Basic Properties of L-fuzzy Quantifiers of the Type $\langle 1 \rangle$ Determined by Fuzzy Measures

Antonín Dvořák, Michal Holčapek

Institute for Research and Applications of Fuzzy Modeling, University of Ostrava
Ostrava, Czech Republic
Email: {antonin.dvorak,michal.holcapek}@osu.cz

Abstract— The aim of this paper is to study monadic **L**-fuzzy quantifiers of the type $\langle 1 \rangle$ determined by fuzzy measures. These fuzzy quantifiers are defined using a novel notion of $\otimes$-fuzzy integral. Several semantic properties of these **L**-fuzzy quantifiers are studied.

Keywords— fuzzy integral, fuzzy logic, fuzzy measure, fuzzy quantifier, generalized quantifier

1 Introduction

This paper studies semantic properties of one class of monadic **L**-fuzzy quantifiers [1, 2] by studying one specific but important class of them, namely fuzzy quantifiers of the type $\langle 1 \rangle$ determined by fuzzy measures.

Quantifiers of the type $\langle 1 \rangle$ are denotations of important noun phrases of natural language, e.g. "something" in "Something is broken.", "everyone" in "Everyone likes Bob.", "nobody" in "Nobody knows everything.", etc. Moreover, classical logical quantifiers "for all" and "there exists" also belong to this type. It is claimed (e.g. in [3]) that from the point of view of natural language semantics, quantifiers of the type $\langle 1, 1 \rangle$ (e.g. "every" in "Every book has leaves.", "most" in "Most birds fly.") are more basic and more important. However, it is usual and advantageous to start with the type $\langle 1 \rangle$ quantifiers, because they are simpler and there are important relationships between them and quantifiers of the type $\langle 1, 1 \rangle$.

Generalized quantifiers evolved, from pioneering works of Mostowski [4], Lindström [5], Barwise and Cooper [6], into quite large research field with deep results. For overview as well as new results see a recent monograph [3]. A quantifier of the type $\langle 1 \rangle$ is usually modeled, given a universe M, as a mapping $Q_M : \mathcal{P}_M \longrightarrow \{true, false\}$ (or, equivalently, as subsets of the power set $\mathcal{P}_M$). It is possible to introduce many properties of (models of) quantifiers, characterizing their behavior from various points of view, for example permutation invariance (PI), isomorphism invariance (ISOM), extension (EXT), and others, see Section 4.

When we think about a definition and properties of generalized quantifiers (like e.g. *many*, *a few* and others), we feel that their truth values should not change abruptly if we gradually change cardinalities of corresponding sets of objects. Consider for example a sentence "Many people read books." If the number of people who read books increase by 1, it would be very strange if the truth value of this sentence changes from false to true. Therefore, it was inevitable that researchers started to consider more than two truth values in this context, and so-called *fuzzy quantifiers* emerged, starting from a generalization of definition from the previous paragraph, where instead of $\{true, false\}$ we consider some other structure of truth values, notably the interval $[0, 1]$.

Research in the field of fuzzy quantifiers started with works of Zadeh [7], Thiele [8], Ralescu [9] and others, see also [10, 11, 12]. An important contribution was made by Hájek in [13]. A comprehensive study of fuzzy quantifiers was undertaken by Glöckner [1] (see also [14]). In the recent paper [15], Novák studies so-called intermediate quantifiers, mainly from the syntactic point of view in the frame of fuzzy type theory [16]. An attempt to model linguistic quantifiers by fuzzy (Sugeno) integral was presented by Ying in [17].

The semantic interpretation of many generalized quantifiers is connected to measurement of "size" of sets in concern. Consider e.g. quantifier "many". The truth value of a proposition "many books have red cover" clearly depends on the "size" of the set of red books. Therefore, it is natural to consider measures (and integrals) of (fuzzy) sets as natural tools for the modeling of important classes of monotonically non-decreasing and monotonically non-increasing generalized quantifiers.

Fuzzy measures and integrals ([18], see also [19, 20]) are important tools allowing us to compare sets with respect to their size. Standardly, fuzzy measures are set functions defined on some algebra of sets which are monotone with respect to inclusion and they assign zero to the empty set. In our approach, fuzzy measures are defined on algebras of fuzzy sets (fuzzy measure spaces) and, generally, they attain values from a complete residuated lattice **L**. Details can be found in [21] in these proceedings, here we present only basic ideas.

Two types of fuzzy integral, namely the $\otimes$-fuzzy integral and the $\rightarrow$-fuzzy integral, are defined on an arbitrary fuzzy measure space. Integrals of $\otimes$ type will be used as models of quantifiers like *all* and *some*, while integrals of $\rightarrow$ type as models of *no* and *not all*, etc. However, in this contribution we will concentrate only on $\otimes$-fuzzy integrals and quantifiers defined by means of them. Nevertheless, if the structure of truth values **L** is a complete MV-algebra, then it is possible to define the $\rightarrow$-fuzzy integral from the $\otimes$-fuzzy integral [21].

2 Preliminaries

For details we refer to our contribution [21] in this volume, here we review only a few necessary notions.

2.1 Structures of truth values

In this paper, we suppose that the structure of truth values is a *complete residuated lattice* (see e.g. [22]), i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top \rangle$ with four binary operations and

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two constants such that $\langle L, \wedge, \vee, \perp, \top\rangle$ is a complete lattice, where $\perp$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \otimes, \top\rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \le b \to c \quad \text{iff} \quad a \otimes b \le c \tag{1}$$

holds for each $a, b, c \in L$, where $\le$ denotes the corresponding lattice ordering. The operations $\otimes$ and $\to$ are usually called multiplication and residuum, respectively.

2.2 **L**-*fuzzy sets*

Let $\mathbf{L} = \langle L, \wedge, \vee, \to, \otimes, \perp, \top\rangle$ be a complete residuated lattice and M be a universe of discourse (possibly empty). A mapping $A : M \to L$ is called an **L**-*fuzzy set on* M. A value $A(m)$ is called a *membership degree of* m *in the* **L**-*fuzzy set* A. The set of all **L**-fuzzy sets on M is denoted by $\mathcal{F}_{\mathbf{L}}(M)$. Obviously, if $M = \emptyset$, then the empty mapping $\emptyset$ is the unique **L**-fuzzy set on $\emptyset$ and thus $\mathcal{F}(\emptyset) = \{\emptyset\}$. An **L**-fuzzy set A on M is called *crisp*, if there is a subset X of M such that $A = 1_X$, where 1_X denotes the characteristic function of X. Particularly, $1_\emptyset$ denotes the empty **L**-fuzzy set on M, i.e., $1_\emptyset(m) = \perp$ for any $m \in M$. This convention will be also kept for $M = \emptyset$. The set of all crisp **L**-fuzzy sets on M is denoted by $\mathcal{P}_{\mathbf{L}}(M)$. An **L**-fuzzy set A is *constant*, if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant **L**-fuzzy set is denoted by the corresponding element of L, e.g., a, b, c.[1]

Let A be an **L**-fuzzy set on M. The *complement* of A is an **L**-fuzzy set $\overline{A}$ on M defined by $\overline{A}(m) = \neg A(m)$ for any $m \in M$. Finally, an extension of the operations $\otimes$ and $\to$ on L to the operations on $\mathcal{F}_{\mathbf{L}}(M)$ is given by

$$(A \otimes B)(m) = A(m) \otimes B(m) \tag{2}$$
$$(A \to B)(m) = A(m) \to B(m) \tag{3}$$

for any $A, B \in \mathcal{F}_{\mathbf{L}}(M)$ and $m \in M$, respectively. A mapping $f^{\to} : \mathcal{F}_{\mathbf{L}}(M) \to \mathcal{F}_{\mathbf{L}}(M')$ defined by $f^{\to}(A)(m) = \bigvee_{m' \in f^{-1}(m)} A(m')$ is called a *fuzzy extension* of the mapping f. Obviously, if f is a bijective mapping, then $f^{\to}(A)(f(m)) = A(m)$ for any $m \in M$.

3 Fuzzy measures and integrals

In this section, we will review a notion of a fuzzy measure of **L**-fuzzy sets and of a $\otimes$-fuzzy integral that will be used to define **L**-fuzzy quantifiers. For details and examples see [21], where also definitions of a complementary fuzzy measure and a $\to$-fuzzy integral can be found. For more information about fuzzy integrals, we refer to [18, 19].

3.1 *Fuzzy measures of* **L**-*fuzzy sets*

For our purposes we will consider algebras of **L**-fuzzy sets as a base for defining fuzzy measures of **L**-fuzzy sets.

Definition 3.1 ([18]). Let M be a non-empty universe of discourse. A subset $\mathcal{M}$ of $\mathcal{F}_{\mathbf{L}}(M)$ is an *algebra of* **L**-*fuzzy sets on* M, if the following conditions are satisfied

(i) $1_\emptyset, 1_M \in \mathcal{M}$,

(ii) if $A \in \mathcal{M}$, then $\overline{A} \in \mathcal{M}$,

(iii) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

A couple $(M, \mathcal{M})$ is called a *fuzzy measurable space*, if $\mathcal{M}$ is an algebra of **L**-fuzzy sets on M.

Let us introduce the concept of fuzzy measure as follows. The first definition is a modification of the definition of a normed measure with respect to truth values (see e.g. [19, 20]).

Definition 3.2. Let $(M, \mathcal{M})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{M} \to L$ is called a *fuzzy measure* on $(M, \mathcal{M})$, if

(i) $\mu(1_\emptyset) = \perp$ and $\mu(1_M) = \top$,

(ii) if $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\mu(A) \le \mu(B)$.

A triplet $(M, \mathcal{M}, \mu)$ is called the *fuzzy measure space*, if $(M, \mathcal{M})$ is a fuzzy measurable space and μ is a fuzzy measure on $(M, \mathcal{M})$.

Example 3.1. Let **L** be a complete residuated lattice with the support $[0, 1]$ and $\mathbb{N}$ be the set of natural numbers with 0. For any non-empty countable (finite or denumerable) universe M, injective mapping $f : M \to \mathbb{N}$, $n \in \mathbb{N}$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, denote

$$A_{f,n}(m) = \begin{cases} A(m), & \text{if } f(m) \le n; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Further, for any injective mapping $f : M \to \mathbb{N}$ and $n \in \mathbb{N}$, define $\mu_{f,n} : \mathcal{F}_{\mathbf{L}}(M) \to [0, 1]$ as follows

$$\mu_{f,n}(A) = \frac{\sum_{m \in \mathrm{Supp}(A_{f,n})} A_{f,n}(m)}{|\mathrm{Supp}(1_{M_{f,n}})|} \tag{5}$$

and, finally, define $\underline{\mu}_f, \overline{\mu}_f : \mathcal{F}_{\mathbf{L}}(M) \to [0, 1]$ as follows

$$\underline{\mu}_f = \liminf_{n \to \infty} \mu_{f,n}(A), \tag{6}$$
$$\overline{\mu}_f = \limsup_{n \to \infty} \mu_{f,n}(A). \tag{7}$$

It is easy to see that $\mu_{f,n}$, $\underline{\mu}_f$ and $\overline{\mu}_f$ are fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ determined by an injective mapping f. If, for example, $M = \mathbb{N}$ and $f = \mathrm{id}$, then $\underline{\mu}_f(A) = \overline{\mu}_f(A) = \perp$ for any **L**-fuzzy set on a finite universe. For the set of all even or odd numbers, both fuzzy measures give $\frac{1}{2}$ and, for the set of all prime numbers, we obtain 0.

If M is finite, then $\underline{\mu}_f = \underline{\mu}_g = \overline{\mu}_f = \overline{\mu}_g$ for any injective mappings $f, g : M \to \mathbb{N}$ and

$$\underline{\mu}_f(A) = \overline{\mu}_f(A) = \frac{\sum_{m \in M} A(m)}{|M|}. \tag{8}$$

Let $(M, \mathcal{M})$ be a fuzzy measurable space and $X \in \mathcal{F}_{\mathbf{L}}(M)$. Denote $\mathcal{M}_X$ the set of all $\mathcal{M}$-measurable sets which are contained in X, i.e.,

$$\mathcal{M}_X = \{A \mid A \in \mathcal{M} \text{ and } A \subseteq X\}. \tag{9}$$

Note that $1_\emptyset \in \mathcal{M}_X$ for each $X \in \mathcal{F}_{\mathbf{L}}(M)$ and if X is $\mathcal{M}$-measurable set, then also $X \in \mathcal{M}_X$. If $X = M$, then we will write only $\mathcal{M}$ instead of $\mathcal{M}_M$.

In the following part we will define an isomorphism between fuzzy measure spaces.

[1] We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant **L**-fuzzy set is assumed.

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Definition 3.3. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between* $(M, \mathcal{M})$ *and* $(M', \mathcal{M}')$, if

(i) g is a bijective mapping with $g(1_\emptyset) = 1_\emptyset$,

(ii) $g(A \cup B) = g(A) \cup g(B)$ and $g(\overline{A}) = \overline{g(A)}$ hold for any $A, B \in \mathcal{M}$,

(iii) there exists a bijective mapping $f : M \to M'$ with $A(m) = g(A)(f(m))$ for any $A \in \mathcal{M}$ and $m \in M$.

Definition 3.4. Let $(M, \mathcal{M})$ and $(M', \mathcal{M}')$ be fuzzy measurable spaces. We say that a mapping $g : \mathcal{M} \to \mathcal{M}'$ is an *isomorphism between* $(M, \mathcal{M}, \mu)$ *and* $(M', \mathcal{M}', \mu')$, if

(i) g is an isomorphism between $(M, \mathcal{M})$ and $(M', \mathcal{M}')$,

(ii) $\mu(A) = \mu'(g(A))$.

If g is an isomorphism between fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$, then we write $g(M, \mathcal{M}, \mu) = (M', \mathcal{M}', \mu')$.

Let $[(M, \mathcal{M}, \mu)]$ denote the class of all fuzzy measure spaces defined on M that are isomorphic with $(M, \mathcal{M}, \mu)$. Obviously, we can write

$$[(M, \mathcal{M}, \mu)] = \{(M, f^{\rightarrow}(\mathcal{M}), \mu_{f^{\rightarrow}}) \mid \\ \mid f : M \to M \text{ is a bijective mapping}\}.$$

3.2 ⊗-fuzzy integral

Definition 3.5. Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $X \in \mathcal{M}$. The *$\otimes$-fuzzy integral of A on X* is given by

$$\int_X^{\otimes} A\, d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in \mathrm{Supp}(Y)} (A(m) \otimes \mu(Y)). \quad (10)$$

If $X = 1_M$, then we write $\int^{\otimes} A\, d\mu$.

Theorem 3.1. *Let $(M, \mathcal{M}, \mu)$ be a fuzzy measure space. If $X \in \mathcal{M}$ is such that $1_{\mathrm{Supp}(Y)} \in \mathcal{M}_X$ for any $Y \in \mathcal{M}_X$, then for any $A \in \mathcal{F}_{\mathbf{L}}(M)$*

$$\int_X^{\otimes} A\, d\mu = \bigvee_{1_Y \in \mathcal{P}_X \setminus \{1_\emptyset\}} \bigwedge_{m \in Y} (A(m) \otimes \mu(1_Y)), \quad (11)$$

where $\mathcal{P}_X = \{1_{\mathrm{Supp}(Z)} \mid Z \in \mathcal{M}_X\}$.

Theorem 3.2. *Let $\mathbf{L}$ be a complete MV-algebra, $(M, \mathcal{M}, \mu)$ be a fuzzy measure space, $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $X \in \mathcal{M}$. Then*

$$\int_X^{\otimes} A\, d\mu = \bigvee_{Y \in \mathcal{M}_X \setminus \{1_\emptyset\}} \big(\mu(Y) \otimes \bigwedge_{m \in \mathrm{Supp}(Y)} A(m)\big). \quad (12)$$

Moreover,

$$\int_X^{\otimes} (c \otimes A)\, d\mu = c \otimes \int_X^{\otimes} A\, d\mu \quad (13)$$

for any $c \in L$.

Theorem 3.3. *Let g be an isomorphism between fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ and $X \in \mathcal{M}$. Then*

$$\int_X^{\otimes} A\, d\mu = \int_{g(X)}^{\otimes} g(A)\, d\mu' \quad (14)$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

In the end of this part we will show that the Sugeno integral is a special case of our proposed integral. Let $\mathbf{L}$ be a complete residuated lattice and $(M, \mathcal{M})$ be a fuzzy measurable space such that $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$. Denote $A_a = \{m \mid m \in M \;\&\; A(m) \geq a\}$. We say that an $\mathbf{L}$-fuzzy set A is *$\mathcal{M}$-Sugeno measurable*, if $1_{A_a} \in \mathcal{M}$ for any $a \in L$. The *Sugeno integral* is given, for any fuzzy measure space $(M, \mathcal{M}, \mu)$ with $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$, for any $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set A and for any $X \in \mathcal{M}$, by

$$\int_X A\, d\mu = \bigvee_{a \in L} (a \wedge \mu(1_{A_a} \cap X)). \quad (15)$$

Theorem 3.4. *Let $\mathbf{L}$ be a complete Heyting algebra, $(M, \mathcal{M}, \mu)$ be a fuzzy measure space with $A \cap B \in \mathcal{M}$ for any $A, B \in \mathcal{M}$, A be an $\mathcal{M}$-Sugeno measurable $\mathbf{L}$-fuzzy set and $X \in \mathcal{M}$. Then $\int_X A\, d\mu = \int_X^{\otimes} A\, d\mu$.*

For a definition and properties of →-fuzzy integral we refer to [21].

4 L-fuzzy quantifiers of the type ⟨1⟩

In [2], we defined the monadic $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1^n, 1\rangle$. Here, we restrict ourselves to their special subclass, namely, to the monadic $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1\rangle$ that can be defined as follows.

Definition 4.1. Let $\mathbf{L}$ be a complete residuated lattice, M be a universe (possibly empty[2]). A mapping $Q_M : \mathcal{F}_{\mathbf{L}}(M) \to L$ is called a *monadic $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ limited to M*.

Definition 4.2. An *unlimited (finite, countable) monadic $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$* is a functional Q assigning to each (finite, countable) universe M a monadic $\mathbf{L}$-fuzzy quantifier Q_M of the type $\langle 1\rangle$ limited to M.

In the following text, we will usually omit the terms "unlimited", "monadic" and "of the type ⟨1⟩" and we will say only "**L**-fuzzy quantifier". Let us demonstrate several examples of unlimited **L**-fuzzy quantifiers that are interpretations of well-known quantifiers in natural language (see [2]). We will use expressions all, some, not all and no as generic expressions which stand for natural language quantifiers of the type ⟨1⟩, e.g. "everything", "someone", "not everyone" and "nothing", respectively.

[2]To define the behavior of generalized quantifiers for the empty universe is important in some situations. It happens, for example, when we study type ⟨1, 1⟩ quantifiers which are obtained from type ⟨1⟩ quantifiers by means of *relativization*. Then it is vital to have values of e.g. $\mathsf{some}_\emptyset(1_\emptyset)$ defined, see discussion in [3], p. 137.

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Example 4.1. Let **L** be a complete residuated lattice. Then

$$(\mathsf{all})_M(A) = \bigwedge_{m\in M} A(m),$$
$$(\mathsf{some})_M(A) = \bigvee_{m\in M} A(m),$$
$$(\mathsf{not\ all})_M(A) = \bigvee_{m\in M} \neg A(m),$$
$$(\mathsf{no})_M(A) = \bigwedge_{m\in M} \neg A(m),$$

where M is an arbitrary universe and $A \in \mathcal{F}_{\mathbf{L}}(M)$, define unlimited **L**-fuzzy quantifiers of the type $\langle 1\rangle$. Obviously, the definitions of all and some (interpretations of quantifiers *all* and *some*) are the same as the interpretations of $\forall$ and $\exists$ in fuzzy logic, respectively. The others are negations of the previous ones. Notice that $(\mathsf{all})_\emptyset(\emptyset) = (\mathsf{no})_\emptyset(\emptyset) = \top$ and $(\mathsf{some})_\emptyset(\emptyset) = (\mathsf{not\ all})_\emptyset(\emptyset) = \bot$.

Now, let us recall some well-known semantics properties that are usually investigated in the case of the **L**-fuzzy quantifiers of the type $\langle 1\rangle$. For more information as well as examples we refer to [1, 2].

Definition 4.3. Let Q, P be **L**-fuzzy quantifiers. Then we say that Q *is less than or equal to* P and denote it by $Q \leq P$, if, for any non-empty universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$,

$$Q_M(A) \leq P_M(A). \tag{16}$$

Further, we say that Q *is equal to* P and denote it by $Q = P$, if $Q \leq P$ and $P \leq Q$.

Definition 4.4. Let Q, P be **L**-fuzzy quantifiers. We say that Q *is identical to* P and denote it by $Q \equiv P$, if for any (possibly empty) universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$,

$$Q_M(A) = P_M(A). \tag{17}$$

Remark 4.2. Note that the behavior of **L**-fuzzy quantifiers for the empty universe is often unpredictable (e.g., $\mathsf{all}_M(A) \leq \mathsf{some}_M(A)$ for all $M \neq \emptyset$, but $\mathsf{some}_\emptyset(1_\emptyset) \leq \mathsf{all}_\emptyset(1_\emptyset)$), therefore, we require only non-empty universes for their comparison in the first definition. Moreover, this restriction seems to be insignificant from the practical point of view. The second definition of identity of **L**-fuzzy quantifiers gives useful denotation.

Definition 4.5. An **L**-fuzzy quantifier Q is *permutation-invariant*, if for arbitrary universe M, bijective mapping $f : M \to M$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$

$$Q_M(A) = Q_M(f^{\to}(A)). \tag{18}$$

The set of all permutation-invariant **L**-fuzzy quantifiers is denoted by PI.

Definition 4.6. An **L**-fuzzy quantifier Q is *isomorphism-invariant*, if for arbitrary universe M, bijective mapping $f : M \to M'$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$,

$$Q_M(A) = Q_{M'}(f^{\to}(A)). \tag{19}$$

The set of all isomorphism-invariant **L**-fuzzy quantifiers is denoted by ISOM.

Definition 4.7. An **L**-fuzzy quantifier Q satisfies *extension*, if for arbitrary universes M, M' with $M \subseteq M'$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$,

$$Q_M(A) = Q_{M'}(A). \tag{20}$$

The set of all **L**-fuzzy quantifiers satisfying extension is denoted by EXT.

Definition 4.8. Let Q be an **L**-fuzzy quantifier. We say that Q is *monotonically non-decreasing*, if for arbitrary universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $A' \in \mathcal{F}_{\mathbf{L}}(M)$ with $A \subseteq A'$,

$$Q_M(A) \leq Q_M(A') \tag{21}$$

and Q is *monotonically non-increasing*, if for arbitrary universe M and $A \in \mathcal{F}_{\mathbf{L}}(M)$ and $A' \in \mathcal{F}_{\mathbf{L}}(M)$ with $A' \subseteq A$,

$$Q_M(A) \leq Q_M(A'). \tag{22}$$

For our purpose we will consider the following stronger definition of **L**-similarity of fuzzy sets. Recall that a mapping $R : \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ is called an **L**-*fuzzy relation on* $\mathcal{F}_{\mathbf{L}}(M)$. Let $[A\ R\ B]$ denote the degree in which **L**-fuzzy sets A and B belongs to **L**-fuzzy relation, i.e., $[A\ R\ B] = R(A, B)$. Let us define an **L**-fuzzy relation $\equiv_M : \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ by $[A \equiv_M B] = \top$, if there is a bijective mapping f of M onto M such that $f^{\to}(A) = B$, and $[A \equiv_M B] = \bot$, otherwise. The following definition generalizes $\equiv_M$.

Definition 4.9. An **L**-fuzzy relation $\approx_M : \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \to L$ is called an **L**-*permutation equivalence* on $\mathcal{F}_{\mathbf{L}}(M)$, if

$$[A \approx_M B] \geq [A \equiv_M B] \tag{23}$$
$$[A \approx_M B] = [B \approx_M A] \tag{24}$$
$$[A \approx_M B] \leq [\overline{A} \approx_M \overline{B}] \tag{25}$$
$$[A \approx_M B] \otimes [B \approx_M C] \leq [A \approx_M C] \tag{26}$$

hold for arbitrary $A, B, C \in \mathcal{F}_{\mathbf{L}}(M)$.

Obviously, (24) and (26) are the common axioms of symmetry and transitivity, respectively. Let $A, B \subseteq M$ are **L**-fuzzy sets which are similar. Then one could wish that the complements of A and B are also similar (at least in the degree in which A and B are **L**-equivalent). This idea is expressed in (25).

Example 4.3 (see [2])**.** Let **L** be a complete residuated lattice, M be any universe and $\mathrm{Perm}(M)$ denote the set of all bijective mappings of M onto M. Then

$$[A \approx^{\wedge}_M B] = \bigvee_{f\in \mathrm{Perm}(M)} \bigwedge_{m\in M} (A(m) \leftrightarrow B(m)) \tag{27}$$

defines the **L**-permutation equivalence $\approx^{\wedge}_M$ on $\mathcal{F}_{\mathbf{L}}(M)$.

Definition 4.10. Let $\approx$ be a class of **L**-permutation equivalences such that for each (finite, countable) universe M there is a unique $\approx_M$ from $\approx$ defined on $\mathcal{F}_{\mathbf{L}}(M)$. A (finite, countable) **L**-fuzzy quantifier Q of the type $\langle 1\rangle$ is *extensional with respect to* $\approx$, if

$$[A \approx_M A'] \leq Q_M(A) \leftrightarrow Q_M(A') \tag{28}$$

for each (finite, countable) universe M and $A, A' \in \mathcal{F}_{\mathbf{L}}(M)$. The set of all extensional **L**-fuzzy quantifiers with respect to $\approx$ is denoted by $\mathrm{EXTENS}(\approx)$.

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5 L-fuzzy quantifiers of the type $\langle 1 \rangle$ determined by fuzzy measures

Let $\mathcal{S}(M)$ denote a set of fuzzy measure spaces defined on M. For better readability, we will denote by

$$\int_{(M,\mathcal{M})}^{\otimes} A\, d\mu \tag{29}$$

the $\otimes$-fuzzy integral $\int^{\otimes} A\, d\mu$ defined over a fuzzy measure space $(M, \mathcal{M}, \mu)$. Now we can define **L**-fuzzy quantifiers limited to M using fuzzy measure spaces from a set $\mathcal{S}(M)$ as follows.

Definition 5.1. Let $\mathcal{S}(M)$ be a (possibly empty) set of fuzzy measure spaces defined on a non-empty universe M. An **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *limited to* M *determined by the fuzzy measure spaces from* $\mathcal{S}(M)$ is a mapping $Q_{\mathcal{S}(M)} : \mathcal{F}_{\mathbf{L}}(M) \to L$ defined by

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{(M,\mathcal{M},\mu)\in\mathcal{S}(M)} \int_{(M,\mathcal{M})}^{\otimes} A\, d\mu. \tag{30}$$

Remark 5.1. It is easy to see that if $\mathcal{S}(M) = \emptyset$ for some (possibly non-empty) set M, then $Q_{\mathcal{S}(M)}(A) = \bot$ for any $A \in \mathcal{F}_{\mathbf{L}}(M)$.

It is easy to see that $\mathcal{S}(\emptyset) = \emptyset$ (there is no fuzzy measure space with $M = \emptyset$). Hence, each unlimited fuzzy quantifier Q based only on the formula (30) has $Q_\emptyset(1_\emptyset) = Q_{\mathcal{S}(\emptyset)}(1_\emptyset) = \bot$. However, for example, it holds that $(\mathsf{all})_\emptyset(1_\emptyset) = \top$. This motivates us to exclude the determination of $Q_\emptyset(1_\emptyset)$ by (30) in the following definition of unlimited **L**-fuzzy quantifier.

Definition 5.2. Let $\mathcal{S}$ be a functional assigning to each universe M a set $\mathcal{S}(M)$ of fuzzy measure spaces defined on M. An *unlimited* **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$ is an unlimited **L**-fuzzy quantifier of the type $\langle 1 \rangle$ assigning an **L**-fuzzy quantifier $Q_{\mathcal{S}(M)}$ determined by the fuzzy measure spaces from $\mathcal{S}(M)$ to each *non-empty* universe M.

Example 5.2. Let M be a non-empty universe and $\mathcal{S}(M)_i = \{(M, \mathcal{F}_{\mathbf{L}}(M), \mu_i)\}$, where, for $i = 1, 2$,

$$\mu_1(A) = \begin{cases} \bot, & \text{if } A = 1_\emptyset, \\ \top, & \text{otherwise} \end{cases} \tag{31}$$

and

$$\mu_2(A) = \begin{cases} \top, & \text{if } A = 1_M, \\ \bot, & \text{otherwise.} \end{cases} \tag{32}$$

If Q is determined by fuzzy measures from $Q_{\mathcal{S}(M)_1}$ for all $M \neq \emptyset$, then $Q = \mathsf{some}$. In fact, if $M \neq \emptyset$ and $A \in \mathcal{F}_{\mathbf{L}}(M)$, then

$$Q_M(A) = Q_{\mathcal{S}(M)_1}(A) = \int_{(M,\mathcal{F}_{\mathbf{L}}(M))}^{\otimes} A\, d\mu_1 =$$
$$= \bigvee_{m\in M} \mu_1(\{m\}) \otimes A(m) =$$
$$= \bigvee_{m\in M} \top \otimes A(m) = \bigvee_{m\in M} A(m) = (\mathsf{some})_M(A).$$

According to Definition 4.3, $Q = \mathsf{some}$. One checks easily that Q determined by $Q_{\mathcal{S}(M)_2}$ for all $M \neq \emptyset$ is equal to all.

Example 5.3. Let **L** be the standard Gödel algebra and

$$\mathcal{S}(M) = \{(M, \mathcal{F}_{\mathbf{L}}(M), \mu)\}.$$

One checks easily, using Theorem 3.1, that

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y\in\mathcal{P}_{\mathbf{L}}(M)\setminus\{1_\emptyset\}} \Big(\mu(1_Y) \wedge \bigwedge_{m\in Y} A(m)\Big),$$

because $\otimes$ coincides with $\wedge$ in Gödel algebras. Notice that because **L** is the Gödel algebra, it is also a Heyting algebra and, according to Theorem 3.4, the $\otimes$-fuzzy integral coincides with the Sugeno integral in this case.

Let M be a non-empty countable universe and μ_f denote one of the fuzzy measures on $(M, \mathcal{F}_{\mathbf{L}}(M))$ defined by (6) and (7) in Example 3.1. Putting

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y\in\mathcal{P}_{\mathbf{L}}(M)\setminus\{1_\emptyset\}} \Big(\mu_f(1_Y) \wedge \bigwedge_{m\in Y} A(m)\Big)$$

for any non-empty countable universe M and $Q_\emptyset(1_\emptyset) = \top$, we obtain a countable **L**-fuzzy quantifier which is an interpretation of the quantifier *many things*. Define

$$\mu_f^{1/2}(A) = \begin{cases} \top, & \text{if } \mu_f(A) \geq \frac{1}{2}, \\ \bot, & \text{otherwise,} \end{cases}$$

for any $A \in \mathcal{F}_{\mathbf{L}}(M)$. Then putting

$$Q_{\mathcal{S}(M)}(A) = \bigvee_{1_Y\in\mathcal{P}_{\mathbf{L}}(M)\setminus\{1_\emptyset\}} \Big(\mu_f^{1/2}(1_Y) \wedge \bigwedge_{m\in Y} A(m)\Big)$$

for any non-empty countable universe M and $Q_\emptyset(1_\emptyset) = \top$, we obtain a countable **L**-fuzzy quantifier which is an interpretation of the quantifier *at least half things*. If we restrict ourselves to the class of all finite **L**-fuzzy quantifiers, then one checks easily (using the equality $\mu_f(A) = \mu_f(h^{\rightarrow}(A))$ from Example 3.1) that both defined quantifiers are PI and ISOM. Moreover, they are EXTENS($\approx^\wedge$) (see Theorem 5.6).

Intuitively, it is obvious that all is the smallest and some the greatest **L**-fuzzy quantifier Q determined by fuzzy measures with respect to the ordering from Definition 4.3.

Theorem 5.1. *For each* **L**-*fuzzy quantifier* Q *determined by fuzzy measures over* $\mathcal{S}$*, it holds that*

$$\mathsf{all} \leq Q \leq \mathsf{some}. \tag{33}$$

In the following part, we will present some results on the semantic properties of **L**-fuzzy quantifiers determined by fuzzy measures. The following theorem states a sufficient condition for **L**-fuzzy quantifiers to be permutation invariant.

Theorem 5.2. *Let* Q *be an unlimited* **L**-*fuzzy quantifier of the type* $\langle 1 \rangle$ *determined by fuzzy measures over* $\mathcal{S}$ *such that for each non-empty universe* M *it holds that* $\mathcal{S}(M) = [(M, \mathcal{M}, \mu)]$*. Then* $Q \in$ PI.

Note that the specification of a necessary condition for **L**-fuzzy quantifiers being permutation invariant seems to be immensely complicated and it is still an open problem. In the following theorem, let us denote fuzzy measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ by $\mathbf{M}$ and $\mathbf{M}'$, respectively.

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Theorem 5.3. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$ such that, for any universes M, M' with the same cardinality,*

(i) if $\mathbf{M} \in \mathcal{S}(M)$ and $f : M \to M'$ is a bijection, then $f^{\to}(\mathbf{M}) \in \mathcal{S}(M')$,

(ii) if $\mathbf{M} \in \mathcal{S}(M)$ and $\mathbf{M}' \in \mathcal{S}(M')$, then $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic.

Then $Q \in$ ISOM.

From the definition of $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1\rangle$ determined by fuzzy measures it should be clear that they possess non-decreasing behavior, which is expressed by the following theorem. Naturally, if we base $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1\rangle$ on complementary fuzzy measures and $\to$-fuzzy integrals [21], they will be non-increasing.

Theorem 5.4. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$. Then Q is a non-decreasing $\mathbf{L}$-fuzzy quantifier.*

The following theorem shows that some is the only type $\langle 1\rangle$ quantifier with the extension (EXT) property which can be successfully modeled by our $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1\rangle$ determined by fuzzy measures. However, the quantifier all and quantifiers which are interesting from the point of view of fuzzy logic, for example “at least half things”, “many things”, “most things” do not possess the extension property, and we can model them, see Examples 5.2 and 5.3. Nevertheless, quantifiers which refer to absolute cardinalities, e.g. “at least three things”, possess the extension property, therefore they cannot be successfully modeled by our quantifiers.

Theorem 5.5. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$. Then $Q \in$ EXT if and only if $Q \equiv$ some.*

Finally, we show that fuzzy quantifiers of type $\langle 1\rangle$ determined by fuzzy measures are behaving well with respect to extensionality (see Definition 4.10).

Theorem 5.6. *Let Q be an unlimited $\mathbf{L}$-fuzzy quantifier of the type $\langle 1\rangle$ determined by fuzzy measures over $\mathcal{S}$ such that $\mathcal{S}(M) = [(M, \mathcal{M}, \mu)]$ for each non-empty universe M. Then $Q \in$ EXTENS($\approx^{\wedge}$).*

6 Conclusion

In the future we will concentrate on studying of $\mathbf{L}$-fuzzy quantifiers of the type $\langle 1, 1\rangle$ (and possibly also of the type $\langle 1^n, 1\rangle$) generated by fuzzy measures. Quantifiers of the type $\langle 1, 1\rangle$ serve as models of very important class of natural language determiners (cf. e.g. [3]), for example “a few X are Y”, “almost all X are Y”, etc. Our definitions of $\otimes$-fuzzy integral and $\to$-fuzzy integral allow us to define these quantifiers, and we believe that they provide an important class of models with interesting properties.

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