Application of Fuzzy Vectors of Normalized Weights in Decision Making Models

Ondřej Pavlačka, Jana Talašová

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University Olomouc
tř. Svobody 26, 771 46 Olomouc, Czech Republic
Email: {pavlacko,talasova}@inf.upol.cz

Abstract— *In this paper, the notion of a fuzzy vector of normalized weights is introduced, and its application in decision making models based on the weighted average operation is studied. The fuzzy vector of normalized weights represents a generalization of normalized fuzzy weights that authors formerly applied for modeling uncertain weights of criteria and/or uncertain probabilities of states of the world. Illustrative examples show that fuzzy vectors of normalized weights extend the possibilities of utilizing the vague expert information concerning weights of criteria. For three particular forms of the fuzzy vectors of normalized weights, fuzzy weighted averages of given values are calculated and the obtained results are compared. Finally, the fuzzy models of multiple criteria decision making and decision making under risk based on the fuzzy weighted average operation are described.*

Keywords— Decision making under risk; Fuzzy vectors; Fuzzy weighted average; Multiple criteria decision making; Normalized fuzzy weights.

1 Introduction

In decision making models, the weighted average is the most often used aggregation operator. In multiple criteria decision making, overall evaluations of alternatives are mostly calculated as weighted averages of their evaluations with respect to particular criteria. The weighted average operator is also used in discrete stochastic models of decision making under risk where the alternatives are usually ranked according to their expected evaluations. The expected evaluations are in fact weighted averages of evaluations of alternatives under possible states of the world; probabilities of the states of the world play the role of normalized weights.

Decision making models based on the weighted average operation can be fuzzified in two aspects: First, uncertain weighted values, i.e. uncertain evaluations with respect to particular criteria in the models of multiple criteria decision making and/or uncertain evaluations under possible states of the world in decision making under risk, can be considered. Such fuzzification is useful because it enables to process correctly e.g. the vague expert evaluations of alternatives with respect to qualitative criteria. This approach is described in detail e.g. in [1]. Second, normalized weights, i.e. weights of criteria and/or probabilities of states of the worlds, can be fuzzified. It makes the mathematical models substantially more complicated. But as the values of normalized weights are commonly set expertly, such fuzzification is very eligible.

The fuzzy extension of the weighted average operation for the case when both the weighted values and the weights are fuzzy has been studied since the second half of 70's. In [2, 3, 4], the fuzzy weights are supposed to be non-negative fuzzy numbers that express the importance of particular criteria; they are normalized only in the process of the fuzzy weighted average calculation. Later, it was shown in [5, 6] that uncertain weights expressing shares of the partial evaluation in the overall one or uncertain probabilities of states of the world have to be modeled by means of a special structure of fuzzy numbers which is called a tuple of normalized fuzzy weights. The tuple of normalized fuzzy weights is formally identical with a so called feasible tuple of fuzzy probabilities mentioned in [7]. The fuzzy weights normalization, i.e. the transformation of fuzzy weights into the normalized fuzzy weights, was studied in [8, 9].

This paper describes a new approach to modeling the uncertain normalized weights of criteria and/or the uncertain probabilities of states of the world - a notion of a fuzzy vector of normalized weights is introduced here. It will be shown that, in comparison with normalized fuzzy weights, fuzzy vectors of normalized weights extend the possibilities of utilizing the vague expert information concerning the weights.

2 Preliminaries

A fuzzy set A on a non-empty set X is characterized by its membership function $A : X \rightarrow [0,1]$. By $Ker\,A$ and $Supp\,A$, we denote a kernel of A, $Ker\ A = \{x \in X \mid A(x) = 1\}$, and a support of A, $Supp\,A = \{x \in X \mid A(x) > 0\}$, respectively. For any $\alpha \in [0,1]$, A_α means an α-cut of A, $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$. The family of all fuzzy sets on X will be denoted by $\mathcal{F}(X)$.

The intersection of two fuzzy sets $A, B \in \mathcal{F}(X)$ is defined as a fuzzy set $A \cap B \in \mathcal{F}(X)$ whose membership function is for all $x \in X$ given by $(A \cap B)(x) = \min\{A(x), B(x)\}$. Since the minimum operation is used, $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$ holds for all $\alpha \in [0,1]$.

Let $X_1, X_2, \ldots, X_m$ be non-empty sets. An m-ary fuzzy relation on $X_1 \times X_2 \times \ldots \times X_m$ is defined as a fuzzy set $R \in \mathcal{F}(X_1 \times X_2 \times \ldots \times X_m)$. A particular case of an m-ary fuzzy relation is the Cartesian product of fuzzy sets, $R^C = R_1 \times R_2 \times \ldots \times R_m$, where $R_i \in \mathcal{F}(X_i)$, $i = 1, 2, \ldots, m$, and $R^C(x_1, x_2, \ldots, x_m) = \min\{R_1(x_1), R_2(x_2), \ldots, R_m(x_m)\}$ for all $(x_1, x_2, \ldots, x_m) \in X_1 \times X_2 \times \ldots \times X_m$. Since the minimum operation is used, $R^C_\alpha = R_{1\alpha} \times R_{2\alpha} \times \ldots \times R_{m\alpha}$ holds for any $\alpha \in [0,1]$.

Let us denote the index set $\{1, 2, \ldots, m\}$ by N_m throughout the paper. For any $i \in N_m$, the i-th projection of an m-ary

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fuzzy relation $R \in \mathcal{F}(X_1 \times X_2 \times \ldots \times X_m)$ is the fuzzy set $[R]_i \in \mathcal{F}(X_i)$ whose membership function is for all $x_i \in X_i$ given by

$$[R]_i(x_i) = \sup_{\substack{y_j \in X_j,\, j \in N_m \\ y_i = x_i}} R(y_1, y_2, \ldots, y_m). \quad (1)$$

A fuzzy number is a fuzzy set C on the set of all real numbers $\mathbb{R}$ that fulfils the following conditions: a) $Ker\ C \neq \emptyset$, b) for all $\alpha \in (0,1]$, C_α are closed intervals, c) $Supp\ C$ is bounded. If $Supp\ C \subseteq [a,b]$, then C is referred to as a fuzzy number on $[a,b]$. The family of all fuzzy numbers will be denoted by $\mathcal{F}_N(\mathbb{R})$, and the family of all fuzzy numbers on $[a,b]$ will be denoted by $\mathcal{F}_N([a,b])$.

In [9, 10], it was shown that any fuzzy number C can be also determined by a couple of functions $\underline{c} : [0,1] \to \mathbb{R}$ and $\overline{c} : [0,1] \to \mathbb{R}$ that describe the minimal and maximal values of the α-cuts of C and of the closure of the support of C. The functions $\underline{c}$ and $\overline{c}$ are left-continuous on $(0,1]$, right-continuous at 0 and satisfy $\underline{c}(\alpha) \leq \underline{c}(\beta) \leq \overline{c}(\beta) \leq \overline{c}(\alpha)$ for all $0 \leq \alpha < \beta \leq 1$. In the sequel, the notation $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)]\}_{\alpha \in [0,1]}$ will be used for the fuzzy number C such that $C_\alpha = [\underline{c}(\alpha), \overline{c}(\alpha)]$ for all $\alpha \in (0,1]$, and $Cl(Supp C) = [\underline{c}(0), \overline{c}(0)]$ where $Cl(Supp C)$ means the closure of the support of C. For the membership function of the fuzzy number $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)]\}_{\alpha \in [0,1]}$, the following relation holds

$$C(x) = \begin{cases} \max\limits_{\alpha \in [0,1]:\, x \in [\underline{c}(\alpha), \overline{c}(\alpha)]} \alpha, & \text{for } x \in [\underline{c}(0), \overline{c}(0)], \\ 0, & \text{elsewhere.} \end{cases} \quad (2)$$

Let us note that if $\underline{c}(\alpha) = \overline{c}(\alpha) = c$ for all $\alpha \in [0,1]$, then C represents a real number c. Furthermore, if $\underline{c}(\alpha) = a$ and $\overline{c}(\alpha) = b$ for all $\alpha \in [0,1]$, where $a < b$, then C represents a closed interval $[a,b]$.

In this paper, a fuzzy number $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)]\}_{\alpha \in [0,1]}$ is called linear if both the functions $\underline{c}$ and $\overline{c}$ are linear. Any linear fuzzy number C can be fully characterized by a quadruple of real numbers $c^1 \leq c^2 \leq c^3 \leq c^4$, where $Cl(Supp\ C) = [c^1, c^4]$ and $Ker\ C = [c^2, c^3]$. In this case, the functions $\underline{c}$ and $\overline{c}$ are given for all $\alpha \in [0,1]$ as follows

$$\underline{c}(\alpha) = c^1 + \alpha(c^2 - c^1) \quad \text{and} \quad \overline{c}(\alpha) = c^4 - \alpha(c^4 - c^3). \quad (3)$$

The notation $C = \langle c^1, c^2, c^3, c^4 \rangle$ will be used for such a linear fuzzy number C. For $c^2 \neq c^3$, the linear fuzzy number C is usually referred to as a trapezoidal fuzzy number; for $c^2 = c^3$, as a triangular fuzzy number.

The extension from functions having crisp arguments to functions with fuzzy set arguments is done according to the extension principle. Let X and Y be non-empty sets and let $f : X \to Y$ be a mapping. Then a fuzzy extension of f is the mapping $f_F : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that for any $A \in \mathcal{F}(X)$, the membership function of $f_F(A)$ is defined for all $y \in Y$ as follows

$$f_F(A)(y) = \begin{cases} \sup\limits_{x \in X:\, f(x) = y} A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

3 Fuzzy Vectors

Since an uncertain value of one continuous variable is expressed by a fuzzy number, then, by means of analogy, a fuzzy vector is used for expressing uncertain values of m continuous variables. An m-dimensional fuzzy vector $\mathbf{V}$ is a fuzzy set on $\mathbb{R}^m$ that fulfils the following conditions: a) $Ker\ \mathbf{V} \neq \emptyset$, b) for all $\alpha \in (0,1]$, $\mathbf{V}_\alpha$ are closed and convex subsets of $\mathbb{R}^m$, c) $Supp\ \mathbf{V}$ is bounded. If $Cl(Supp\ \mathbf{V}) \subseteq Q \subset \mathbb{R}^m$, then $\mathbf{V}$ is called a fuzzy vector on Q. Let us remark that the α-cuts $\mathbf{V}_\alpha$ are compact subsets of $\mathbb{R}^m$ for all $\alpha \in (0,1]$. The family of all m-dimensional fuzzy vectors will be denoted by $\mathcal{F}_V(\mathbb{R}^m)$, and the family of all m-dimensional fuzzy vectors on $Q \subset \mathbb{R}^m$ will be denoted by $\mathcal{F}_V(Q)$. Obviously, since a closed and convex subset of $\mathbb{R}$ is a closed interval, for $m = 1$ a fuzzy number is obtained, i.e. $\mathcal{F}_V(\mathbb{R}) = \mathcal{F}_N(\mathbb{R})$.

In literature, the requirement of convexity of $\mathbf{V}_\alpha$ is often weakened, e.g. in [11], the α-cuts of fuzzy vectors are supposed to be closed and simply connected subsets of $\mathbb{R}^m$. As convexity is the characteristic property of expertly given uncertain values, the convexity of α-cuts of fuzzy vectors will be considered in the paper.

For any $i \in N_m$, the i-th projection of the m-dimensional fuzzy vector $\mathbf{V}$ will be denoted by $[\mathbf{V}]_i$. It was proved in [9, 10] that $[\mathbf{V}]_i$ is a fuzzy number and its membership function is defined for all $y \in \mathbb{R}$ by the following formula

$$[\mathbf{V}]_i(y) = \max_{\substack{x_j \in \mathbb{R},\, j \in N_m \\ x_i = y}} \mathbf{V}(x_1, x_2, \ldots, x_m). \quad (5)$$

In the fuzzy models of decision making discussed in this paper, the expert setting of uncertain input data by fuzzy vectors is based on the following results that were proved in [9, 10].

If $X_1, X_2, \ldots, X_m$ are fuzzy numbers, then the Cartesian product $\mathbf{X} = X_1 \times X_2 \times \ldots \times X_m$ is an m-dimensional fuzzy vector, and $[\mathbf{X}]_i = X_i$ for all $i \in N_m$. Similarly, if $\mathbf{X}_i \in \mathcal{F}_V(\mathbb{R}^{n_i})$ for all $i \in N_m$, then the Cartesian product $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_m$ is an n-dimensional fuzzy vector where $n = \sum_{i=1}^m n_i$.

If the uncertain values of m variables are expressed by fuzzy numbers $X_1, X_2, \ldots, X_m$, and the set of all admissible combinations of their values is given by a crisp nonseparable m-ary relation $Q \subset \mathbb{R}^m$, then the maximal fuzzy relation (in the sense of ordering given by inclusion) expressing available information concerning the values of the m-tuple of variables is given by $\mathbf{X}_Q = (X_1 \times X_2 \times \ldots \times X_m) \cap Q$. If Q is closed and convex, then the fuzzy set $\mathbf{X}_Q$ is an m-dimensional fuzzy vector on Q if and only if

$$(Ker\ X_1 \times Ker\ X_2 \times \ldots \times Ker\ X_m) \cap Q \neq \emptyset. \quad (6)$$

Moreover, $[\mathbf{X}_Q]_i \subseteq X_i$ for all $i \in N_m$. The equality $[\mathbf{X}_Q]_i = X_i$ holds for all $i \in N_m$ if and only if for all $i \in N_m$ and for all $\alpha \in (0,1]$ the following holds:

$$\text{For any } x_i \in X_{i\alpha}, \text{ there exist } x_j \in X_{j\alpha},\ j \in N_m,\ j \neq i, \text{ such that } (x_1, \ldots, x_i, \ldots, x_m) \in Q. \quad (7)$$

Calculations with fuzzy vectors are based on the following well-known result, the proof of which can be found e.g. in [11]. Let $D \subseteq \mathbb{R}^m$, $m \geq 1$, let $f : D \to \mathbb{R}$ be a continuous function, and let $\mathbf{X} \in \mathcal{F}_V(D)$. Then $f_F(\mathbf{X})$ is a fuzzy number whose membership function is given by

$$f_F(\mathbf{X})(y) = \begin{cases} \max\limits_{\mathbf{x} \in D:\, f(\mathbf{x}) = y} \mathbf{X}(\mathbf{x}), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

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and for all $\alpha \in (0, 1]$ the following holds

$$f_F(\mathbf{X})_\alpha = f(\mathbf{X}_\alpha). \tag{9}$$

Employing the notation $f_F(\mathbf{X}) = \{[\underline{y}(\alpha), \overline{y}(\alpha)]\}_{\alpha\in[0,1]}$, then it follows from (9) that for any $\alpha \in (0, 1]$, the values $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ can be obtained by solving the following problems of mathematical programming

$$\underline{y}(\alpha) = \min_{\mathbf{x}\in\mathbf{X}_\alpha} f(\mathbf{x}), \tag{10}$$

$$\overline{y}(\alpha) = \max_{\mathbf{x}\in\mathbf{X}_\alpha} f(\mathbf{x}). \tag{11}$$

Furthermore, since the functions $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ are right-continuous at 0, the values $\underline{y}(0)$ and $\overline{y}(0)$ are given as follows

$$\underline{y}(0) = \lim_{\alpha\to 0+} \underline{y}(\alpha) \quad \text{and} \quad \overline{y}(0) = \lim_{\alpha\to 0+} \overline{y}(\alpha). \tag{12}$$

If $\mathbf{X} = X_1 \times X_2 \times \ldots \times X_m$ where $X_i \in \mathcal{F}_N(\mathbb{R})$ for any $i \in N_m$, $X_i = \{[\underline{x}_i(\alpha), \overline{x}_i(\alpha)]\}_{\alpha\in[0,1]}$, then for all $\alpha \in [0, 1]$ the following holds

$$\underline{y}(\alpha) = \min_{x_i\in[\underline{x}_i(\alpha),\overline{x}_i(\alpha)],\ i\in N_m} f(x_1, x_2, \ldots, x_m), \tag{13}$$

$$\overline{y}(\alpha) = \max_{x_i\in[\underline{x}_i(\alpha),\overline{x}_i(\alpha)],\ i\in N_m} f(x_1, x_2, \ldots, x_m). \tag{14}$$

If $\mathbf{X}_Q = (X_1\times X_2\times\ldots\times X_m)\cap Q$, where the crisp relation $Q \subset \mathbb{R}^m$ and fuzzy numbers $X_i = \{[\underline{x}_i(\alpha), \overline{x}_i(\alpha)]\}_{\alpha\in[0,1]}$, $i \in N_m$, fulfil the condition (6), then the values $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ can be obtained for all $\alpha \in [0, 1]$ as follows

$$\underline{y}(\alpha) = \min_{\substack{x_i\in[\underline{x}_i(\alpha),\overline{x}_i(\alpha)],\ i\in N_m \\ (x_1,x_2,\ldots,x_m)\in Q}} f(x_1, x_2, \ldots, x_m), \tag{15}$$

$$\overline{y}(\alpha) = \max_{\substack{x_i\in[\underline{x}_i(\alpha),\overline{x}_i(\alpha)],\ i\in N_m \\ (x_1,x_2,\ldots,x_m)\in Q}} f(x_1, x_2, \ldots, x_m). \tag{16}$$

The formulas (15) and (16) correspond to the concept of constrained fuzzy arithmetics that was studied e.g. in [12].

4 Fuzzy Vectors of Normalized Weights

An m-dimensional fuzzy vector of normalized weights is an arbitrary m-dimensional fuzzy vector $\mathbf{W}$, whose support $Supp\,\mathbf{W}$ is a subset of the m-dimensional probability simplex $\mathcal{S}_m = \{(w_1, w_2, \ldots, w_m) \in \mathbb{R}^m \mid w_i \geq 0 \text{ for all } i \in N_m,\ \sum_{i=1}^m w_i = 1\}$. The family of all m-dimensional fuzzy vectors of normalized weights will be denoted by $\mathcal{F}_V(\mathcal{S}_m)$. Figure 1 illustrates an example of a 3-dimensional fuzzy vector of normalized weights.

From the general point of view, an m-dimensional fuzzy vector of normalized weights expresses an uncertain division of a unit into m fraction. In this paper, we will apply the fuzzy vectors of normalized weights to model the uncertain shares of partial objectives of evaluation in the overall one in fuzzy models of multiple criteria decision making, and to model the uncertain probabilities of states of the world in fuzzy-stochastic models of decision making under risk.

In the decision making models, direct expert setting of a fuzzy vector of normalized weights could be quite complicated. Therefore, three procedures of setting of fuzzy vectors of normalized weights will be proposed now.

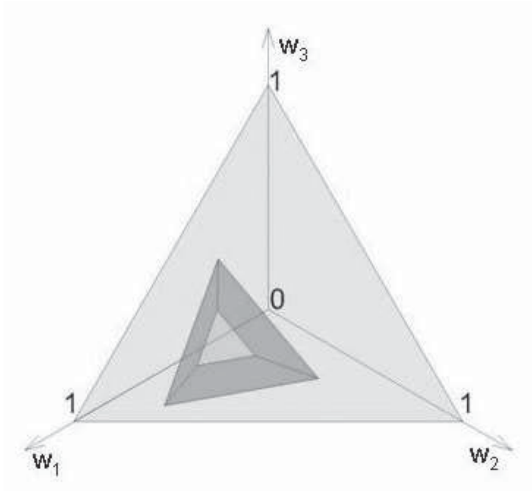

Figure 1: A 3-dimensional fuzzy vector of normalized weights.

Firstly, an expert can describe the uncertain values of normalized weights by a special structure of fuzzy numbers called normalized fuzzy weights. The normalized fuzzy weights are defined as fuzzy numbers $W_1, W_2, \ldots, W_m \in \mathcal{F}_N([0, 1])$ that satisfy for all $\alpha \in (0, 1]$ and for all $i \in N_m$ the following condition:

$$\text{For any } w_i \in W_{i\alpha}, \text{ there exist } w_j \in W_{j\alpha},\ j \in N_m,\ j \neq i, \quad \text{such that } w_i + \sum_{j=1,\ j\neq i}^{m} w_j = 1. \tag{17}$$

The condition (17) represents the special case of (7) for $Q = \mathcal{S}_m$. Since the probability simplex $\mathcal{S}_m$ is closed and convex, the fuzzy set $\mathbf{W} = (W_1 \times W_2 \times \ldots \times W_m) \cap \mathcal{S}_m$ is a fuzzy vector of normalized weights, and $[\mathbf{W}]_i = W_i$ for all $i \in N_m$. The procedures of practical setting of normalized fuzzy weights were described in [6, 9].

Example 1 Let $m = 3$. An expert describes the uncertain values of normalized weights by the triple of linear normalized fuzzy weights $W_1 = \langle 0.4, 0.5, 0.65, 0.8\rangle$, $W_2 = \langle 0.15, 0.2, 0.35, 0.55\rangle$ and $W_3 = \langle 0.05, 0.15, 0.3, 0.45\rangle$. The corresponding fuzzy vector of normalized weights is the fuzzy set $\mathbf{W} = (W_1 \times W_2 \times W_3)\cap\mathcal{S}_3$. For all $\alpha \in (0, 1]$, the α-cuts $\mathbf{W}_\alpha$ are given as follows

$$\begin{aligned} \mathbf{W}_\alpha = \{(w_1, w_2, w_3) \in \mathcal{S}_3 \mid\ & \\ & w_1 \in [0.4 + 0.1\alpha, 0.8 - 0.15\alpha], \\ & w_2 \in [0.15 + 0.05\alpha, 0.55 - 0.2\alpha], \\ & w_3 \in [0.05 + 0.1\alpha, 0.45 - 0.15\alpha]\}. \end{aligned} \tag{18}$$

The normalized fuzzy weights W_1, W_2, W_3 and the fuzzy vector of normalized weights $\mathbf{W} = (W_1 \times W_2 \times W_3) \cap \mathcal{S}_3$ are depicted in Figure 2. □

Secondly, an expert can give the information about normalized weights by a crisp relation $Q \subseteq \mathcal{S}_m$. If Q is non-empty, closed and convex, then it represents a special type of the fuzzy vector of normalized weights. For any $i \in N_m$, the

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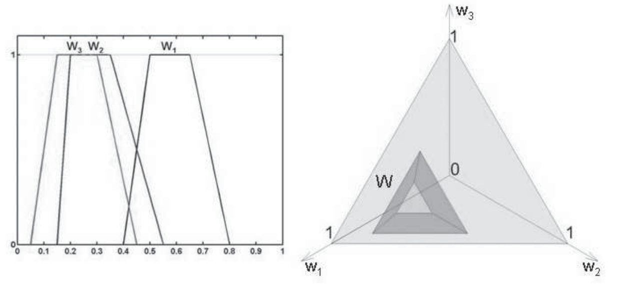

Figure 2: Normalized fuzzy weights W_1, W_2 and W_3 and the fuzzy vector of normalized weights $\mathbf{W} = (W_1 \times W_2 \times W_3) \cap \mathcal{S}_3$.

uncertain value of the i-th weight is expressed by

$$[Q]_i = \big\{w_i \in [0,1] \mid w_i \text{ is the } i\text{-th component of at least one vector } \mathbf{w} \in Q\big\}. \tag{19}$$

Example 2 Let $m = 3$. An expert gives only the ordinal information about the normalized weights by the crisp relation $T = \{(w_1, w_2, w_3) \in \mathcal{S}_3 \mid w_1 \geq w_2 \geq w_3\}$. Since T is closed and convex, T forms a 3-dimensional fuzzy vector of normalized weights $\mathbf{T}$ (see Fig. 3). Obviously, it follows from (19) that $[\mathbf{T}]_1 = [\frac{1}{3}, 1]$, $[\mathbf{T}]_2 = [0, \frac{1}{2}]$ and $[\mathbf{T}]_3 = [0, \frac{1}{3}]$. □

Thirdly, an expert can give both information about the uncertain values of weights, i.e. normalized fuzzy weights $W_1, W_2, \ldots, W_m$, and information about the crisp relations among normalized weights, i.e. the crisp relation $Q \subset \mathcal{S}_m$. If Q is non-empty, closed and convex and if $(Ker\ W_1 \times Ker\ W_2 \times \ldots \times Ker\ W_m) \cap Q \neq \emptyset$, then $\mathbf{W}_Q = (W_1 \times W_2 \times \ldots \times W_m) \cap Q$ is the corresponding fuzzy vector of normalized weights. For any $i \in N_m$, the true uncertain value of the i-th weight is expressed by $[\mathbf{W}_Q]_i \subseteq W_i$; the α-cuts of $[\mathbf{W}_Q]_i$ can be obtained for all $\alpha \in (0,1]$ in the following way

$$[\mathbf{W}_Q]_{i\alpha} = \big\{w_i \in W_{i\alpha} \mid (W_{1\alpha} \times \ldots \times W_{i-1\alpha} \times \{w_i\} \times W_{i+1\alpha} \times \ldots \times W_{m\alpha}) \cap Q \neq \emptyset\big\}. \tag{20}$$

Example 3 Let $m = 3$. First, an expert gives the ordinal information about the normalized weights by a crisp relation $T = \{(w_1, w_2, w_3) \in \mathcal{S}_3 \mid w_1 \geq w_2 \geq w_3\}$. Later on, he/she adds an uncertain cardinal information by a triple of linear normalized fuzzy weights $W_1 = \langle 0.4, 0.5, 0.65, 0.8\rangle$, $W_2 = \langle 0.15, 0.2, 0.35, 0.55\rangle$ and $W_3 = \langle 0.05, 0.15, 0.3, 0.45\rangle$. As $W_1 \geq W_2 \geq W_3$, i.e. $W_{1\alpha} \geq W_{2\alpha} \geq W_{3\alpha}$ for all $\alpha \in (0,1]$, it holds that $(KerW_1 \times KerW_2 \times KerW_3) \cap T \neq \emptyset$. The fuzzy set $\mathbf{W}_T = (W_1 \times W_2 \times W_3) \cap T$ represents the corresponding fuzzy vector of normalized weights (see Fig. 3). For all $\alpha \in (0,1]$, the following holds

$$\begin{aligned} \mathbf{W}_{T\alpha} = \big\{&(w_1, w_2, w_3) \in \mathcal{S}_3 \mid w_1 \geq w_2 \geq w_3, \\ &w_1 \in [0.4 + 0.1\alpha, 0.8 - 0.15\alpha], \\ &w_2 \in [0.15 + 0.05\alpha, 0.55 - 0.2\alpha], \\ &w_3 \in [0.05 + 0.1\alpha, 0.45 - 0.15\alpha]\big\}. \end{aligned} \tag{21}$$

According to (20), the projections of $\mathbf{W}_T$ are given as follows: $[\mathbf{W}_T]_1 = W_1$, $[\mathbf{W}_T]_2 = \{[\underline{w}_{T2}(\alpha), \overline{w}_{T2}(\alpha)]\}_{\alpha \in [0,1]}$, where $\underline{w}_{T2}(\alpha) = 0.15 + 0.05\alpha$, $\overline{w}_{T2}(\alpha) = 0.475 - 0.05\alpha$ for $0 \leq \alpha \leq 0.5$ and $\overline{w}_{T2}(\alpha) = 0.55 - 0.2\alpha$ for $0.5 < \alpha \leq 1$, and $[\mathbf{W}_T]_3 = \langle 0.05, 0.15, 0.25, 0.3\rangle$.

Let us note that although the fuzzy numbers W_1, W_2 and W_3 satisfy $W_1 \geq W_2 \geq W_3$, the crisp relation T represents an additional information for the model. Since the normalized fuzzy weights W_1, W_2 and W_3 satisfy with respect to T only the condition (6), but not (7), the projections $[\mathbf{W}_T]_1$, $[\mathbf{W}_T]_2$ and $[\mathbf{W}_T]_3$ are less uncertain then the normalized fuzzy weights W_1, W_2 and W_3. □

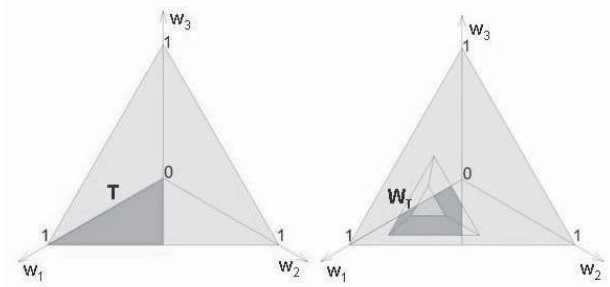

Figure 3: Fuzzy vectors of normalized weights $\mathbf{T} = \{(w_1, w_2, w_3) \in \mathcal{S}_3 \mid w_1 \geq w_2 \geq w_3\}$ and $\mathbf{W}_T = (W_1 \times W_2 \times W_3) \cap T$.

5 Fuzzy Weighted Average of Fuzzy Numbers with a Fuzzy Vector of Normalized Weights

A weighted average of real numbers $u_1, u_2, \ldots, u_m$ with a vector of normalized weights $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ is defined by the following formula

$$a(\mathbf{w}, u_1, u_2, \ldots, u_m) = \sum_{i=1}^{m} w_i u_i. \tag{22}$$

Since the weighted average operation is a continuous real function defined on $\mathcal{S}_m \times \mathbb{R}^m$, its fuzzy extension a_F has the property that $a_F(\mathbf{X})$ is a fuzzy number for all $\mathbf{X} \in \mathcal{F}_V(\mathcal{S}_m \times \mathbb{R}^m)$. In decision making models, the vector of normalized weights is usually supposed to be independent of the weighted values, and the weighted values to be mutually independent as well. Therefore, only the fuzzy vectors $\mathbf{X} = \mathbf{W} \times U_1 \times U_2 \times \ldots \times U_m$, where $\mathbf{W} \in \mathcal{F}_V(\mathcal{S}_m)$ and $U_i \in \mathcal{F}_N(\mathbb{R})$ for all $i \in N_m$, will be considered in this paper.

According to (8), the fuzzy weighted average of fuzzy numbers $U_1, U_2, \ldots, U_m$ with a fuzzy vector of normalized weights $\mathbf{W}$ is a fuzzy number $U = a_F(\mathbf{W}, U_1, U_2, \ldots, U_m)$ whose membership function is for all $u \in \mathbb{R}$ given by the formula

$$U(u) = \max_{\substack{\mathbf{w} \in \mathcal{S}_m \\ u_i \in \mathbb{R},\ i \in N_m \\ \sum_{i=1}^m w_i \cdot u_i = u}} \min\big\{\mathbf{W}(\mathbf{w}), U_1(u_1), \ldots, U_m(u_m)\big\}. \tag{23}$$

Since the weighted average operation is an increasing function in variables $u_1, u_2, \ldots, u_m$, it was proved in [9] that the general mathematical programming problems (10) and (11) can be simplified. Let us denote $U_i = \{[\underline{u}_i(\alpha), \overline{u}_i(\alpha)]\}_{\alpha \in [0,1]}$ for any $i \in N_m$. For the fuzzy weighted average U, $U =$

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$\{[\underline{u}(\alpha), \overline{u}(\alpha)]\}_{\alpha\in[0,1]}$, the values of the functions $\underline{u}$ and $\overline{u}$ can be calculated for all $\alpha \in (0, 1]$ as follows

$$\underline{u}(\alpha) = \min_{(w_1,w_2,\ldots,w_m)\in\mathbf{W}_\alpha} \sum_{i=1}^{m} w_i \cdot \underline{u}_i(\alpha), \quad (24)$$

$$\overline{u}(\alpha) = \max_{(w_1,w_2,\ldots,w_m)\in\mathbf{W}_\alpha} \sum_{i=1}^{m} w_i \cdot \overline{u}_i(\alpha). \quad (25)$$

Hence, complexity of the calculation of U depends only on the form of the α-cuts $\mathbf{W}_\alpha$.

It is worth noting that in the case of fuzzy weighted average of fuzzy numbers with normalized fuzzy weights $W_1, W_2, \ldots, W_m$, i.e. when $\mathbf{W} = (W_1 \times W_2 \times \ldots \times W_m) \cap \mathcal{S}_m$, it is not necessary to solve the mathematical programming problems (24) and (25). For such a case, the following effective algorithm for computing the values $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$, $\alpha \in [0, 1]$, was described in [6]. Let us denote $W_i = \{[\underline{w}_i(\alpha), \overline{w}_i(\alpha)]\}_{\alpha\in[0,1]}$ for any $i \in N_m$. For each $\alpha \in [0, 1]$, let $\{i_k\}_{k=1}^{m}$ be such a permutation on an index set N_m that $\underline{u}_{i_1}(\alpha) \le \underline{u}_{i_2}(\alpha) \le \ldots \le \underline{u}_{i_m}(\alpha)$. For $k \in N_m$, let us denote

$$w_{i_k}(\alpha) = 1 - \sum_{j=1}^{k-1} \overline{w}_{i_j}(\alpha) - \sum_{j=k+1}^{m} \underline{w}_{i_j}(\alpha). \quad (26)$$

Let $k^* \in N_m$ be such an index that $\underline{w}_{i_{k^*}}(\alpha) \le w_{i_{k^*}}(\alpha) \le \overline{w}_{i_{k^*}}(\alpha)$. Then

$$\underline{u}^N(\alpha) = \sum_{j=1}^{k^*-1} \overline{w}_{i_j}(\alpha) \cdot \underline{u}_{i_j}(\alpha) + w_{i_{k^*}}(\alpha) \cdot \underline{u}_{i_{k^*}}(\alpha) + \sum_{j=k^*+1}^{m} \underline{w}_{i_j}(\alpha) \cdot \underline{u}_{i_j}(\alpha). \quad (27)$$

Let $\{i_h\}_{h=1}^{m}$ be such a permutation on an index set N_m that $\overline{u}_{i_1}(\alpha) \ge \overline{u}_{i_2}(\alpha) \ge \ldots \ge \overline{u}_{i_m}(\alpha)$. For $h \in N_m$, let us denote

$$w_{i_h}(\alpha) = 1 - \sum_{j=1}^{h-1} \overline{w}_{i_j}(\alpha) - \sum_{j=h+1}^{m} \underline{w}_{i_j}(\alpha). \quad (28)$$

Let $h^* \in N_m$ be such an index that $\underline{w}_{i_{h^*}}(\alpha) \le w_{i_{h^*}}(\alpha) \le \overline{w}_{i_{h^*}}(\alpha)$. Then

$$\overline{u}^N(\alpha) = \sum_{j=1}^{h^*-1} \overline{w}_{i_j}(\alpha) \cdot \overline{u}_{i_j}(\alpha) + w_{i_{h^*}}(\alpha) \cdot \overline{u}_{i_{h^*}}(\alpha) + \sum_{j=h^*+1}^{m} \underline{w}_{i_j}(\alpha) \cdot \overline{u}_{i_j}(\alpha). \quad (29)$$

Example 4 Let us consider the fuzzy vectors of normalized weights $\mathbf{W}$ from Example 1, $\mathbf{T}$ from Example 2 and $\mathbf{W}_T$ from Example 3. Let the weighted values be, for simplicity, given by real numbers $u_1 = 0.3$, $u_2 = 0.1$ and $u_3 = 0.9$. Let us denote $U_W = a_F(\mathbf{W}, u_1, u_2, u_3)$, $U_T = a_F(\mathbf{T}, u_1, u_2, u_3)$ and $U_{W_T} = a_F(\mathbf{W}_T, u_1, u_2, u_3)$.

Since $\mathbf{W} = (W_1 \times W_2 \times W_3) \cap \mathcal{S}_3$, where W_1, W_2, W_3 are normalized fuzzy weights, by the special algorithm mentioned above we obtain that $U_W = \langle 0.22, 0.32, 0.44, 0.54\rangle$.

As $\mathbf{T} = \{(w_1, w_2, w_3) \in \mathcal{S}_3 \mid w_1 \ge w_2 \ge w_3\}$ is a crisp relation on $\mathcal{S}_3$ and weighted values are real numbers, the mathematical programming problems (24) and (25) does not depend on α. Therefore, the fuzzy weighted average U_T is equal to a closed interval, $U_T = \{[\underline{u}_T, \overline{u}_T]\}_{\alpha\in[0,1]}$. The values $\underline{u}_T = 0.2$ and $\overline{u}_T = 0.433$ were obtained by solving the following linear programming problems

$$\underline{u}_T = \min_{\substack{\sum_{i=1}^{3} w_i = 1\\ w_1 - w_2 \ge 0\\ w_2 - w_3 \ge 0\\ w_3 \ge 0}} 0.3w_1 + 0.1w_2 + 0.9w_3, \quad (30)$$

$$\overline{u}_T = \max_{\substack{\sum_{i=1}^{3} w_i = 1\\ w_1 - w_2 \ge 0\\ w_2 - w_3 \ge 0\\ w_3 \ge 0}} 0.3w_1 + 0.1w_2 + 0.9w_3. \quad (31)$$

The fuzzy weighted average of u_1, u_2 and u_3 with the fuzzy vector of normalized weights $\mathbf{W}_T$, $\mathbf{W}_T = (W_1 \times W_2 \times W_3) \cap T$, is the fuzzy number $U_{W_T} = \{[\underline{u}_{W_T}(\alpha), \overline{u}_{W_T}(\alpha)]\}_{\alpha\in[0,1]}$ where

$$\underline{u}_{W_T} = \min_{\substack{w_i \in W_{i\alpha},\, i \in N_3\\ \sum_{i=1}^{3} w_i = 1\\ w_1 - w_2 \ge 0\\ w_2 - w_3 \ge 0\\ w_3 \ge 0}} 0.3w_1 + 0.1w_2 + 0.9w_3, \quad (32)$$

$$\overline{u}_{W_T} = \max_{\substack{w_i \in W_{i\alpha},\, i \in N_3\\ \sum_{i=1}^{3} w_i = 1\\ w_1 - w_2 \ge 0\\ w_2 - w_3 \ge 0\\ w_3 \ge 0}} 0.3w_1 + 0.1w_2 + 0.9w_3. \quad (33)$$

From (32) and (33), we obtain that $\underline{u}_{W_T}(\alpha) = 0.235 + 0.07\alpha$ for $0 \le \alpha < 0.5$, $\underline{u}_{W_T}(\alpha) = 0.22 + 0.1\alpha$ for $0.5 \le \alpha \le 1$, and $\overline{u}_{W_T}(\alpha) = 0.42 - 0.02\alpha$.

Fig. 4 shows the fuzzy weighted averages U_W, U_T and U_{W_T}. □

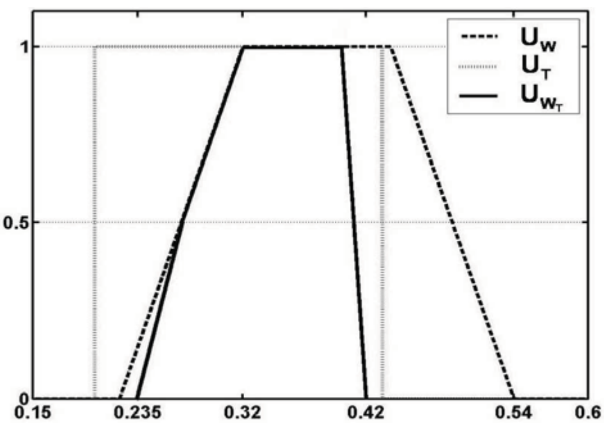

Figure 4: Fuzzy weighted averages U_W, U_T and U_{W_T}.

6 Fuzzy Models of Decision Making Based on the Fuzzy Weighted Average

In this section, a multiple criteria decision making model and a model of decision making under risk, where the fuzzy weighted average of fuzzy numbers with fuzzy vector of normalized weights is applied, will be described.

First, let us consider a problem of multiple criteria decision making, where the best of alternatives $x_1, x_2, \ldots, x_n$ is to be chosen. The alternatives are being evaluated with respect to a given overall objective that is partitioned into m partial objectives associated with criteria $C_1, C_2, \ldots, C_m$. Let the uncertain information about the shares of the partial objectives in the overall one be given by an m-dimensional fuzzy vector of normalized weights $\mathbf{W}$. Let uncertain partial fuzzy evaluations of alternatives x_i, $i \in N_n$, with respect to the criteria C_j, $j \in N_m$, be expressed by fuzzy numbers $U_{i,j}$ defined on $[0,1]$ that represent the fuzzy degrees of satisfaction of the corresponding partial objectives of evaluation.

Then the overall fuzzy evaluation U_i of the alternative x_i, $i \in N_n$, can be expressed as follows

$$U_i = a_F(\mathbf{W}, U_{i,1}, U_{i,2}, \ldots, U_{i,m}). \quad (34)$$

The overall fuzzy evaluations $U_1, U_2, \ldots, U_n$ express the fuzzy degrees of satisfaction of the overall objective of evaluation. The best alternative is the first alternative in an ordering of the fuzzy numbers $U_1, U_2, \ldots, U_n$ or the closest to the ideal alternative whose evaluation is equal to 1. For more details on metrics and ordering of fuzzy numbers see e.g. [13]. The overall fuzzy evaluations of alternatives can also be approximated linguistically by linearly ordered elements of a proper linguistic evaluation scale defined on $[0,1]$ (see [1]).

Second, let us consider a problem of decision making under risk that is described by the following fuzzy decision matrix (see Table 1), where $x_1, x_2, \ldots, x_n$ are alternatives, $S_1, S_2, \ldots, S_r$ states of the world, $\mathbf{P}$ is an r-dimensional fuzzy vector of normalized weights that expresses the uncertain probabilities of the states of the world, and for any $i \in N_n$ and $k \in N_r$, $U_{i,k} \in \mathcal{F}_N([0,1])$ denotes the fuzzy degree in which the alternative x_i satisfies a given decision objective if the state S_k occurs.

Table 1: Fuzzy decision matrix.

	S_1	S_2	$\ldots$	S_r	$\mathbf{P}$
x_1	$U_{1,1}$	$U_{1,2}$	$\ldots$	$U_{1,r}$	FEU_1
x_2	$U_{2,1}$	$U_{2,2}$	$\ldots$	$U_{2,r}$	FEU_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
x_n	$U_{n,1}$	$U_{n,2}$	$\ldots$	$U_{n,r}$	FEU_n

It was shown in [5] that the fuzzy expected value of evaluations of alternatives $x_1, x_2, \ldots, x_n$ are for all $i \in N_n$ expressed by

$$FEU_i = a_F(\mathbf{P}, U_{i,1}, U_{i,2}, \ldots, U_{i,r}). \quad (35)$$

The best alternative is usually determined by the rule of the maximum fuzzy expected value of evaluation. The maximum fuzzy expected value of evaluation is selected from $FEU_1, FEU_2, \ldots, FEU_n$ analogously as it was shown in the case of multiple criteria decision making.

A similar approach can be applied also to multiple criteria decision making under risk (see [14]).

7 Conclusion

In this paper, a new fuzzification of the weighted average operation where uncertain normalized weights are modeled by a general fuzzy vector defined on the probability simplex was presented. Several ways of expert setting the fuzzy vector of normalized weights in fuzzy models were proposed. It was shown that, in comparison with a tuple of normalized fuzzy weights, the fuzzy vector of normalized weights extends the possibilities of modeling the expert's knowledge concerning the weights. Besides the formal definition of the fuzzy weighted average of fuzzy numbers with a fuzzy vector of normalized weights, also a general algorithm for computing the fuzzy weighted average was presented; its three particular forms were illustrated by examples. Finally, applications of the fuzzy weighted average operation in models of multiple criteria decision making and decision making under risk were described.

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