

As the fuzzy implication $\Rightarrow$ for IWA operators, we choose the Reichenbach implication, $\Rightarrow_{\text{R}}$, as defined by $(v \Rightarrow_{\text{R}} a) = \overline{v\overline{a}} = 1 - v(1 - a)$. The reasons for this choice are, first, that it allows us easily to obtain the WAM at $\rho = \frac{1}{2}$, and, second, that the choice of the fuzzy implication appears not to be critical for the behavior of such operators. The latter was tested empirically; the paper doesn't leave space for presenting details from the test.

A key difference between between MWA and IWA operators is that in MWA operators, as opposed to IWA operators, the effect of the importance weights decreases from full effect to no effect as the andness goes from $\frac{1}{2}$ to one of its extremes, 0 or 1. For an appropriate choice of the fuzzy implication operator, the IWA operators represent, like the MWA operators, the WAM at andness $\frac{1}{2}$. These behaviors of the two kind of importance weighted averaging operators are illustrated by Figure 1. We notice, as the figure indicates, that in cases where

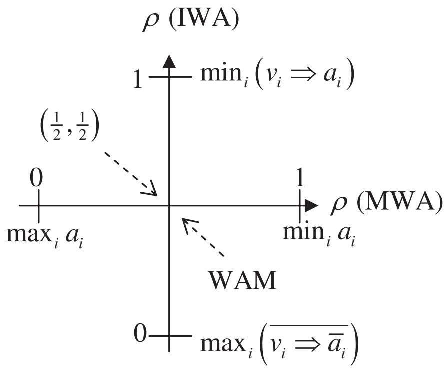

Figure 1: Illustration of MWA and IWA operators

we are completely uncertain about the proper kind of importance weighting and andness for an application, WAM may be a good choice as a starting point.

2.3 *The two classes of operators generalized*

In the following, we shall in particular consider importance weighted generalizations of two classes of operators, namely the Power Means (PM) and the Ordered Weighted Averaging (OWA) operators, each in a De Morgan dual form for increased efficacy. For each class, we propose two andness-directing functions, $\psi(\rho)$ and $\psi(\rho, n)$, each yielding a value ($\in [1, \infty[$) of the andness controlling parameter of the class. While $\psi(\rho)$ directly yields an andness of the operator that, in general, is somewhat close to the target ρ, $\psi(\rho, n)$ yields the target rather accurately, due to it also being dependent of n, as applied in the definition by (1).

For each class, we provide a common expression of the two kinds of importance weighting generalization, such that the choice of a kind is controlled by a parameter $\gamma \in \{0, 1\}$, yielding the multiplicative generalization by $\gamma = 0$ and the implicative generalization by $\gamma = 1$. The operators are in their common generalized form referred to as, respectively, Andness-directed importance Weighted averaging (AWA) operators and andness-directed importance Weighted OWA operators (W-OWA). Figure 2 gives an overview over the operator name acronyms applied.

Basis	*Without* importance weighting	*With* importance weighting		
		Common class	**M**ultipl. $\gamma = 0$ **M**WA	**I**mpl. $\gamma = 1$ **I**WA
Power Means	AA	AWA	A**M**WA	A**I**WA
OWA	OWA	W-OWA	**M**W-OWA	**I**W-OWA

Figure 2: Operator and operator class name acronyms applied

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3 AWA operators

3.1 Basis in the Power Means

The AWA (Andness-directed importance Weighted Averaging) operators are based on the Power Means (PM) that (in their unweighted form) are defined by $h_\alpha^{\text{PM}}(\mathbf{a}) = (\frac{1}{n}\sum_i a_i^\alpha)^{\frac{1}{\alpha}}$, $\alpha \in \mathbb{R}$. When α goes from $-\infty$ to $+\infty$, the andness of h_α^{PM} goes from 1 down to 0. It has an asymmetric behavior around andness $= \frac{1}{2}$, as obtained at $\alpha = 1$; in particular, it is "mandatory" for $\alpha \le 0$, corresponding to andness $\tilde{>}\frac{2}{3}$, in the sense that $h_\alpha^{\text{PM}}(\mathbf{a}) = 0$, if $\min_i(a_i) = 0$, which normally is an undesired property. [1] However, by the De Morgan dual variant PM', defined by (i) $h_\alpha^{\text{PM}}(\mathbf{a})$, $\alpha \ge 1$, for andness $\le \frac{1}{2}$, and (ii) $1 - h_\alpha^{\text{PM}}(\overline{\mathbf{a}})$, $\alpha \ge 1$, for andness $\ge \frac{1}{2}$, we obtain a symmetric behavior around andness $= \frac{1}{2}$ and avoid the "mandatory" property. For both (i) and (ii), the arithmetic mean is obtained at $\alpha = 1$; when α goes to $+\infty$, the andness of $h_\alpha^{\text{PM}'}$ goes in case (i) to 0 and in case (ii) to 1.

The AWA operators comprise the multiplicative and implicative importance weighted generalizations of PM$'$, namely the AMWA (Andness-directed Multiplicative importance Weighted Averaging) operators and the AIWA (Andness-directed Implicative importance Weighted Averaging) operators. The AIWA operators were introduced in [3] where we analyzed their properties and showed that they implement implicative importance weighting by the Riechenbach implication. The AMWA operators is the multiplicative weighting variant of the AIWA operators.

3.2 Definition of AWA operators

The AWA operators are defined by the following semi-recursive expression, where the case of $\rho > \frac{1}{2}$ is defined by its duality to the case of $\rho \le \frac{1}{2}$:

$$h_\rho^{\text{AWA}}(\mathbf{v}, \mathbf{a}) = \begin{cases} (\sum_i (y_i a_i^\alpha))^{\frac{1}{\alpha}} & \rho \le \frac{1}{2} \\ 1 - h_{\overline{\rho}}^{\text{AWA}}(\mathbf{v}, \overline{\mathbf{a}}) & \rho > \frac{1}{2} \end{cases} \quad (2)$$

where $\alpha = \psi^{\text{AA}}(\rho) = \overline{\rho}/\rho$, and for all i:

$$y_i = \frac{v_i^{\alpha^\gamma}}{\sum_i v_i^{\alpha^\gamma}} \quad (3)$$

with $\gamma \in \{0, 1\}$, $\gamma = 0$ (by which $y_i = \frac{v_i}{\sum_i v_i} = w_i$) yielding the AMWA operators, and $\gamma = 1$ (by which $y_i = \frac{v_i^\alpha}{\sum_i v_i^\alpha}$) yielding the AIWA operators.

By $\mathbf{v} = (1, 1, \ldots, 1)$ we obtain the common unweighted case of AMWA and AIWA that we shall refer to as Andness-directed Averaging (AA) operators; thus, these operators are, in the case of $\rho \le \frac{1}{2}$, defined by: $h_\rho^{\text{AA}}(\mathbf{a}) = h_\rho^{\text{AWA}}((1, 1, \ldots, 1), \mathbf{a}) = (\frac{1}{n}\sum_i a_i^\alpha)^{\frac{1}{\alpha}}$.

3.3 Accurate control of AWA andness

Above, the parameter α is, for a targeted andness ρ, defined by $\alpha = \psi^{\text{AA}}(\rho) = \overline{\rho}/\rho$ in the case $\rho \le \frac{1}{2}$ (and, by duality, as implemented by the semi-recursive call, $\alpha = \psi^{\text{AA}}(\overline{\rho}) = \rho/\overline{\rho}$ in the case $\rho > \frac{1}{2}$). While by $\psi^{\text{AA}}(\rho)$, andness(h^{AWA}) is equal to ρ, if $\psi^{\text{AA}}(\rho) \in \{0, \frac{1}{2}, 1\}$ and, otherwise, somewhat close to ρ, a much more precise approximation can be obtained by also considering the dimension n, namely through replacing $\psi^{\text{AA}}(\rho)$ by $\psi^{\text{AA}}(\rho, n)$ as defined by:

$$\psi^{\text{AA}}(\rho, n) = (-\log_2 \rho)^{1+(\log_2 n)^{1/3}} \quad (4)$$

An empirical test of the error $\|\rho - \text{andness}(h^{\text{AWA}})\|$, showed that by $\psi^{\text{AA}}(\rho)$ the average error over $\rho \in I$ increases from about 0.03 to about 0.09 as n increases from 2 to 100, while it by $\psi^{\text{AA}}(\rho, n)$ remains around 0.005.

[1] The mandatory threshold of andness $\tilde{>}\frac{2}{3}$ enforced by the PM is, from a practical point of view, rather arbitrary.

3.4 Decomposability of AWA operators

A property of particular interest for applications of AWA operators is their decomposability, allowing us to update the aggregate if the value of an argument has changed or a new argument has to be considered, without re-computing the whole aggregate. [3] Thus, considering the common case by AWA (2) for $\rho \le \frac{1}{2}$, let $h_\rho^{\text{AWA}}(\mathbf{v}, \mathbf{a}) = c$ to be updated with a new argument a_{n+1} with the importance weight v_{n+1}. Let $\cup$ denote the concatenation of vectors, such that $\mathbf{x} \cup (x_{n+1}) = (x_1, \ldots, x_n) \cup (x_{n+1}) = (x_1, \ldots, x_{n+1})$. Then, the updated aggregate is computed as the AWA aggregate of c and the new argument, as defined by: $h_\rho^{\text{AWA}}(\mathbf{v} \cup (v_{n+1}), (\mathbf{a} \cup (a_{n+1})) = h_\rho^{\text{AWA}}((s, v_{n+1}), (c, a_{n+1})) = \left(\frac{s^{\alpha^\gamma} c^\alpha + v_{n+1}^{\alpha^\gamma} a_{n+1}^\alpha}{s^{\alpha^\gamma} + v_{n+1}^{\alpha^\gamma}}\right)^{1/\alpha}$, with $s = \left(\sum_{i=1}^n v_i^{\alpha^\gamma}\right)^{1/\alpha^\gamma}$.

4 W-OWA operators

4.1 Basis in the OWA operators

The properties of OWA operators [2] are controlled by a vector of sum-normalized position weights $\mathbf{u}$ (called OWA weights) such that the OWA aggregate of an argument vector $\mathbf{a}$ is defined by $h^{\text{OWA}}(\mathbf{a}) = \sum_i (u_i, a_{(i)})$ where $(\cdot)$ is an index permutation such that $a_{(1)} \ge \cdots \ge a_{(n)}$. Two key properties of OWA operators are their andness, defined by andness$(\mathbf{u}) = \frac{1}{n-1}\sum_i((i-1)u_i)$, and their normalized dispersion (or entropy), defined by $\text{ndisp}(\mathbf{u}) = -\frac{1}{\ln n}\sum_i(u_i \ln u_i)$. This andness measure is consistent with (1).

In [5, 6], Yager proposed an importance weighted generalization of quantifier guided OWA operators that essentially is a multiplicative importance weighting generalization, which we here shall refer to as MW-OWA operators. They are defined by $h_f^{\text{MW-OWA}}(\mathbf{v}, \mathbf{a}) = \sum_i (u_i, a_{(i)})$, where $u_i = f(s_i) - f(s_{i-1})$, $i = 1, \ldots, n$, with $s_0 = 0$ and $s_i = \frac{\sum_{k=1}^i v_{(k)}}{\sum_{k=1}^n v_k}$ for $i > 0$, $f : I \to I$ is a regular increasing monotone quantifier (*i.e.*, $f(0) = 0$, $f(1) = 1$, and $f(x) \ge f(y)$ if $x > y$), such as $f(x) = x^\beta$, $\beta \ge 0$, and $(\cdot)$ as before is a permutation of the index such that $a_{(1)} \ge \cdots \ge a_{(n)}$. It is easily seen that when β goes from 0 to $+\infty$, andness($h^{\text{MW-OWA}}$) goes from 0 to 1, with $h_f^{\text{MW-OWA}}$ representing the max (with andness $= 0$) at $\beta = 0$, the WAM (with andness $= \frac{1}{2}$) at $\beta = 1$, and the min (with andness $= 1$) at $\beta \to +\infty$.

In the case $v_i = 1$, $i = 1, \ldots, n$, we get $s_i = \frac{i}{n}$, and, hence, $\mathbf{u}$ representing the underlying OWA weights (for the OWA operator without importance weighting) that by $f(x) = x^\beta$ are somewhat close to the MEOWA weights [7, 8], *i.e.*, the unique OWA weighting vector with the maximum dispersion at the given andness and dimension n. However, the behavior is not symmetric around andness $= \frac{1}{2}$; in fact, while the dispersion

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of the OWA weights for andness $> \frac{1}{2}$ (*i.e.*, $\beta > 1$) is rather close to the maximum, it is much less for andness $< \frac{1}{2}$ (*i.e.*, $\beta < 1$). For instance, for $n = 5$ the maximum normalized dispersion at andness $= 0.8$ and andness $= 0.2$ is 0.755 in both cases, while the OWA weights with $f(x) = x^\beta$ have the normalized dispersion 0.744 in the first case, but only 0.717 in the second case. Therefore, an improvement with a near optimal dispersion for all degrees of andness, is obtained by defining $h_f^{\text{MW-OWA}}$ as above for andness $\geq \frac{1}{2}$, and by its De Morgan duality for andness $< \frac{1}{2}$, namely (with a change of notation to express the andness ρ) $h_\rho^{\text{MW-OWA}}(\mathbf{v}, \mathbf{a}) = 1 - h_{\overline{\rho}}^{\text{MW-OWA}}(\mathbf{v}, \overline{\mathbf{a}})$. By this improvement, the normalized dispersion in the second case (andness $= 0.2$) above is increased to 0.744, as in the first case, *i.e.*, in both cases only 0.011 lower than the maximum.

4.2 Definition of the W-OWA operators

In the following, we present an extension of the duality based MW-OWA operators to a common class, W-OWA, comprising both the multiplicative importance weighting generalization, MW-OWA, and an implicative importance weighting generalization, IW-OWA, with parametric control of the kind of generalization. The W-OWA operators are defined by:

$$h_\rho^{\text{W-OWA}}(\mathbf{v}, \mathbf{a}) = \begin{cases} \left(\sum_i (u_i b_{(i)})\right) & \rho \geq \frac{1}{2} \\ 1 - h_{\overline{\rho}}^{\text{W-OWA}}(\mathbf{v}, \overline{\mathbf{a}}) & \rho < \frac{1}{2} \end{cases} \tag{5}$$

where, for all i, b_i is defined by:

$$b_i = (v_i^{\gamma(2\rho-1)} \Rightarrow_{\text{R}} a_i) = 1 - v_i^{\gamma(2\rho-1)}(1 - a_i) \tag{6}$$

with $\gamma \in \{0, 1\}$, $\gamma = 0$ (by which $b_i = a_i$) yielding the class of MW-OWA operators, and $\gamma = 1$ (by which $b_i = (v_i^{(2\rho-1)} \Rightarrow_{\text{R}} a_i)$) yielding the class of IW-OWA operators; $u_i = f(s_i) - f(s_{i-1}), i = 1, \ldots, n$, with $s_0 = 0$ and

$$s_i = \frac{\sum_{k=1}^{i} v_{(k)}}{\sum_{k=1}^{n} v_k} \tag{7}$$

for $i > 0$, and $f(x) = x^\beta$, with $\beta = \psi^{\text{W-OWA}}(\rho) = \rho/\overline{\rho}$; finally, $(\cdot)$ is an index permutation such that $b_{(1)} \geq \cdots \geq b_{(n)}$. We notice that the Reichenbach implication applied above may be replaced by any fuzzy implication.

By $\mathbf{v} = (1, 1, \ldots, 1)$ we obtain the common case of MW-OWA and IW-OWA, without importance weighting, namely the duality based, quantifier guided OWA operators that, in the case of $\rho \geq \frac{1}{2}$, are defined by: $h_\rho^{\text{OWA}}(\mathbf{a}) = h_\rho^{\text{W-OWA}}((1, 1, \ldots, 1), \mathbf{a}) = \sum_i (u_i a_{(i)})$ where $u_i = f(\frac{i}{n}) - f(\frac{i-1}{n})$, with $f(x) = x^\beta$.

4.3 Accurate control of W-OWA andness

Above, the parameter β is, for a targeted andness ρ, defined by $\beta = \psi^{\text{W-OWA}}(\rho) = \rho/\overline{\rho}$ in the case $\rho \geq \frac{1}{2}$ (and, by duality, as implemented by the semi-recursive call, $\beta = \psi^{\text{W-OWA}}(\overline{\rho}) = \overline{\rho}/\rho$ in the case $\rho < \frac{1}{2}$). While by $\psi^{\text{W-OWA}}(\rho)$, andness$(h^{\text{W-OWA}})$ is equal to ρ, if $\rho \in \{0, \frac{1}{2}, 1\}$ or $n \to +\infty$, and, otherwise, close to ρ, a more precise approximation can be obtained by also considering the dimension n, namely through replacing $\psi^{\text{W-OWA}}(\rho)$ by $\psi^{\text{W-OWA}}(\rho, n)$ as defined by:

$$\psi^{\text{W-OWA}}(\rho, n) = \frac{0.5 + n\rho}{0.5 + n\overline{\rho}} \tag{8}$$

For instance, by $\rho = 0.8$ (and, through the duality, by $\rho = 0.2$), we get $\beta = \psi^{\text{W-OWA}}(0.8) = \frac{0.8}{1-0.8} = 4$. For $n = 5$, this value of β gives an andness of about of about 0.86, i.e., 0.06 more than the targeted andness, while we by $\beta = \psi^{\text{W-OWA}}(0.8, 5) = \frac{0.5+5\cdot0.8}{0.5+5(1-0.8)} = 3.0$ obtain the desired andness of 0.8. As n increases from 2 to 100, the average over $\rho \in I$ of the error $\|\rho - \text{andness}(h_\rho^{\text{W-OWA}})\|$ decreases from about 0.07 to about 0.002 by $\psi^{\text{W-OWA}}(\rho)$, and from about 0.0003 to about 0.00001 by $\psi^{\text{W-OWA}}(\rho, n)$.

5 On the multiplicative importance weighting

MWA operators are essentially weighted means that, qua means, are symmetric, monotonic increasing, continuous, and idempotent. They produce means in the interval $[\min_i a_i, \max_i a_i]$, and are monotonically increasing with the orness. If an MWA operator has the andness $= \frac{1}{2}$, it represents the WAM.

5.1 Applications of MWA operators

A common application is for *estimation of a utility variable*. In this case, each argument represents the estimation by an expert (or the measure by some source), while its importance weight represents the decision maker's confidence in the experts's ability to estimate the correct value. The andness and the orness represent decision maker's risk attitude, namely, the degree of, respectively, pessimism and optimism. The aggregate is in this case an estimate of the utility variable, considering the decision maker's risk attitude and, for each expert, the estimate by the expert and the decision masker's confidence in the expert.

Another application is for *selection between (alternative) options*, where the option with the highest score (weighted mean) is winning. In this case, each argument represents the degree to which the option considered has a particular property of interest, while its importance weight represents the importance of having the property to a high degree. The andness represents the degree to which all properties of interest must be present to a high degree. The outcome of the averaging is a ranking of options, possible with a threshold distinguishing acceptable options from unacceptable options.

5.2 Discontinuity property of MWA operators

While MWA and IWA operators both represent the WAM at andness $= \frac{1}{2}$, a key difference is that for MWA, as opposed to IWA, the effect of the importance weights decreases as the andness converges to one of its extremes, 1 or 0. In these cases, MWA evaluates to the smallest argument with a positive weight, yielding a discontinuity, since a small change in an importance weight, from, say, 0.01 to 0, may give a drastic change in the aggregate.

For instance, let h_ρ^{MWA} be an MWA operator (like AMWA or MW-OWA) at andness ρ, and assume $\mathbf{a} = (0.9, 0.1)$. Then $h_1^{\text{MWA}}((1, 0.01), \mathbf{a}) = 0.1$, while $h_1^{\text{MWA}}((1, 0), \mathbf{a}) = 0.9$. Similarly, $h_0^{\text{MWA}}((0.01, 1), \mathbf{a}) = 0.9$, while $h_0^{\text{MWA}}((0, 1), \mathbf{a}) = 0.1$. In both cases, the small decrease from 0.01 to 0 of an importance weight, caused a drastic change (of size 0.8) in the aggregate.

This behavior is an effect of MWA operators as weighted means. It models the attitude of the decision maker, in the cases of extreme pessimism and extreme optimism, to select

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the most pessimistic, respectively, most optimistic, estimate (argument) by any source with a positive confidence (importance weight). Such behavior is not acceptable in multicriteria aggregation for object recognition; for instance, it is not acceptable that the unimportant second criterion determines the overall satisfaction as by aggregate $h_1^{\text{MWA}}((1, 0.01), \mathbf{a}) = 0.1$. IWA operators avoid this by only considering criteria to the degree they are important; thus $h_1^{\text{IWA}}((1, 0.01), \mathbf{a}) = h_1^{\text{IWA}}((1, 0), \mathbf{a}) = 0.9$.

5.3 Other MWA operators

The PM, on which the AMWA and AIWA operators are based, are in the family of the quasi-arithmetic means (QM) [9, 10]. The weighted QM (WQM) forms the family of MWA operators defined by $h^{\text{WQM}}(\mathbf{v}, \mathbf{a}) = \phi^{-1}(\sum_i (w_i \phi(a_i)))$ where ϕ is a continuous strictly monotonic function, and ϕ^{-1} is the inverse of ϕ. The Weighted PM (WPM), $h^{\text{WPM}}(\mathbf{v}, \mathbf{a}) = (\sum_i (w_i a_i^\alpha))^{1/\alpha}$, $\alpha \in \mathbb{R}$, belong to this family from which they are derived by $\phi_\alpha(x) = x^\alpha$ (with $\phi_\alpha^{-1}(x) = x^{1/\alpha}$).

By $\phi_\alpha(x) = e^{\alpha x}$ (with $\phi_\alpha^{-1}(x) = \frac{\ln x}{\alpha}$), we obtain the Weighted Exponential Means (WEM), $h_\alpha^{\text{WEM}}(\mathbf{v}, \mathbf{a}) = \frac{1}{\alpha} \ln(\sum_i (w_i e^{\alpha a_i}))$, that converges to the WAM for $\alpha \to 0$, and to $\min_i a_i$ and $\max_i a_i$ for α going to, respectively, $-\infty$ and $+\infty$. A nice property of the WEM is its symmetric behavior around andness $= \frac{1}{2}$ as obtained for $\alpha \to 0$. Unlike the WPM, the WEM do not impose the mandatory property.

6 On the implicative importance weighting

6.1 Applications of IWA operators

Implicative importance weighting is applied by IWA operators for pattern matching inference in multicriteria recognition or classification problems, where an observed object (*e.g.*a document or some physical object) is compared to a goal concept (e.g., a query or a class). The goal concept is intensionally characterized by a set of criteria that, in general, are importance weighted. Each criterion expresses a constraint on the values of an object attribute and may, in general, be represented by a fuzzy subset. The importance weight of a criterion expresses the importance of satisfying the criterion in recognizing or classifying an object as an instance of the goal concept.

This scheme applies for querying of the two main kinds, namely object querying and concept querying. [11, 12] In *object querying*, *e.g.*, document retrieval and database querying, the goal concept forms a query. When posed to an object base, the query determines a fuzzy subset of the set of objects, representing the query's fuzzy extension in that set. An object's degree of membership in the extension can be seen as the degree to which the object is an instance of the goal concept and is also referred to as the object's *score* in the goal concept. In *concept querying*, *e.g.*, object recognition and classification, the characterization of the observed object forms a query. When posed to a concept base, *i.e.*, a base of possible goal concepts or classes, the query determines in this case a fuzzy subset of the set of concepts. The answer may in both cases be presented by a ranked list of the objects or concepts (depending on the query type) for which the score is above a given threshold.

6.2 IWA weighting and reasoning scheme

Let C be a goal concept, characterized by the criteria $C_1, \ldots, C_n$ with the importance weights $v_1, \ldots, v_n$, with v_i being the importance of satisfying C_i, and let $a_i = C_i(x) = \mu_{C_i}(x)$ be the degree to which the object x (or, actually, the constrained attribute of x) satisfies C_i. Then, the inference implemented by IWA operators has a Modus Ponens form that in the case of andness = 1 is expressed by:

$$\frac{\begin{array}{c} a_1 = C_1(x), \ldots, a_n = C_n(x) \\ (v_1 \Rightarrow C_1) \wedge \cdots \wedge (v_n \Rightarrow C_n) \to C \end{array}}{C(x) = \wedge_i (v_i \Rightarrow C_i) \ (= h_{\rho=1}^{\text{IWA}}(\mathbf{v}, \mathbf{a}))} \tag{9}$$

Equation 9 expresses that, for an AND aggregation, the goal concept is satisfied by an object to the degree that the requirement *'for all criteria, the criterion is satisfied if it is important'* is met. This may be expressed by $\forall i(\text{important}(\mathrm{C_i}) \Rightarrow \text{satisfied}(\mathrm{C_i}))$ that, by $v_i = \text{important}(\mathrm{C_i})$ and $a_i = \text{satisfied}(\mathrm{C_i})$, may be written $\forall i(v_i \Rightarrow a_i)$, or, by the standard evaluation of $\forall$ by the min operator, $\min_i(v_i \Rightarrow a_i)$. By other equivalent expressions of $\forall i(v_i \Rightarrow a_i)$, the requirement can be expressed by other words, for instance, by $\neg \exists i(v_i \wedge \neg a_i)$, *'there does not exist a criterion that is important but not satisfied'*.

The transformation of an argument a by an implicative importance weight v for an averaging aggregation at andness ρ is modeled by a function $g_\rho : I^2 \to I : (v, a) \mapsto g_\rho(v, a)$. Thus, for a fuzzy implication $\Rightarrow$, $g_1(v, a) = (v \Rightarrow a)$ is a transformation for an AND aggregation, and $g_0(v, a) = \overline{g_{\overline{0}}(v, \overline{a})} = \overline{v \Rightarrow \overline{a}}$ is its De Morgan dual transformation for an OR aggregation. An important property of g_1 and g_0 is that they for $v = 0$ evaluate to the neutral element for the aggregation, namely, respectively, 1 and 0. In general, for $\rho \in I$, we define g_ρ as the andness-orness weighted sum of g_1 and g_0:

$$g_\rho(v, a) = \rho g_1(v, a) + \overline{\rho} g_0(v, a) \tag{10}$$

which in the case of the Reichenbach implication $\Rightarrow_{\mathrm{R}}$, with $g_1^{\mathrm{R}}(v, a) = (v \Rightarrow_{\mathrm{R}} a) = \overline{v\overline{a}}\ (= 1 - v(1-a))$ and $g_0^{\mathrm{R}}(v, a) = \overline{v \Rightarrow_{\mathrm{R}} \overline{a}} = va$, evaluates to $g_\rho^{\mathrm{R}}(v, a) = \rho - v(\rho - a)$. In [13], we presented an (implicative) importance weighting generalization of OWA operators and showed that transformation by g_ρ^{R} satisfies the requirements for such an importance weighting.

6.3 Other IWA operators

In [3], we introduced another approach to implicative importance weighting generalization of OWA operators, in that case the MEOWA operators [7, 8], namely the IW-MEOWA operators that, by the current notation, are defined by:

$$h_\rho^{\text{IW-MEOWA}}(\mathbf{v}, \mathbf{a}) = \frac{h_\rho^{\text{IW-MEOWA}}(\mathbf{u}, \mathbf{v}, \mathbf{a}) - l}{u - l} \tag{11}$$

where $h_\rho^{\text{IW-MEOWA}}(\mathbf{u}, \mathbf{v}, \mathbf{a}) = \sum_i (u_i b_{(i)})$, $\mathbf{u}$ is the MEOWA weighting vector for the given ρ and n, $b_i = g_\rho^{\mathrm{R}}(v, a) = \rho - v(\rho - a)$, and $(\cdot)$ is an index permutation, such that $b_{(1)} \geq \cdots \geq b_{(n)}$. Finally, l and u are the lower and upper bounds for $h_\rho^{\text{IW-MEOWA}}(\mathbf{v}, \mathbf{a})$, namely, repectively, $h_\rho^{\text{IW-MEOWA}}(\mathbf{u}, \mathbf{v}, (0, \ldots, 0))$ and $h_\rho^{\text{IW-MEOWA}}(\mathbf{u}, \mathbf{v}, (1, \ldots, 1))$. It was shown that (11) provides a linear transformation of $h_\rho^{\text{IW-MEOWA}}(\mathbf{u}, \mathbf{v}, \mathbf{a})$, such that $h_\rho^{\text{IW-MEOWA}}(\mathbf{v}, \mathbf{a})$ represents the WAM at andness $= \rho = \frac{1}{2}$.

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7 Comparison of MWA and IWA aggregates

While operators of the same kind, MWA or IWA, tend to behave similarly—in the sense that they provide the same ordering of a given set of options at the same andness and the same importance weights—this is, as we may expect, not the case for operators of difference kinds. This is illustrated by the following examples, where the andness of the MWA and W-OWA classes are targeted by, respectively, $\psi^{\text{AA}}(\rho, n)$ and $\psi^{\text{W-OWA}}(\rho, n)$, as denoted by the aggregation operator symbol; thus, $h^{\text{AMWA}}_{\frac{2}{3},2}$ denotes that h^{AMWA} is applied with $\psi^{\text{AA}}(\frac{2}{3}, 2)$.

Assume that the decision problem is characterized by andness $= \frac{2}{3}$, $n = 2$, $\mathbf{v} = (0.4, 1)$ that is sum-normalized to $\mathbf{w} = (0.286, 0.714)$, and consider the two options represented by the argument vectors $\mathbf{a}_1 = (0.1, 0.7)$ and $\mathbf{a}_2 = (0.9, 0.4)$.

By *MWA aggregation*, the second option, as represented by $\mathbf{a}_2$, get the highest score and is therefore ranked higher than the first option, as the following evaluations, all at andness $= \frac{2}{3}$, show: $h^{\text{AMWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_1) = 0.420 < h^{\text{AMWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_2) = 0.474$ and $h^{\text{MW-OWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_1) = 0.454 < h^{\text{MW-OWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_2) = 0.470$. This is also the case for MWA operators of other classes; thus, for the Weighted Geometric Mean (WGM), $h^{\text{WGM}}(\mathbf{v}, \mathbf{a}) = \prod_i a_i^{w_i}$, that at $n = 2$ has andness $= \frac{2}{3}$, we get $h^{\text{WGM}}(\mathbf{v}, \mathbf{a}_1) = 0.401 < h^{\text{WGM}}(\mathbf{v}, \mathbf{a}_2) = 0.504$; similarly for the Weighted Exponential Means (WEM), $h^{\text{WEM}}_{\alpha}(\mathbf{v}, \mathbf{a}) = \frac{1}{\alpha} \ln\left(\sum_i (w_i e^{\alpha a_i})\right)$, that at $(\alpha, n) = (-3, 2)$ has andness $= \frac{2}{3}$, we get $h^{\text{WEM}}_{-3}(\mathbf{v}, \mathbf{a}_1) = 0.402 < h^{\text{WEM}}_{-3}(\mathbf{v}, \mathbf{a}_2) = 0.484$.

By *IWA aggregation*, the first option, as represented by $\mathbf{a}_1$, get the highest score and is therefore ranked higher than the second option: $h^{\text{AIWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_1) = 0.579 > h^{\text{AIWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_2) = 0.422$ and $h^{\text{IW-OWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_1) = 0.551 > h^{\text{IW-OWA}}_{\frac{2}{3},2}(\mathbf{v}, \mathbf{a}_2) = 0.473$; similarly, for the IW-MEOWA operator (11), we get $h^{\text{IW-MEOWA}}_{\frac{2}{3}}(\mathbf{v}, \mathbf{a}_1) = 0.530 > h^{\text{IW-MEOWA}}_{\frac{2}{3}}(\mathbf{v}, \mathbf{a}_2) = 0.515$.

This behavior may be explained as follows. By IWA aggregation, the first criterion (argument) is ignored to some degree, due to its rather low importance (0.4); the higher satisfaction of the second, highly important criterion by the first option is the sufficient to give this option the highest rank. By MWA aggregation, the very low value of the first argument in the first option contributes to give this option the lowest rank, despite the rather low importance of the argument.

These observations are supported by the experiment in [3], where the comparison of the IWA operators AIWA and IW-MEOWA for a small data set and a set of implicatively importance weighted queries, indicated a quite similar behavior of the two operators.

8 Conclusion

Importance weighting of two kinds, namely multiplicative and implicative, were proposed as generalizations of two classes of averaging aggregation operators, namely the Power Means (PM) and the Ordered Weighted Averaging (OWA) operators. Each class is applied in a De Morgan dual version, yielding symmetric behavior on both side of andness $= \frac{1}{2}$, and, at the same time, avoiding the mandatory property of the PM and obtaining a near maximum dispersion of OWA at all degrees of andness. The two generalizations of a class are represented by a common expression, where the kind of importance weighting is controlled through a binary parameter.

For each class, were proposed an andness-directing function that allows us to obtain a targeted andness for an operator rather accurately through also considering the number of arguments.

For each kind of importance weighting, we presented and discussed how it works in the operator under different degrees of andness, and the kind of application problems they apply to. Operators of different classes, but with the same kind of importance weighting, appear to have rather similar behavior at the same andness.

For a given choice of the kind of importance weighting, computational issues may affect the choice of operator class. While the PM based operators (AWA) in particular require much computationally rather heavy power functions, OWA based operators (W-OWA) in particular require an ordering (sorting) of the arguments. If reevaluation of a set of options by a modified query occurs frequently, AWA operators have an advantage through their decomposability property.

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