On possibilistic sequencing problems with fuzzy parameters

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Abstract— *In this paper a wide class of sequencing problems with imprecise parameters is discussed. The imprecision is modeled by using closed intervals and fuzzy intervals, whose membership functions are regarded as possibility distributions for the values of unknown parameters. A possibilistic interpretation of fuzzy problems is provided, some solution concepts are proposed and some algorithms for computing the solutions are designed.*

Keywords— Sequencing, min-max regret, possibility theory, fuzzy interval, fuzzy optimization and design

1 Introduction

In a sequencing problem we wish to find a feasible order of elements, called jobs, to achieve some goal. This goal typically depends on job completion times and may also depend on some other job parameters such as due dates or weights. There are a lot of deterministic sequencing problems with different computational properties and a comprehensive description of them can be found for instance in [1]. Unfortunately, most of sequencing problems turned out to be NP-hard but there are also some important problems for which efficient polynomial algorithms exist.

Sequencing problems involve many parameters whose exact values are often unknown. For instance, a job processing time, which is a crucial parameter in all sequencing problems, is rarely precisely known. In order to model such a situation a stochastic approach can be applied (see e.g. [2]), where for every unknown parameter a probability distribution is specified and, typically, the expected cost of a solution is minimized. The stochastic approach has several drawbacks. Namely, it may be hard or expensive to estimate the probability distributions for the parameters. Also, minimizing the expected performance may be not reasonable if the obtained solution is used only once.

In practice, decision makers are often interested in minimizing the cost of a decision in the worst case, that is under the worst realization of parameters that may occur. In recent years a *robust* approach do discrete optimization has attracted a considerable attention. In this approach we specify a *scenario set* containing all possible realizations of parameters which may occur. No probability distribution over the scenario set is given. One of methods of determining the scenario set consists of specifying an interval of possible values for every unknown parameter. A scenario set is then the Cartesian product of all the uncertainty intervals. A natural criterion for choosing a solution under imprecision is the *maximal regret*, which expresses the maximum distance of a solution from optimality over all scenarios [3]. A deeper discussion on using the maximal regret criterion in decision making under uncertainty can be found in [4]. The class of min-max regret sequencing problem with interval parameters has been discussed in a number of papers, for instance in [5, 6, 7, 8, 9, 10, 11, 12]. We briefly describe the known results in this area later in this paper.

In recent years, theory of fuzzy sets has been applied to model the imprecision in optimization, in particular in sequencing problems. A good review of different concepts in fuzzy optimization can be found in [13]. In papers [14, 15, 16] some single machine sequencing problems with fuzzy processing times, fuzzy due dates and fuzzy precedence constraints have been discussed. In these papers a fuzzy due date expresses a degree of satisfaction with job completion time and a sequence is computed, which maximizes the minimum satisfaction or the sum of satisfactions over all jobs. In [17] an alternative approach has been proposed, where the possibility or necessity of job delays is minimized. Also, in [18] an optimality evaluation of sequences under fuzzy parameters has been investigated. We recall and extend this approach here.

In this paper we wish to propose a new approach to sequencing problems with fuzzy parameters. We generalize the min-max regret problems with interval parameters to the fuzzy case by extending the interval uncertainty representation to the fuzzy interval one. The fuzzy parameters induce a possibility distribution over the scenario set, which becomes then reacher in information than the classical scenario set. We provide a possibilistic interpretation of the obtained fuzzy problem and describe a solution concept, which is an adaptation of the elegant solution method proposed in [19] for fuzzy linear programming. This solution concept has been recently adopted in [20] for a class of combinatorial optimization problems. Apart from showing a general framework, we also point out some difficulties which arise when one try to solve a particular problem. Contrary to the class of combinatorial optimization problems described in [20], the sequencing problems are typically more complex to solve and there are very few general properties that are valid for all problems. However, as for the problems described in [20], the main computational difficulties are in the classical interval case and the algorithms known for the interval uncertainty representation can be generalized to the fuzzy case.

This paper is organized as follows. In Section 2 we recall the definition of the classical deterministic sequencing problem. In Section 3 we present the min-max regret approach to sequencing problems with interval parameters - we provide a

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general formulation and we recall some known results in this area. Finally, in Section 4 we introduce the class of sequencing problems with fuzzy parameters. We give a link between the fuzzy problems and the min-max regret ones. We construct some general methods of solving a fuzzy problem and we illustrate them using a sample problem.

2 Deterministic sequencing problems

We are given a set of *jobs* $J = \{J_1, \ldots, J_n\}$, which may be partially ordered by some *precedence constraints* of the form $i \to j$, where $i, j \in J$. For the simplicity of notations we will identify every job $J_i \in J$ with its index $i \in \{1, \ldots, n\}$. A solution is a *sequence* (permutation) $\sigma = (\sigma(1), \ldots, \sigma(n))$ of J and it represents an order in which the jobs are processed. A sequence σ is *feasible* if $i \to j$, $i \neq j$, implies that job j appears after i in σ. We will denote by $\boldsymbol{\mathcal{S}}$ the set of all feasible sequences. We use $C_i(\sigma)$ to denote the *completion time* of job i in sequence σ. In a *single machine* case, a processing time p_i is given for every job $i \in J$ and if $i = \sigma(k)$, then $C_i(\sigma) = \sum_{j=1}^{k} p_{\sigma(j)}$. If every job must be processed on $m > 1$ machines, first on machine 1 next on machine 2 and so on, then we get the *permutation flow shop problem*. In this case p_{ij} is a processing time of job i on machine j and $C_i(\sigma)$ is the time when job i is finished on the m-th machine (see Figure 1).

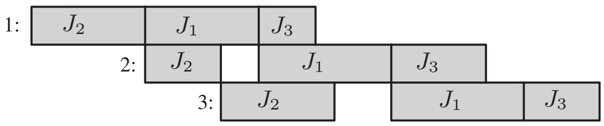

Figure 1: A permutation flow show problem with $m = 3$, $n = 3$ and the schedule that corresponds to the sequence $(2, 1, 3)$.

For every job $i \in J$, there is a cost function $f_i : R \to R$, which measures the cost of completing i at time t. The value of this function may depend on some other parameters associated with job i such as *due date* d_i or *weight* w_i. Finally, $F(\sigma)$ denotes a cost of sequence σ. There are two general types of the cost function $F(\sigma)$, namely a *bottleneck objective* $F(\sigma) = \max_{i=1,\ldots,n} f_i(C_i(\sigma))$ and a *sum objective* $F(\sigma) = \sum_{i=1}^{n} f_i(C_i(\sigma))$.

In a deterministic sequencing problem we seek a feasible sequence of the minimal cost, that is we solve the following optimization problem:

$$\min_{\sigma \in \boldsymbol{\mathcal{S}}} F(\sigma). \tag{1}$$

Our analysis in the next sections of this paper will be based on the concept of a *deviation*. In the deterministic case the deviation of a sequence σ expresses its distance to optimum and is defined as follows:

$$\delta_\sigma = F(\sigma) - \min_{\rho \in \boldsymbol{\mathcal{S}}} F(\rho). \tag{2}$$

Obviously, σ is optimal if and only if $\delta_\sigma = 0$. If the deviation is positive, then it indicates a distance of σ from optimality.

Sequencing problems are usually described by using a convenient Graham's notation. So, every sequencing problem can be denoted by a triple $\alpha|\beta|\gamma$, where α is the machine environment ($\alpha = 1$ for the single machine case), β specifies the job characteristic and γ describes the objective function. The following examples of sequencing problems are well known and will be explored later in this paper:

- $1|prec|L_{max}$. In this problem we seek a sequence, which minimizes the maximum *lateness*. So, the bottleneck objective is $\max_{i \in J}\{C_i(\sigma) - d_i\}$. We can generalize this problem by minimizing the maximal weighted lateness or modify it to minimize the maximal weighted *tardiness* (see [1]).
- $1||\sum C_i$. In this problem there are no precedence constraints between jobs and we seek a sequence for which the sum of completion times of all jobs, i.e. the *total flow time* $\sum_{i \in J} C_i(\sigma)$, is minimal.
- $1|p_i = 1|\sum w_i U_i$. In this problem there are no precedence constraints between jobs and all jobs have unit processing times. A job is *late* in σ if $C_i(\sigma) > d_i$. In this case we write $U_i(\sigma) = 1$ and $U_i(\sigma) = 0$ otherwise. The cost of σ is $\sum_{i \in J} w_i U_i(\sigma)$, so it expresses the weighted number of late jobs.
- $Fm||C_{max}$. This is the permutation flow shop problem with $m > 1$ machines. There are no precedence constraints between jobs and the bottleneck cost of σ is the *makespan* $\max_{i \in J} C_i(\sigma)$, that is the completion time of the last job on the last machine. The problem is polynomially solvable only if $m = 2$ and becomes NP-hard for $m \geq 3$ (see [1]).

The above examples illustrate a large variety of basic sequencing models. As we will see in the next section, they have quite different computational properties under imprecision and the state of the art in this area is still far from being complete.

3 Minmax regret sequencing problems

In practice, the exact values of parameters in a sequencing problem such as processing times, due dates or weights may be not precisely known. Assume that we have l parameters and the value of a parameter ξ_i, $i = 1, \ldots, l$, may fall within a closed interval $[\underline{\xi}_i, \overline{\xi}_i]$ independently of the values of the other parameters. A parameter ξ_i is *precise* if $\underline{\xi}_i = \overline{\xi}_i$. Every vector $S = (s_1, \ldots, s_l) \in R^l$ such that $s_i \in [\underline{\xi}_i, \overline{\xi}_i]$ is called a *scenario* and it expresses a possible state of the world, where $\xi_i = s_i$ for $i = 1, \ldots, l$. A scenario is called *extreme* if all parameters take the lower or upper bounds in their uncertainty intervals. We use Γ to denote the set of all possible scenarios. Hence Γ is the Cartesian product of all the uncertainty intervals. Now the cost of a sequence σ depends on scenario $S \in \Gamma$ and we will denote it as $F(\sigma, S)$. We will also denote by $F^*(S)$ the cost of an optimal sequence under scenario S. In order to obtain the value of $F^*(S)$ we need to solve problem (1) under the fixed realization of parameters S. It is clear that deviation of σ also depends on scenario S and we will denote it as $\delta_\sigma(S) = F(\sigma, S) - F^*(S)$.

Now the optimality of a sequence σ can be characterized by a *deviation interval* $[\underline{\delta}_\sigma, \overline{\delta}_\sigma]$, where $\underline{\delta}_\sigma = \min_{S \in \Gamma} \delta_\sigma(S)$ is the minimal deviation and $\overline{\delta}_\sigma = \max_{S \in \Gamma} \delta_\sigma(S)$ is the maximal deviation over all scenarios. The quantity $\overline{\delta}_\sigma$ is called in

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literature the *maximal regret* of σ and it expresses the largest distance of σ from optimum over the scenario set Γ. A scenario S_σ, for which the deviation of σ attains maximum, is called a *worst case scenario* for σ. So, under the interval uncertainty representation, we only know that $\delta_\sigma \in [\underline{\delta}_\sigma, \overline{\delta}_\sigma]$ and we can give the following characterization of a feasible sequence: a sequence σ is *possibly optimal* if $\underline{\delta}_\sigma = 0$ and it is *necessarily optimal* if $\overline{\delta}_\sigma = 0$. Notice that a sequence is possibly optimal if and only if it is optimal under some scenario $S \in \Gamma$ and it is necessarily optimal if and only if it is optimal for all scenarios $S \in \Gamma$.

Now the question arises which sequence of $\boldsymbol{\mathcal{S}}$ should be chosen. A necessarily optimal one is a natural choice. However, it rarely exists in most practical situations, because the necessary optimality is very strong criterion. In a more reasonable approach we can minimize the the maximal regret, that is the quantity $\overline{\delta}_\sigma$, which may be viewed as a distance to the necessary optimality. As a result we get the following *min-max regret sequencing problem*:

$$\min_{\sigma \in \boldsymbol{\mathcal{S}}} \overline{\delta}_\sigma. \tag{3}$$

We call an optimal solution to (3) a *min-max regret sequence*. In the next section we briefly review some known facts on problem (3).

3.1 Complexity of min-max regret sequencing problems

The first problem that arises while analyzing a particular min-max regret sequencing problem is the computation of the maximal regret for a given sequence σ, that is the quantity $\overline{\delta}_\sigma$. Unfortunately, contrary to the class of problems discussed in [20], there is no general method of performing this task. For the problems considered in [20], it is possible to find two extreme scenarios that maximize and minimize the deviation and, consequently, the computation of the maximal regret has the same complexity as the deterministic problem. For sequencing problems the situation is much more complex. First of all, for some problems there may be no extreme scenario that maximizes (minimizes) the deviation [21]. Furthermore, computing the maximal regret may be much more time consuming than solving a deterministic problem. For instance, in the min-max regret $1||\sum C_i$ problem with interval processing times, computing $\overline{\delta}_\sigma$ requires solving an assignment problem while an optimal sequence under a given scenario can be computed in $O(n \log n)$ time. An extreme case has been described in [9], where a permutation flow shop problem with m machines and with interval processing times and with only 2 jobs has been discussed. Since the solution set contains only two sequences, the total computational effort is focused on computing the maximal regret of a given sequence.

It is not surprising that min-max regret sequencing problems are typically hard to solve. There are only several problems that are known to be polynomially solvable. A polynomial algorithm for $1|prec|L_{max}$ with interval processing times and interval due dates has been constructed in [6]. A polynomial algorithm for $1|prec|\max w_i T_i$ with interval weights and precise processing times and precise due dates has been proposed in [8]. Apart from this two problems, only some very special cases are known to be polynomially solvable, for instance $Fm||C_{max}$ with interval processing times and only two jobs [9] and $1|p_i = 1|\sum w_i U_i$ with interval weights and a precise common due date ($d_1 = d_2 = \cdots = d_n$) [21]. This latter problem is equivalent to the min-max regret version of the selecting items problem, which is known to be polynomially solvable [21, 22].

Among the known negative results, the most important one has been obtained in [10], where it has been shown that the min-max regret $1||\sum C_i$ problem with interval processing times is NP-hard. This problem is also known to be approximable within 2 [12] and can be solved by using a mixed integer programming formulation proposed in [11]. However, the complexity status of a number of basic problems is still unknown. We do not know whether $F2||C_{max}$ with interval processing times is NP-hard. We can, however, compute the maximal regret of a given sequence and solve the problem by using a branch and bound algorithm [7, 3]. Also the problem $1|p_i = 1|\sum w_i U_i$ with interval weights and arbitrary precise due dates and the problem $1|prec|\max w_i L_i$ with interval processing times, interval due dates and precise weights are open. The former problem can be solved by a mixed integer programming formulation [21]. There are also a large number of different sequencing problems whose min-max regret versions have never been investigated and they should be the subjects of further research.

4 Fuzzy sequencing problems

In this section we propose an extension of the min-max regret approach to sequencing problems. We will apply fuzzy intervals to model the imprecise parameters and we will use possibility theory to define a solution concept. A comprehensive description of possibility theory can be found in [23].

4.1 Basic notions on possibility theory

A *fuzzy interval* $\tilde{A}$ is a fuzzy set in the space of reals whose membership function $\mu_{\tilde{A}}$ is normal, quasi concave and upper semicontinuous. It is typically assumed that the support of a fuzzy interval is compact. The main property of a fuzzy interval is the fact that all its λ-cuts, that is the sets $\tilde{A}^\lambda = \{x : \mu_{\tilde{A}}(x) \geq \lambda\}$, $\lambda \in (0, 1]$, are closed intervals. We will assume that $\tilde{A}^0$ is the smallest closed set containing the support of $\tilde{A}$. So, every fuzzy interval $\tilde{A}$ can be represented as a family of closed intervals $\tilde{A}^\lambda = [\underline{a}^\lambda, \overline{a}^\lambda]$, parametrized by the value of $\lambda \in [0, 1]$. In many practical applications, the class of *trapezoidal fuzzy intervals* is used. A trapezoidal fuzzy interval, denoted by a quadruple $(\underline{a}, \overline{a}, \alpha, \beta)$, can be represented as the family $[\underline{a} - \alpha(1 - \lambda), \overline{a} + \beta(1 - \lambda)]$ for $\lambda \in [0, 1]$. Notice that this representation contains classical intervals ($\alpha = \beta = 0$) and real numbers (additionally $\underline{a} = \overline{a}$) as special cases.

In this paper we adopt a possibilistic interpretation of a fuzzy interval [23]. Assume that for an imprecise real quantity ξ a fuzzy interval with membership function μ_ξ is given. This membership function expresses a *possibility distribution* for the values of ξ, namely $\Pi(\xi = x) = \mu_\xi(x)$ is the possibility of the event that ξ will take the value of x. It is easily seen that the closed interval $[\underline{\xi}^\lambda, \overline{\xi}^\lambda]$, $\lambda \in [0, 1]$, contains all values of ξ, whose possibility of occurrence is not less than λ. In particular, the interval $[\underline{\xi}^0, \overline{\xi}^0]$ should contain all possible values of ξ, while the interval $[\underline{\xi}^1, \overline{\xi}^1]$ should contain the most plausible ones. Some methods of obtaining the possibility distribution of an unknown quantity can be found in [23].

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Let $\tilde{G}$ be a fuzzy set in the space of reals with membership function $\mu_{\tilde{G}}$. Then $\xi \in \tilde{G}$ is a *fuzzy event* and the necessity that $\xi \in \tilde{G}$ holds is defined in the following way:

$$\begin{aligned} \mathrm{N}(\xi \in \tilde{G}) &= 1 - \Pi(\xi \notin \tilde{G}) = \\ &= 1 - \sup_{x \in R} \min\{\mu_\xi(x), 1 - \mu_{\tilde{G}}(x)\} \end{aligned} \tag{4}$$

where $1 - \mu_{\tilde{G}}(x)$ is the membership function of the complement of the fuzzy set $\tilde{G}$. It is not difficult to see that if $\tilde{G} = (0, \overline{g}, 0, \beta) = (\overline{g}, \beta)$, then the following equality is true:

$$\mathrm{N}(\xi \in \tilde{G}) = 1 - \inf_{\lambda \in [0,1]} \{\overline{\xi}^{\lambda} \le \overline{g}^{1-\lambda}\} \tag{5}$$

and $\mathrm{N}(\xi \in \tilde{G}) = 0$ if $\overline{\xi}^1 > \overline{g}^0$. Equality (5) is illustrated in Figure 2.

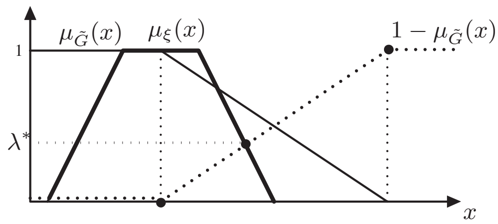

Figure 2: $\mathrm{N}(\xi \in \tilde{G}) = 1 - \lambda^*$.

4.2 Possibilistic sequencing problem

Assume that for every unknown parameter ξ_i, $i = 1, \ldots, l$, in a sequencing problem a fuzzy interval with membership function μ_{ξ_i} is specified. According to the interpretation given in the previous section μ_{ξ_i} is a possibility distribution for the values of ξ_i. Under the assumption that all parameters are unrelated, there is a possibility distribution over all scenarios $S = (s_1, \ldots, s_l) \in R^l$ defined as follows:

$$\begin{aligned} \pi(S) = \Pi\left(\bigwedge_{i=1}^{l} [\xi_i = s_i]\right) = \min_{i=1,\ldots,l} \Pi(\xi_i = s_i) = \\ = \min_{i=1,\ldots,l} \mu_{\xi_i}(s_i). \end{aligned} \tag{6}$$

So, the value of $\pi(S)$ express the possibility of the event that scenario S will occur. Notice that we generalize in this way the scenario set $\Gamma \subset R^l$ described in Section 3. Indeed, for the interval uncertainty representation $\pi(S) = 1$ if $S \in \Gamma$ and $\pi(S) = 0$ otherwise. In the fuzzy case $\pi(S)$ may take any value in the interval $[0, 1]$ so fuzzy intervals allow us to model the imprecision in a more sophisticated manner.

In the interval case the value of deviation δ_σ falls within a closed interval. Analogously, in the fuzzy case it falls within a fuzzy interval with membership function μ_{δ_σ}. Of course, μ_{δ_σ} is a possibility distribution for the values of δ_σ and, according to possibility theory, it is defined as follows:

$$\mu_{\delta_\sigma}(x) = \Pi(\delta_\sigma = x) = \sup_{\{S:\, \delta_\sigma(S) = x\}} \pi(S). \tag{7}$$

Recall that the statement "'σ is optimal"' is equivalent to the assertion $\delta_\sigma = 0$. Consequently, we can define the degrees of possible and necessary optimality of a given sequence as follows:

$$\Pi(\sigma \text{ is optimal}) = \Pi(\delta_\sigma = 0) = \mu_{\delta_\sigma}(0), \tag{8}$$

$$\mathrm{N}(\sigma \text{ is optimal}) = \mathrm{N}(\delta_\sigma = 0) = 1 - \sup_{x > 0} \mu_{\delta_\sigma}(x). \tag{9}$$

As in the interval case, a question arises which sequence should be chosen. In order to perform this task we adopt a concept first applied to fuzzy linear programming in [19]. Assume that a decision maker knows his/her preference about the sequence deviation and expresses it by using a fuzzy interval $\tilde{G} = (\overline{g}, \beta)$. So, the values of deviation in $[0, \overline{g}]$ are fully accepted, the values in $[\overline{g} + \beta, \infty)$ are not at all accepted and the degree of acceptance decreases from 1 to 0 in $[\overline{g}, \overline{g} + \beta]$. Our aim is to compute a feasible sequence $\sigma \in \boldsymbol{\mathcal{S}}$, for which the necessity of the event $\delta_\sigma \in \tilde{G}$ is maximal, namely we wish to solve the following optimization problem:

$$\max_{\sigma \in \boldsymbol{\mathcal{S}}} \mathrm{N}(\delta_\sigma \in \tilde{G}). \tag{10}$$

This solution concept has been first proposed for linear programming problem with fuzzy objective function coefficients [19]. Observe that (10) can also be expressed as $\min_{\sigma \in \boldsymbol{\mathcal{S}}} \Pi(\delta_\sigma \in \tilde{G}^d)$, where $\tilde{G}^d$ is the complement of the fuzzy goal $\tilde{G}$ with membership function $1 - \mu_{\tilde{G}}(x)$. If we fix $\tilde{G} = (0, 0)$ in (10), then we get a special case, in which we seek a feasible sequence that maximizes the degree of necessary optimality:

$$\max_{\sigma \in \boldsymbol{\mathcal{S}}} \mathrm{N}(\delta_\sigma = 0) = \max_{\sigma \in \boldsymbol{\mathcal{S}}} \mathrm{N}(\sigma \text{ is optimal}). \tag{11}$$

We focus now on a method of solving the problem (10). According to equality (5), the problem (10) is equivalent to the following optimization problem:

$$\begin{array}{l} \min \lambda \\ \overline{\delta}_\sigma^{\lambda} \le \overline{g}^{1-\lambda} \\ \sigma \in \boldsymbol{\mathcal{S}} \\ \lambda \in [0, 1] \end{array} \tag{12}$$

If λ^* is the optimal objective value and σ^* is an optimal solution to (12), then $\mathrm{N}(\delta_{\sigma^*} \in \tilde{G}) = 1 - \lambda^*$. If (12) is infeasible, then $\mathrm{N}(\delta_\sigma \in \tilde{G}) = 0$ for all feasible sequences σ. Of course, if we replace expression $\overline{g}^{1-\lambda}$ with 0 in (12), then we get an equivalent formulation of the problem (11).

Let us focus now on the quantity $\overline{\delta}_\sigma^{\lambda}$. The closed interval $[\underline{\delta}_\sigma^{\lambda}, \overline{\delta}_\sigma^{\lambda}]$, $\lambda \in [0, 1]$, contains all values of deviation δ_σ, whose possibility of occurrence is not less than λ. So

$$\overline{\delta}_\sigma^{\lambda} = \sup_{\{S\,:\pi(S) \ge \lambda\}} \{F(\sigma, S) - F^*(S)\} \tag{13}$$

is the greatest deviation of σ, whose possibility of occurrence is not less than λ. From the definition of $\pi(S)$ it is easy to see that $\{S : \pi(S) \ge \lambda\} = [\underline{\xi}_1^{\lambda}, \overline{\xi}_1^{\lambda}] \times \cdots \times [\underline{\xi}_l^{\lambda}, \overline{\xi}_l^{\lambda}] = \Gamma^{\lambda}$. Consequently, the quantity $\overline{\delta}_\sigma^{\lambda}$ is the maximal regret of σ in the min-max regret version of the problem with scenario set Γ^{λ}. In particular, the condition $\overline{\delta}_\sigma^{\lambda} \le 0$ means that σ is necessarily optimal under Γ^{λ}.

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Let us point out that the fuzzy problem is not simpler than the corresponding min-max regret one. Indeed, the interval uncertainty representation is a special case of the fuzzy one. If we additionally fix $\tilde{G} = (0, M)$ for a sufficiently large M, then the fuzzy problem is equivalent to the min-max regret one.

Observe that $\overline{\delta}^{\lambda}_{\sigma}$ is a nonincreasing and $\overline{g}^{1-\lambda}$ is a nondecreasing function of λ. In consequence, the problem (12) can be solved be a standard binary search technique if we only can decide somehow whether there is a feasible sequence $\sigma \in \boldsymbol{\mathcal{S}}$ fulfilling inequality $\overline{\delta}^{\lambda}_{\sigma} \leq \overline{g}^{1-\lambda}$ for a fixed $\lambda \in [0, 1]$. This task is easy if we have an algorithm for the corresponding min-max regret sequencing problem with interval parameters. Solving this problem for scenario set Γ^{λ} we get a min-max regret sequence σ^*. Then $\overline{\delta}^{\lambda}_{\sigma} \leq \overline{g}^{1-\lambda}$ for some $\sigma \in \boldsymbol{\mathcal{S}}$ if and only if $\overline{\delta}^{\lambda}_{\sigma^*} \leq \overline{g}^{1-\lambda}$. Notice that in the problem (11) it is enough to detect a necessarily optimal sequence for scenario set Γ^{λ}, which may be computationally easier than solving the min-max regret problem. The binary search algorithm is shown in Figure 3.

```
Find a min-max regret sequence σ under Γ^1
if δ̄^1_σ > ḡ^0 then return ∅
λ1 ← 0.5, k ← 1, λ2 ← 0
while |λ1 − λ2| ≥ ε do
    λ2 ← λ1
    Find a min-max regret sequence ρ under Γ^{λ1}
    if δ̄^{λ1}_ρ ≤ ḡ^{1−λ1} then
        λ1 ← λ1 − 1/2^{k+1}, σ ← ρ
    else
        λ1 ← λ1 + 1/2^{k+1}
    end if
    k ← k + 1
end while
return σ
```

Figure 3: Solving the problem (10) with a given precision $\epsilon \in (0, 1)$. Algorithm returns $\emptyset$ if $\mathrm{N}(\delta_{\sigma} \in \tilde{G}) = 0$ for all $\sigma \in \boldsymbol{\mathcal{S}}$.

It is easy to check that if the min-max regret problem with interval parameters can be solved in $f(n)$ time, then the corresponding fuzzy problem is solvable in $O(f(n) \log \epsilon^{-1})$ time. W thus can see that the problem with fuzzy parameters boils down to solving a small number of min-max regret sequencing problems. So, every exact algorithm for the min-max regret problem can easily be adopted for the more general fuzzy case. In [6] a polynomial algorithm for the min-max regret $1|prec|L_{max}$ problem with interval processing times and interval due dates has been constructed. Consequently, the fuzzy $1|prec|L_{max}$ problem is solvable efficiently as well because we can use this algorithm as a subroutine in the binary search.

The binary search algorithm is the most general method of solving the fuzzy problem which, however, requires a given precision of calculations. The formulation (12) sometimes allows us to design an exact algorithm based on a mixed integer programming (MIP) formulation. The obtained MIP model can be then solved by using some standard packages such as CPLEX or GLPK. We will show an example in the next section.

4.3 *The fuzzy problem* $1||\sum C_i$

Assume that for every job $i \in J$, a fuzzy processing time with possibility distribution μ_{p_i} is given. A scenario in this problem is a particular realization of the processing times. The cost of a sequence σ under S is the total flow time, that is the sum of job completion times. Under a given processing times scenario S, an optimal sequence can be obtained in $O(n \log n)$ time by ordering the jobs with respect to nondecreasing processing times [1].

We first show that the problem of computing the most necessarily optimal sequence, i.e. the problem (11), is efficiently solvable. Let us fix $\lambda \in [0, 1]$ and consider the min-max regret $1||\sum C_i$ problem with interval processing times specified by scenario set $\Gamma^{\lambda} = [\underline{p}^{\lambda}_i, \overline{p}^{\lambda}_i] \times \cdots \times [\underline{p}^{\lambda}_n, \overline{p}^{\lambda}_n]$. Consider a scenario $S \in \Gamma^{\lambda}$ such that under this scenario the processing times are $\frac{1}{2}(\underline{p}^{\lambda}_i + \overline{p}^{\lambda}_i)$ for $i \in J$. We can compute in $O(n \log n)$ time and optimal sequence ρ under S. It turns out that $\overline{\delta}^{\lambda}_{\rho} \leq 2\overline{\delta}^{\lambda}_{\sigma}$ for all sequences σ (see [20]). In consequence, if there is a necessarily optimal sequence σ such that $\overline{\delta}^{\lambda}_{\sigma} = 0$, then ρ must also be necessarily optimal. We thus have an efficient method of detecting a necessarily optimal sequence and the problem (11) can be solved in $O(n \log n \log \epsilon^{-1})$ time by using the binary search shown in Figure 3.

We now focus on the more general problem (10). Unfortunately, this problem is NP-hard because the corresponding min-max regret problem with interval processing times is NP-hard [10]. Using the formulation (12) we will show that it is possible to design a mixed integer linear programming model to solve the fuzzy problem. Let $\mathcal{A}$ be the set of all binary vectors (x_{ij}), $i = 1, \dots, n$, $j = 1 \dots, n$, fulfilling the so called *assignment constraints*, that is $\sum_{i=1}^{n} x_{ij} = 1$ for all $j = 1, \dots, n$ and $\sum_{j=1}^{n} x_{ij} = 1$ for all $i = 1, \dots, n$. A vector $(x_{ij}) \in \mathcal{A}$ represents a sequence σ in which $x_{ij} = 1$ if job $i \in J$ occupies position j in σ. Obviously, there is one to one correspondence between the sequences of the set of jobs J and the vectors in $\mathcal{A}$ (recall that there are no precedence constraints in J). If $(x_{ij}) \in \mathcal{A}$ corresponds to sequence σ, then the maximal regret of σ under scenario set Γ^{λ} can be computed in the following way [3, 11]:

$$\overline{\delta}^{\lambda}_{\sigma} = \max_{(z_{ij}) \in \mathcal{A}} \sum_{i=1}^{n} \sum_{j=1}^{n} c^{\lambda}_{ij} z_{ij}, \qquad (14)$$

where

$$c^{\lambda}_{ij} = \overline{p}^{\lambda}_i \sum_{k=1}^{j} (j-k) x_{ik} + \underline{p}^{\lambda}_i \sum_{k=j+1}^{n} (j-k) x_{ik}. \qquad (15)$$

Observe that (14) is an assignment problem. We can construct the dual to (14) and it is well known that this dual has the same optimal objective function value as (14). So, it holds:

$$\begin{aligned} \overline{\delta}^{\lambda}_{\sigma} = \quad & \min \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i \\ & \alpha_i + \beta_j \geq c^{\lambda}_{ij} \text{ for } i, j = 1, \dots, n \end{aligned} \qquad (16)$$

where α_i and β_i, $i = 1, \dots, n$, are unrestricted dual variables associated with the assignment constraints. Now, using formulation (12), we can represent the fuzzy problem in the fol-

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lowing way:

$$\begin{array}{l} \min \lambda \\ \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i \leq \overline{g}^{1-\lambda} \\ \alpha_i + \beta_j \geq c_{ij}^{\lambda} \text{ for } i,j = 1,\ldots,n \\ (x_{ij}) \in \mathcal{A} \\ \lambda \in [0,1] \end{array} \tag{17}$$

where c_{ij}^{λ} are given as (15). If trapezoidal fuzzy intervals $(\underline{p}_i, \overline{p}_i, \alpha_i, \beta_i)$, $i = 1,\ldots,n$, are used to model the processing times, then we can substitute $\overline{p}_i^{\lambda} = \overline{p}_i + (1-\lambda)\beta_i$ and $\underline{p}_i^{\lambda} = \underline{p}_i - (1-\lambda)\alpha_i$ in (15). The resulting model will be still not linear because some expressions of the form λx_{ij} will appear. However, we can easily linearize the model by substituting $t_{ij} = \lambda x_{ij}$ and adding additional constraints $t_{ij} - x_{ij} \leq 0$, $\lambda - t_{ij} + x_{ij} \leq 0$, $-\lambda + t_{ij} \leq 0$, $t_{ij} \geq 0$ for all $i,j = 1,\ldots,n$. Hence the resulting model is a mixed integer linear one and can be solved by using an available software. We refer the reader to [11] for some techniques that allow us to refine the formulation and speed up the computations.

5 Conclusions

In this paper we have adopted a general approach to sequencing problems with fuzzy parameters. We have applied possibility theory to model the imprecise parameters and to define a solution concept. We have extended the approach used in [19] and [20] to another class of problems. As in many fuzzy problems, the main computational difficultly is in the classical interval case. Namely, every exact algorithm that computes a min-max regret sequence under interval data can be easily adopted to solve the fuzzy problem. One can use a binary search method or try to design a mixed integer programming model.

Unfortunately, the min-max regret sequencing problems with interval data are mostly hard to solve. There are only few problems that are known to be polynomially solvable and the complexity of some important problems, such as the permutation flow shop on two machines, remains open. Contrary to the class of problems discussed in [20], there is also lack of general properties of the min-max regret sequencing problems. Specifically, there is no general method of computing the maximal regret of a given sequence and there are no general relationships between the optimality evaluation and the min-max regret sequences. In other words, every sequencing problem possesses its own properties and the area of research in this field is still not fully explored.

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