Generalized Atanassov's Intuitionistic Fuzzy Index. Construction Method

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Abstract— In this work we introduce the concept of *Generalized Atanassov's Intuitionistic Fuzzy Index*. We characterize it in terms of fuzzy implication operators and we propose a construction method with automorphisms. Finally, we study some special properties of the generalized Atanassov's intuitionistic fuzzy index.

Keywords— Atanassov's Intuitionistic Fuzzy Index, Fuzzy Implication Operator, Automorphisms.

1 Introduction

In 1983 Atanassov [1] introduced Atanassov's intuitionistic fuzzy sets ($A - IFSs$) in such a way that each element of the set has two values assigned, the membership degree and the non-membership degree. In this direction, Atanassov defined these sets also indicating that the index of intuitionism of each element obtained by subtracting the sum of membership and non-membership from one (a subtraction that should be positive and less than or equal to one), it is a measurement of the effect of working with Atanassov's intuitionistic fuzzy sets. We consider this intuitionistic index (or condition of intuitionism) a very important characteristic of $A - IFSs$ since from it we can obtain very valuable information of each element and taking on advantage of this potentiality in different applications. For example, in image processing the task of divide into disjoint parts a digital image is denoted as segmentation. The most commonly used strategy for segmenting images is global thresholding that refers to the process of dividing the pixels in an image on the basis of their intensity levels of gray. The experts have uncertainty when assigning the pixels either to the background or to the object trough the choice of the membership functions. Moreover, this choice has proven to be of uttermost importance regarding the algorithms performance. In order to overcome this problem, we consider using the Atanassov's intuitionistic fuzzy index values for representing the uncertainty of the expert in determining that the pixel belongs to the background or that it belongs to the object. From this point of view, we can consider the expert provides the degree of membership of an element to an $A - IFS$ and also the degree of intuitionism the expert has in given this membership degree (see [9]). This fact has led us to present the new concept of *Generalized Atanassov's Intuitionistic Fuzzy Index* that generalize the expression given by Atanassov. We also provide a characterization method by means of fuzzy implication operators. Moreover, we study a construction method using automorphisms that allow us to present simple expressions of said index.

Pankowska and Wygralak ([16, 17, 21]) proposed another generalization of the intuitionistic index based on strong negations and triangular norms. This approach is used to construct flexible algorithms of group decision making which involve relative scalar cardinalities defined by means of generalized sigma counts of fuzzy sets.

2 Preliminary definitions

Let U be an ordinary finite non-empty set. An *Atanassov's intuitionistic fuzzy set* ($A - IFS$) [1] in U is an expression A given by

$$A = \{(u, \mu_A(u), \nu_A(u)) | u \in U\} \tag{1}$$

where

$$\mu_A : U \longrightarrow [0,1]$$
$$\nu_A : U \longrightarrow [0,1]$$

satisfy the condition $0 \leq \mu_A(u) + \nu_A(u) \leq 1$ for all u in U.

The numbers $\mu_A(u)$ and $\nu_A(u)$ denote respectively the degree of membership and the degree of non-membership of the element u in set A. We will also use the notation $A(u) = (\mu_A(u), \nu_A(u))$. We will represent as $A - IFSs(U)$ the set of all the Atanassov's intuitionistic fuzzy sets in U.
Atanassov defined the *Atanassov's intuitionistic fuzzy index* of the element u in $A \in A - IFSs(U)$ as:

$$\Pi_A(u) = 1 - \mu_A(u) - \nu_A(u). \tag{2}$$

We know fuzzy sets are represented exclusively by the membership function degree, that is,

$$A = \{(u, \mu_A(u)) | u \in U\}. \tag{3}$$

Hereinafter, fuzzy sets have associated a non-membership degree given by one minus the membership degree:

$$\begin{aligned} A = \; & \{(u, \mu_A(u), \nu_A(u)) | u \in U\} = \\ & \{(u, \mu_A(u), 1 - \mu_A(u)) | u \in U\}. \end{aligned} \tag{4}$$

Since $\mu_A(u) + \nu_A(u) = \mu_A(u) + 1 - \mu_A(u) = 1$, in this sense fuzzy sets are considered as a particular case of Atanassov's intuitionistic fuzzy sets. We will represent as $FSs(U)$ the set of all the fuzzy sets in U.

We will call automorphism of the unit interval every function $\varphi : [0,1] \to [0,1]$ that is continuous and strictly increasing such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

A function $n : [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$ is called a *strong negation* whenever it is strictly decreasing, continuous and involutive. Trillas ([19, 20]) proved that $n : [0,1] \to [0,1]$ is a strong negation if and only if there exists an automorphism φ of the unit interval such that

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$n(x) = \varphi^{-1}(1-\varphi(x))$. In this work we will only consider strong negations.

We denote as L^* the following set:

$$L^* = \{(x,y)|(x,y) \in [0,1]\times[0,1] \text{ and } x+y \leq 1\} \quad (5)$$

and the elements $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

For every $(x,y),(z,t) \in L^*$ the following expressions are known ([1]-[8],[12]-[14]):

- $(x,y) \leq_{L^*} (z,t)$ *if and only if* $x \leq z$ *and* $y \geq t$. This relation is transitive, reflexive and antisymmetric.
- $(x,y) = (z,t)$ *if and only if* $(x,y) \leq_{L^*} (z,t)$ *and* $(z,t) \leq_{L^*} (x,y)$.
- $(x,y) \preceq (z,t)$ *if and only if* $x \leq z$ *and* $y \leq t$.

  In [14] is proven $(L^*, \leq_{L^*})$ is a complete lattice.

Therefore, if $A \in FSs(U)$, then

$$A_c = \{(u, 1-\mu_A(u), \mu_A(u))|u \in U\}. \quad (6)$$

Next, we recall the definition of Atanassov's intuitionistic fuzzy t-norm and t-conorm and also the notion of t-representability, in such a way, Deschrijver *et al.* gave a construction method of Atanassov's intuitionistic fuzzy t-norms and t-conorms by means of t-norms and t-conorms on $[0,1]$ (see [14]).

Definition 1 *A function* $\mathbf{T} : (L^*)^2 \to L^*$ *is said to be an Atanassov's intuitionistic fuzzy t-norm if it is commutative, associative, and increasing (in both arguments with respect to the order* $\leq_{L^*}$*), with neutral element* 1_{L^*}*. In the same way, a function* $\mathbf{S} : (L^*)^2 \to L^*$ *is said to be an Atanassov's intuitionistic fuzzy t-conorm if it is commutative, associative, increasing and with neutral element* 0_{L^*}*.*

Definition 2 *An Atanassov's intuitionistic fuzzy t-norm* $\mathbf{T}$ *is called t-representable if and only if there exist a t-norm* T *and a t-conorm* S *on* $[0,1]$ *such that, for all* $(x,y),(z,t) \in L^*$

$$\mathbf{T}((x,y),(z,t)) = (T(x,z), S(y,t)) \in L^*. \quad (7)$$

An Atanassov's intuitionistic fuzzy t-conorm $\mathbf{S}$ *is called t-representable if and only if there exist a t-norm* T *and a t-conorm* S *on* $[0,1]$ *such that, for all* $(x,y),(z,t) \in L^*$

$$\mathbf{S}((x,y),(z,t)) = (S(x,z), T(y,t)) \in L^*. \quad (8)$$

Example 1 *Let be* $(x,y),(z,t) \in L^*$, $T = min$ *and* $S = max$ *on* $[0,1]$

$$(a) \quad \mathbf{min}((x,y),(z,t)) = (min(x,z), max(y,t)). \quad (9)$$

$$(b) \quad \mathbf{max}((x,y),(z,t)) = (max(x,z), min(y,t)). \quad (10)$$

Definition 3 *([10, 14]) An Atanassov's intuitionistic fuzzy negation is a function* $\mathbf{n} : L^* \to L^*$ *that is decreasing (with respect to* $\leq_{L^*}$*) such that* $\mathbf{n}(0_{L^*}) = 1_{L^*}$ *and* $\mathbf{n}(1_{L^*}) = 0_{L^*}$*. If for all* $(x,y) \in L^*$ $\mathbf{n}(\mathbf{n}((x,y))) = (x,y)$ *it is said that* $\mathbf{n}$ *is involutive.*

The characterization of the Atanassov's intuitionistic fuzzy negations and the following result are presented in [14], for interval-valued fuzzy sets are proven in [10].

Theorem 1 *A function* $\mathbf{n} : L^* \to L^*$ *is an involutive Atanassov's intuitionistic fuzzy negation if and only if there exists an involutive fuzzy negation* n *such that*

$$\mathbf{n}((x,y)) = (n(1-y), 1-n(x)). \quad (11)$$

Throughout this work we will restrict to involutive Atanassov's intuitionistic fuzzy negations $\mathbf{n}$ generated from a given negation n, as in Theorem 1.

3 Generalized Atanassov's Intuitionistic Fuzzy Index

In this section we propose the definition of generalized Atanassov's intuitionistic fuzzy index and we characterize such index by means of fuzzy implication operators and automorphisms.

Definition 4 *A function* $\Pi_G : L^* \to [0,1]$ *is called a generalized Atanassov's intuitionistic fuzzy index associated with the strong negation* n*, if it satisfies the following conditions:*

(i) $\Pi_G((x,y)) = 1$ *if and only if* $x = 0$ *and* $y = 0$*;*

(ii) $\Pi_G((x,y)) = 0$ *if and only if* $x + y = 1$*;*

(iii) If $(z,t) \preceq (x,y)$*, then* $\Pi_G((x,y)) \leq \Pi_G((z,t))$*;*

(iv) $\Pi_G((x,y)) = \Pi_G(\mathbf{n}((x,y)))$ *for all* $(x,y) \in L^*$ *such that* $\mathbf{n}$ *is generated from an involutive fuzzy negation* n*, as in Theorem 1.*

Example 2

$$(a) \quad \Pi_G((x,y)) = 1-y-x, \quad (12)$$

with $n(x) = 1-x$ *for all* $x \in [0,1]$*. As we can observe, this expression is equal to the expression (2) given by Atanassov.*

$$(b) \quad \Pi_G((x,y)) = ((1-y)^{0.5} - x^{0.5})^2 \quad (13)$$

with $n(x) = (1-x^{0.5})^2$ *for all* $x \in [0,1]$.

Fig. 1 depicts the expressions given by Example 2.

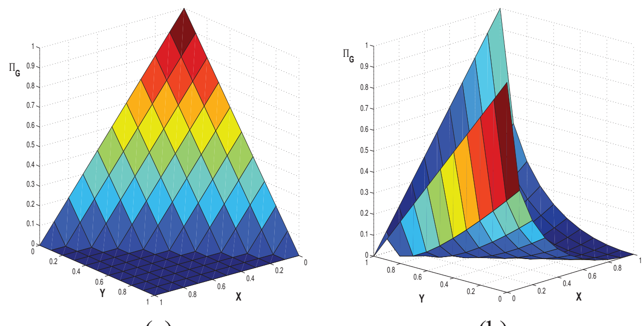

Figure 1: (a) $\Pi_G((x,y)) = 1-y-x$ (b) $\Pi_G((x,y)) = ((1-y)^{0.5} - x^{0.5})^2$

Next, we study the symmetry property of Π_G and we prove that only it is satisfied if we take as n the standard negation. Later, we introduce a construction method of the generalized Atanassov's intuitionistic fuzzy index using two automorphisms. In this way, it is quite simple to give different expressions of the intuitionistic index. Moreover, we analyze the case if the two automorphisms that we are taking are equal.

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Proposition 1 *Let Π_G be a generalized Atanassov's intuitionistic fuzzy index associated with the strong negation n. Then,*

$$\Pi_G((x,y)) = \Pi_G((y,x))$$

if and only if

$$n(x) = 1 - x;\ that\ is, \mathbf{n}((x,y)) = (y,x).$$

Proof. If $\Pi_G((x,y)) = \Pi_G((y,x))$ and by means of item (iv) of Definition 4 and Theorem 1, we obtain, $\Pi_G((x,y)) = \Pi_G((y,x)) = \Pi_G(\mathbf{n}((x,y))) = \Pi_G((n(1-y), 1-n(x)))$. Therefore, $n(1-y) = y$ and $n(x) = 1-x$.
On the other side, if n is the standard negation, then $\Pi_G((x,y)) = \Pi_G(\mathbf{n}((x,y))) = \Pi_G((n(1-y), 1-n(x))) = \Pi_G((y,x))$.

Proposition 2 *If φ_1, φ_2 are two automorphisms of the unit interval, then*

$$\Pi_G((x,y)) = \varphi_1^{-1}(\varphi_2(1-y) - \varphi_2(x)) \quad (14)$$

with $n(x) = \varphi_2^{-1}(1-\varphi_2(x))$ is a generalized Atanassov's intuitionistic fuzzy index associated with the strong negation n.

Proof. We must proof the four properties of a generalized Atanassov's intuitionistic fuzzy index. The proof of properties (i) and (ii) is direct bearing in mind the bound conditions of automorphisms definition. (iii) if $(z,t) \preceq (x,y)$ then, $z \leq x$ and $t \leq y$. Automorphisms are strictly increasing therefore, $\varphi_2(z) \leq \varphi_2(x)$ and $\varphi_2(1-y) \leq \varphi_2(1-t)$, so we obtain $\Pi_G((x,y)) \leq \Pi_G((z,t))$. (iv) Bearing in mind Teorem 1 and also $n(x) = \varphi_2^{-1}(1-\varphi_2(x))$, we have $\Pi_G(\ \mathbf{n}((x,y))) = \Pi_G((n(1-y), 1-n(x))) = \varphi_1^{-1}(\varphi_2(n(x)) - \varphi_2(n(1-y))) = \varphi_1^{-1}(\varphi_2(\varphi_2^{-1}(1-\varphi_2(x))) - \varphi_2(\varphi_2^{-1}(1-\varphi_2(1-y)))) = \Pi_G((x,y))$.

Proposition 3 *In the conditions of Proposition 2:*

$$\Pi_G((0,y)) = 1 - y$$

if and only if

$$\varphi_1(x) = \varphi_2(x)\ \ for\ all\ \ x \in [0,1].$$

Proof. Direct.

For us, a fuzzy implication operator will be an implication in the sense of Fodor and Roubens [15], that is, a function $I : [0,1]^2 \to [0,1]$ that satisfies the following properties:

I_1. If $x \leq z$ then $I(x,y) \geq I(z,y)$ for all $y \in [0,1]$;

I_2. If $y \leq t$ then $I(x,y) \leq I(x,t)$ for all $x \in [0,1]$;

I_3. $I(0,x) = 1$ (dominance of falsity) for all $x \in [0,1]$;

I_4. $I(x,1) = 1$ for all $x \in [0,1]$.

I_5. $I(1,0) = 0$.

Depending on the application the following properties can also be demanded to a fuzzy implication operator:

I_6. $I(1,x) = x$ (neutrality of truth).

I_7. $I(x, I(y,z)) = I(y, I(x,z))$ (exchange property).

I_8. $I(x,y) = 1$ if and only if $x \leq y$.

I_9. $I(x,y) = I(n(y), n(x))$ (contraposition) with a strong negation n.

I_{10}. I is a continuous function.

In [6] is provide the following important result.

Proposition 4 *Let $I : [0,1]^2 \to [0,1]$. For all $x,y,z \in [0,1]$, the following properties hold:*

(i) *I satisfies I_1 if and only if $I(max(x,y),z) = min(I(x,z), I(y,z))$.*

(ii) *I satisfies I_1 if and only if $I(min(x,y),z) = max(I(x,z), I(y,z))$.*

(iii) *I satisfies I_2 if and only if $I(x, min(y,z)) = min(I(x,y), I(x,z))$.*

(iv) *I satisfies I_2 if and only if $I(x, max(y,z)) = max(I(x,y), I(x,z))$.*

Next, we will give a characterization of the generalized Atanassov's intuitionistic fuzzy index Π_G by means of a function $I : [0,1]^2 \to [0,1]$ that satisfies some properties of fuzzy implication operators.

Theorem 2 *Let n be a strong negation. A function $\Pi_G : L^* \to [0,1]$ is a generalized Atanassov's intuitionistic fuzzy index associated with the strong negation n*

if and only if

there exists a function $I : [0,1]^2 \to [0,1]$ satisfying $I1, I8, I9$, and $I(x,y) = 0$ if and only if $x = 1$ and $y = 0$, such that

$$\Pi_G((x,y)) = n(I(1-y,x)).$$

Proof. Sufficiency. We need to prove the four properties of Definition 4. Properties (i) and (ii) are direct. Property (iii) we must consider the result proven in [6] that if I satisfies I_1 and I_9, then I satisfies I_2. Property (iv) is direct bearing in mind that I satisfies I_9 and n is a strong negation. In order to prove necessity, let us suppose that Π_G is a generalized Atanassov's intuitionistic fuzzy index associated with the strong negation n. Define I

$$I(x,y) = \begin{cases} 1 \text{ if } x \leq y \\ n(\Pi_G((y, 1-x))) \text{ if } x > y. \end{cases} \quad (15)$$

Now it is easy to prove that I satisfies I_1, I_8, I_9 and also $I(x,y) = 0$ if and only if $x = 1$ and $y = 0$.

Example 3 *We build these generalized Atanassov's intuitionistic fuzzy indices associated with the strong negation $n(x) = 1 - x$ for all $x \in [0,1]$.*

(a) Lukasiewicz implication: $I(x,y) = min(1, 1-x+y)$. Then,

$$\Pi_G((x,y)) = n(I(1-y,x)) =$$
$$n(min(1, 1-1+y+x)) = n(y+x) = 1-y-x. \quad (16)$$

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(b) Fodor implication:

$$I(x,y) = \begin{cases} 1 & if\, x \leq y \\ max(1-x,y) & if\, x > y \end{cases} \quad (17)$$

In this case we have

$$\Pi_G((x,y)) = \begin{cases} 0 & if\, x+y=1 \\ n(max(y,x)) & if\, x+y<1. \end{cases} \quad (18)$$

Corollary 1 *Let n be a strong negation. A continuous function $\Pi_G : L^* \to [0,1]$ such that $\Pi_G((x,0)) = n(x)$ for all $x \in [0,1]$ is a generalized Atanassov's intuitionistic fuzzy index associated with the function $I : [0,1]^2 \to [0,1]$ with I satisfying $I7$ and $I10$*

if and only if

there exists an automorphism φ of the unit interval such that

$$\Pi_G((x,y)) = \varphi^{-1}(\varphi(1-y) - \varphi(x)) \text{ and}$$

$$n(x) = \varphi^{-1}(1-\varphi(x))$$

Proof. It's enough to take into account Theorem 2, the relations between the properties of I studied in [6] and the following theorem proved in 1987 by Smets and Magrez (see [18]): a function $I : [0,1]^2 \to [0,1]$ verifies $I2$, $I7, I8$ and $I10$ if and only if there exists an automorphism φ of the unit interval such that $I(x,y) = \varphi^{-1}(min(1, 1-\varphi(x)+\varphi(y)))$.

Example 4 *if $\varphi(x) = x^2$ for all $x \in [0,1]$, then*

$$\Pi_G((x,y)) = ((1-y)^2 - x^2)^{0.5}, \quad (19)$$

with $n(x) = (1-x^2)^{0.5}$ for all $x \in [0,1]$.

Next, we provide the expressions of Π_G when it is applied to the meet operator min and the join operator max on L^*.

Theorem 3 *Under conditions of Theorem 2 the following items hold:*

(1) $\Pi_G(\mathbf{max}((x,y),(z,t))) = min\{max(\Pi_G((x,y)),\Pi_G((x,t))), max(\Pi_G((z,y)),\Pi_G((z,t)))\}$

(2) $\Pi_G(\mathbf{min}((x,y),(z,t))) = max\{min(\Pi_G((x,y)),\Pi_G((x,t))), min(\Pi_G((z,y)),\Pi_G((z,t)))\}$

Proof. (1) We must take into account the definition of meet operator min and the join operator max on L^*, also if I satisfies I_1 and I_9, then I satisfies I_2 and finally the items (i), (iv) of Proposition 4.

$\Pi_G(\mathbf{max}((x,y),(z,t))) = \Pi_G((max(x,z), min(y,t))) = n(I(1-min(y,t), max(x,z)))$. Therefore,

$$\begin{aligned}
& n(\Pi_G((max(x,z),min(y,t)))) = \\
& I(1-min(y,t),max(x,z)) = \\
& I(max(1-y,1-t),max(x,z)) = \\
& max\{I(max(1-y,1-t),x), I(max(1-y,1-t),z)\} = \\
& max\{min(I(1-y,x),I(1-t,x)), min(I(1-y,z),I(1-t,z))\} = \\
& max\{min(n(\Pi_G((x,y))),n(\Pi_G((x,t)))), min(n(\Pi_G((z,y))),n(\Pi_G((z,t))))\} = \\
& max\{n(max(\Pi_G((x,y)),\Pi_G((x,t)))), (n(max(\Pi_G((z,y)),\Pi_G((z,t))))\} = \\
& n(min\{max(\Pi_G((x,y)),\Pi_G((x,t))), max(\Pi_G((z,y)),\Pi_G((z,t)))\}).
\end{aligned}$$

(2) It is quite similar to the previous one but we must take into account items (ii) and (iii) of Proposition 4.

Note the result presented in Theorem 3 is not valid for any t-norm, t-conorm in L^*. The reason is that we have not defined the order relation $\leq_{L^*}$ in Definition 4, only we have defined the item (iii) If $(z,t) \preceq (x,y)$, then $\Pi_G((x,y)) \leq \Pi_G((z,t))$. That is, $\mathbf{max}((x,y),(z,t)) = (max(x,z), min(y,t)) \leq_{L^*} (S(x,z), T(y,t)) = \mathbf{S}((x,y),(z,t))$ and we obtain that $\Pi_G(\mathbf{max}((x,y),(z,t)))$ and $\Pi_G(\mathbf{S}((x,y),(z,t)))$ are incomparable.

4 Conclusions

We define the concept of *generalized Atanassov's intuitionistic fuzzy index* and we give different construction methods. We consider this concept could be applied to image processing and we want to relate it with the concept of local contrast of a window of an image, that is, the local variations in brightness. In [11] we proposed an expression of the local contrast constructed from Atanassovs intuitionistic index by means of Atanassov's intuitionistic fuzzy S-implications. Local contrast, from our point of view, must satisfy some specific properties and we would like to study the performance of the new concept presented in this work with regard to our definition of local contrast of an image.

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