

# Early Start Can Inhibit Learning: Towards A New Explanation

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**Abstract**— *The age at which we teach different topics change. If it turns out that students do not learn, say, reading by the time they should, a natural idea is to start teaching them earlier. Several decades ago, reading and writing started in the first grade, now they start at kindergarten and even earlier. At first glance, the earlier we start, the better the students will learn. Yes, they may play less, their childhood may be not as careless as it used to be, but at least they will learn better. In practice, however, this does not always work this way: early start often inhibits learning.*

*In this paper, we propose a simple model that explains why this can happen, and how we can avoid such situations and enhance the student learning. In particular, we explain why, contrary to the traditional concrete-first way of teaching abstracts concepts of school mathematics, abstract-first approach is often beneficial.*

**Keywords**— Abstract-first, early start, education, optimal education, uncertainty

## 1 Formulation of the Problem

### 1.1 Early Start: A Seemingly Natural Idea

The age at which we teach different topics change. If it turns out that students do not learn, say, reading by the time they should, a natural idea is to start teaching them earlier.

Several decades ago, reading and writing started in the first grade, now they start at kindergarten and even earlier. At first glance, the earlier we start, the better the students will learn.

### 1.2 Early Start: Known Side Effects

With the early start, children may play less, their childhood may be not as careless as it used to be – but a usual expectation is that with an early start, children will learn better.

### 1.3 Early Start: Serious Problems

In practice, however, early start does not always help: often, early start inhibits learning.

For example, according to [18],

- human infants who started learning to turn their heads to specific sounds at the age of 31 days mastered this task, on average, at the age of 71 days, while
- infants who started learning this task at birth mastered this task, on average, at the age of 128 days.

This phenomenon is not limited to human infants: according to [8], an early start in training rhesus monkeys to discriminate objects decreased their peak performance level.

Numerous examples when an early start inhibits learning are presented and discussed in [2, 3, 4, 6, 19].

### 1.4 Natural Questions

The empirical fact that an early start often inhibits learning leads to the following natural question: how do take this phenomenon into account when enhancing student learning?

To be able to take this phenomenon into account in the learning process, we must be able to understand this phenomenon – and ideally, understand on the quantitative level.

### 1.5 These Questions Are Still Largely Open

In [2, 3, 4, 6], an attempt is made to understand why early start can inhibit learning. However,

- the existing understanding is still mostly on the qualitative level, and
- even on this level, the proposed explanations are still not fully satisfactory; see, e.g., [19].

### 1.6 More General Questions

The above questions about the efficiency of the early start can be viewed as a particular case of more general questions: what is the best order of presenting the material, the order that leads to the best possible learning?

### 1.7 These More General Questions Are Often Very Important

Many empirical studies have shown that a change in the order in which different parts of the material are presented often drastically changes the learning efficiency; see, e.g., [5, 9, 10, 13, 14, 16, 21, 22].

This is not only about using common sense: sometimes, the empirical results are counter-intuitive. For example:

- it is usually assumed that most students learn mathematical concepts better if
  - they are first presented with concrete examples of these concepts, and
  - they only learn abstract ideas later on;
- however, it turns out that empirically, the abstract-first approach for presenting the material often enhances learning; see, e.g., [9, 10, 14, 21].

### 1.8 What We Do in This Paper

In this paper, we propose a simple model that attempts:

- to explain the negative effect of early start and,

- more generally, to explain the reasons why a change in presentation order can drastically change the efficiency of learning.

We then show how this explanation can be used to avoid inhibition of learning – and to enhance the student learning.

## 2 Towards A Theoretical Explanation

### 2.1 Learning: A Natural Geometric Representation

To facilitate reasoning about learning, let us start with a simple geometric representation of learning.

The process of learning means that we change the state of a student:

- from a state in which the student did not know the material (or does not have the required skill)
- to a state in which the student has (some) knowledge of the required material (or has the required skill).

Let  $s_0$  denote the original state of a student, and let  $S$  denote the set of all the states corresponding to the required knowledge or skill.

- We start with a state which is not in the set  $S$  ( $s_0 \notin S$ ), and
- we end up in a state  $s$  which is in the set  $S$ .

On the set of all possible states, it is natural to define a metric  $d(s, s')$  as the difficulty (time, effort, etc.) needed to go from state  $s$  to state  $s'$ . Our objective is to help the students learn in the easiest (fastest, etc.) way. In terms of the metric  $d$ , this means that we want to go from the original state  $s_0 \notin S$  to the state  $s \in S$  for which the effort  $d(s_0, s)$  is the smallest possible.

In geometric terms, the smallest possible effort means the shortest possible distance. Thus, our objective is to find the state  $s \in S$  which is the closest to  $s_0$ . Such closest state is called the *projection* of the original state  $s_0$  on the set  $S$ .

### 2.2 Learning Complex Material: Geometric Interpretation

The above geometric description of learning as a transition from the original state  $s_0$  to its projection on the desired set  $S$  describes learning *as a whole*. Our objective is to find out which order of presenting information is the best. Thus, our objective is to analyze the *process* of learning, i.e., learning as a multi-stage phenomenon. For this analysis, we must explicitly take into account that the material to be learned consists of several pieces.

Let  $S_i$ ,  $1 \leq i \leq n$ , denote the set of states in which a student has learned the  $i$ -th part of the material. Our ultimate objective is to make sure that the student learns all the parts of the material. In terms of states, learning the  $i$ -th part of the material means belonging to the set  $S_i$ . Thus, in terms of states, our objective means that the student should end up in a state which belongs to all the sets  $S_1, \dots, S_n$  – i.e., in other words, in a state which belongs to the intersection

$$S \stackrel{\text{def}}{=} S_1 \cap \dots \cap S_n \quad (1)$$

of the corresponding sets  $S_i$ .

In these terms, if we present the material in the order  $S_1, S_2, \dots, S_n$ , this means that:

- we first project the original state  $s_0$  onto the set  $S_1$ , resulting is a state  $s_1 \in S_1$  which is the closest to  $s_0$ ;
- then, we project the state  $s_1$  onto the set  $S_2$ , resulting is a state  $s_2 \in S_2$  which is the closest to  $s_1$ ;
- ...
- at the last stage of the cycle, we project the state  $s_{n-1}$  onto the set  $S_n$ , resulting is a state  $s_n \in S_n$  which is the closest to  $s_{n-1}$ .

In some cases, we end up learning all the material – i.e., in a state  $s_n \in S_1 \cap \dots \cap S_n$ . However, often, by the time the students have learned  $S_n$ , they have somewhat forgotten the material that they learned in the beginning. So, it is necessary to repeat this material again (and again). Thus, starting from the state  $s_n$ , we again sequentially project onto the sets  $S_1, S_2$ , etc.

### 2.3 The Above Geometric Interpretation Makes Computational Sense

The above “sequential projections” algorithm is actually actively used in many applications; see, e.g., [7, 12, 20]. In the case when all the sets  $S_i$  are convex, the resulting Projections on Convex Sets (POCS) method actually guarantees (under certain reasonable conditions) that the corresponding projections converge to a point from the intersection  $S_1 \cap \dots \cap S_n$  – i.e., in our terms, that the students will eventually learn all parts of the necessary material.

In the more general non-convex case, the convergence is not always guaranteed – but the method is still efficiently used, and often converges.

### 2.4 The Simplest Case: Two-Part Knowledge

Let us start with the simplest case when knowledge consists of two parts. In this simplest case, there are only two options:

- The first option is that:
  - we begin by studying  $S_1$ ;
  - then, we study  $S_2$ ,
  - then, if needed, we study  $S_1$  again, etc.
- The second option is that:
  - we begin by studying  $S_2$ ;
  - then, we study  $S_1$ ,
  - then, if needed, we study  $S_2$  again, etc.

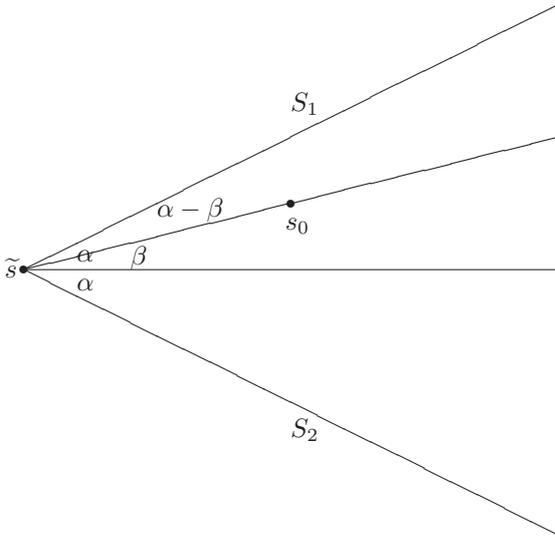
We want to get from the original state  $s_0$  to the state  $\tilde{s} \in S_1 \cap S_2$  which is the closest to  $s_0$ . The effectiveness of learning is determined by how close we get to the desired set  $S = S_1 \cap S_2$  in a given number of iterations.

In the case of two-part knowledge, it is natural to conclude that the amount of this knowledge is reasonably small – otherwise, we would have divided into a larger number of easier-to-learn pieces.

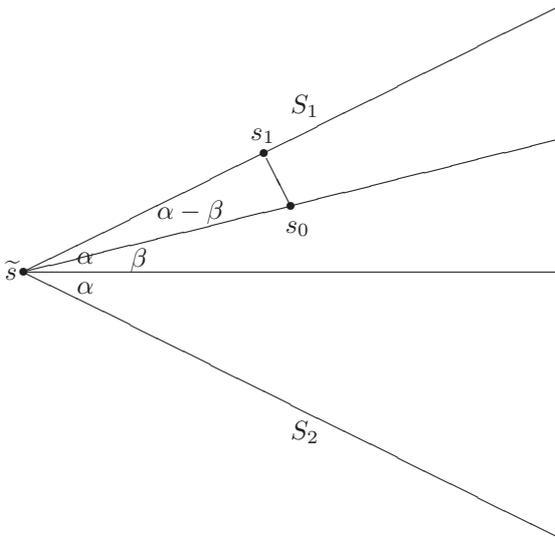
In geometric terms, this means that the original state  $s_0$  is close to the desired intersection set  $S_1 \cap S_2$ , i.e., that the distance  $d_0 \stackrel{\text{def}}{=} d(s_0, \tilde{s})$  is reasonably small.

Since all the states are close to each other, in the vicinity of the state  $\tilde{s}$ , we can therefore expand the formulas describing the borders of the sets  $S_i$  into Taylor series and keep only terms which are linear in the (coordinates of the) difference  $s - \tilde{s}$ . Thus, it is reasonable to assume that the border of each of the two sets  $S_i$  is described by a linear equation – and is hence a (hyper-)plane: a line in 2-D space, a plane in 3-D space, etc.

As a result, we arrive at the following configuration. Let  $2\alpha$  denote the angle between the borders of the sets  $S_1$  and  $S_2$ , so that the angles between each of these borders and the midline is exactly  $\alpha$ . Let  $\beta$  denote the angle between the direction from  $\tilde{s}$  to  $s_0$  and the midline. In this case, the angle between the border of  $S_1$  and the midline is equal to  $\alpha - \beta$ . So, we arrive at the following configuration:



In the first option, we first project  $s_0$  onto the set  $S_1$ . As a result, we get the following configuration:



Here, the projection line  $s_0s_1$  is orthogonal to the border of  $S_1$ . From the right triangle  $\triangle \tilde{s}s_0s_1$ , we therefore conclude that the distance  $d_1 \stackrel{\text{def}}{=} d(\tilde{s}, s_1)$  from the projection point  $s_1$  to the desired point  $\tilde{s}$  is equal to

$$d_1 = d_0 \cdot \cos(\alpha - \beta). \quad (2)$$

On the next step, we project the point  $s_1$  from  $S_1$  onto the line  $S_2$  which is located at the angle  $2\alpha$  from  $S_1$ . Thus, for the projection result  $s_2$ , we will have

$$d_2 = d(s_2, \tilde{s}) = d_1 \cdot \cos(2\alpha) = d_0 \cdot \cos(\alpha - \beta) \cdot \cos(2\alpha). \quad (3)$$

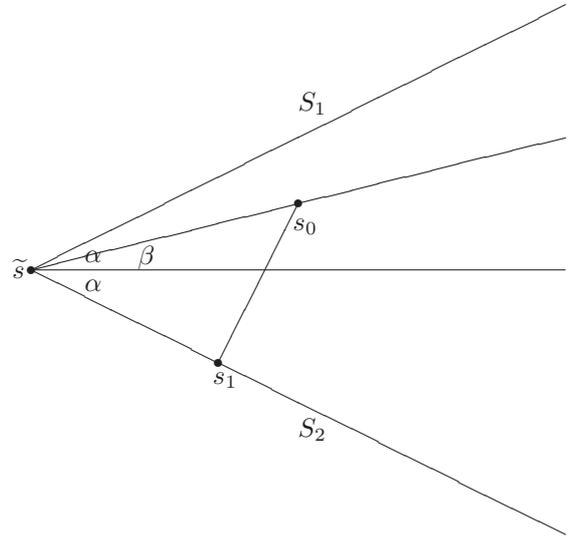
After this, we may again project onto  $S_2$ , then again project onto  $S_1$ , etc. For each of these projections, the angle is equal to  $2\alpha$ , so after each of them, the distance from the desired point  $\tilde{s}$  is multiplied the same factor  $\cos(2\alpha)$ .

As a result, after  $k$  projection steps, we get a point  $s_k$  at a distance

$$d_k = d(s_k, \tilde{s}) = d_0 \cdot \cos(\alpha - \beta) \cdot \cos^{k-1}(2\alpha) \quad (4)$$

from the desired state  $\tilde{s}$ .

In the second option, we start with teaching  $S_2$ , i.e., if we first project the state  $s_0$  into the set  $S_2$ . In this option, we get the following configuration:



Here, we have

$$d_1 = d_0 \cdot \cos(\alpha + \beta), \quad (5)$$

$$d_2 = d(s_2, \tilde{s}) = d_1 \cdot \cos(2\alpha) = d_0 \cdot \cos(\alpha + \beta) \cdot \cos(2\alpha), \quad (6)$$

...

$$d_k = d(s_k, \tilde{s}) = d_0 \cdot \cos(\alpha + \beta) \cdot \cos^{k-1}(2\alpha). \quad (7)$$

Since, in general,  $\cos(\alpha - \beta) \neq \cos(\alpha + \beta)$ , we can see that a change in the presentation order can indeed drastically change the success of the learning procedure.

### 2.5 Conclusion: Dependence Explained

Thus, our simple geometric model explains why the effectiveness of learning depends on the order in which the material is presented.

## 3 Specific Recommendations: Case of Two-Part Knowledge

### 3.1 Analysis

Let us extract more specific recommendations from our model. According to the above formulas, starting with  $S_1$

leads to a more effective learning than starting with  $S_2$  if and only if

$$d_0 \cdot \cos(\alpha - \beta) \cdot \cos^{k-1}(2\alpha) < d_0 \cdot \cos(\alpha + \beta) \cdot \cos^{k-1}(2\alpha), \quad (8)$$

i.e., equivalently, if and only if

$$\cos(\alpha - \beta) < \cos(\alpha + \beta). \quad (9)$$

Since for the angles  $x \in [0, \pi]$ , the cosine  $\cos(x)$  is a decreasing function, we conclude that projection of  $S_1$  is better if and only if

$$\alpha - \beta > \alpha + \beta. \quad (10)$$

Thus, we arrive at the following recommendation:

### 3.2 Recommendation

To make learning more efficient, we should start with studying the material which is further away from the current state of knowledge. In other words, we should start with a material that we know the least.

This ties in nicely with a natural commonsense recommendation that to perfect oneself, one should concentrate on one's deficiencies.

This recommendation is also in a very good accordance:

- with the seemingly counter-intuitive conclusion from [9, 10, 14, 21], that studying more difficult (abstract) ideas first enhances learning, and
- with the human infant studies [18] according to which a concentration on teaching, to human infants, skills that they can easily learn is detrimental in the long run.

## 4 General Case

### 4.1 Analysis of the Problem

What happens in the general case, when instead of only two knowledge components, we have a large number of different components? In the beginning, it still makes sense to project to the set  $S_{i_1}$  which is the farthest from the original state  $s_0$ .

After this original projection, in the general case, we still have a choice. We can project to any set  $S_{i_2}, i_2 \neq i_1$ , in which case the current distance  $d_1$  to the desired state is multiplied by the cosine  $\cos(\alpha_{i_1 i_2})$  of the angle between the corresponding sets  $S_{i_1}$  and  $S_{i_2}$ . After  $k$  steps, we get the original distance multiplied by the product of the corresponding cosines.

Our objective is to find the best order, i.e., the sequence  $S_{i_1}, S_{i_2}, \dots, S_{i_n}$  that covers all  $n$  sets  $S_1, \dots, S_n$  and for which the corresponding product

$$\cos(\alpha_{i_1 i_2}) \cdot \cos(\alpha_{i_1 i_2}) \cdot \dots \cdot \cos(\alpha_{i_n i_1}) \quad (11)$$

attains the smallest possible value.

Usually, it is easier to deal with the sums than with the products. To transform the product into a sum, we can use the fact that:

- minimizing the product is equivalent to minimizing its logarithm, and
- the logarithm of the product is equal to the sum of the logarithms.

Thus, minimizing the product (11) is equivalent to minimizing the sum

$$D(i_1, i_2) + \dots + D(i_n, i_1), \quad (12)$$

where

$$D(i, j) \stackrel{\text{def}}{=} \log(\cos(\alpha_{ij})). \quad (13)$$

In other words, we arrive at the following problem:

- we have  $n$  objects with known distances

$$D(i, j), \quad 1 \leq i, j \leq n;$$

- we must find a way to traverse all the objects and come back in such a way that the overall traveled distance is the smallest possible.

This is a well-known problem called a *traveling salesman problem*.

It is known that

- in general, this problem is NP-hard (see, e.g., [17]), and
- in many cases, there exist reasonable algorithms for solving this problem; see, e.g., [1].

### 4.2 Recommendations and the Need for Expert (Fuzzy) Knowledge

Based on the above analysis, we can make the following recommendations:

- To find the optimal order of presenting the material, we must solve the corresponding instance of the traveling salesman problem, with the distances determined by the formula (13).
- Since in general, the traveling salesman problem is computationally difficult (NP-hard), to efficiently solve this problem, we must use expert knowledge – the knowledge for which fuzzy technique have been invented; see, e.g., [11, 15].

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