Valuations on the Algebra of Intervals

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Abstract— *Interval-valued fuzzy sets are based on the algebra of subintervals of the unit interval* $[0,1]$. *We study valuations as a special type of measures on this algebra. We present a description of all valuations which preserve the standard fuzzy negation and extend the identity on the elements of the form* (x,x). *Consequences for sublattices are formulated.*

Keywords— Girard algebra, interval-valued fuzzy set, intuitionistic fuzzy set, measure, valuation

1 Introduction

In order to work with vague quantities, fuzzy sets were suggested [13]; their membership degrees are in the unit interval $I = [0,1]$. Sometimes this approach appeared too restrictive; instead of one value, the membership degree of a point was restricted to an interval in I. This led to the notion of *interval-valued fuzzy sets*. The original term was *intuitionistic fuzzy sets* [1, 2], but it causes misunderstandings because there is no relevance to the intuitionistic logic. Other synonyms found in the literature are *IFS events* [8], etc. The idea of using more general membership degrees is rather old, cf. [4].

Another motivation of interval-valued fuzzy sets is the following: If we determine the membership degree of a point p by an interval $(x,y) \in I^2$, $x \le y$, we say that p *belongs* to the fuzzy set with the degree at least x and *does not belong* to it with the degree at least $z = 1 - y$; the remainder $1-(x+z) = y - x \ge 0$ represents missing knowledge. This approach works with pairs $(x,z) \in I^2$, $x+z \le 1$, representing *bipolar* information. It is distinguished for philosophical, psychological, and methodological reasons; from the mathematical point of view, the descriptions by the pair (x,y) or (x,z) lead to isomorphic models and results can be easily translated from one context to the other.

We refer to [3, 5, 6, 7, 8] for recent investigation of properties of interval-valued fuzzy sets, operations and measures on them.

2 Standard interval algebra

Here we define the standard algebra of all intervals in I. In the sequel, we shall also consider its subalgebras.

We shall work with the set

$$I_2 = \left\{(x,y) \in I^2 \mid x \le y\right\} . \tag{1}$$

Equipped with the usual pointwise ordering,

$$(x_1,y_1) \le (x_2,y_2) \iff (x_1 \le x_2 \text{ and } y_1 \le y_2) , \tag{2}$$

I_2 becomes a lattice with the least element $\mathbf{0} = (0,0)$ and the greatest element $\mathbf{1} = (1,1)$ and the lattice operations

$$(x_1,y_1) \wedge (x_2,y_2) = (\min(x_1,y_1), \min(x_2,y_2)) , \tag{3}$$

$$(x_1,y_1) \vee (x_2,y_2) = (\max(x_1,y_1), \max(x_2,y_2)) . \tag{4}$$

The sublattice

$$D = \{(x,x) \mid x \in I\} \subset I_2 \tag{5}$$

consists of the elements $(x,x) \in I_2$ which represent real membership degrees corresponding to $x \in I$ in standard fuzzy sets. Thus I_2 represents an extension of $D \cong I$ by elements $(x,y) \in I^2$, $x < y$.

The standard fuzzy negation $'\colon I \to I$, $x' = 1 - x$, has a natural extension to the negation $'\colon I_2 \to I_2$,

$$(x,y)' = (y',x') . \tag{6}$$

Thus we have a bounded lattice with an antitone involutive operation $'$. We call this structure the *standard interval algebra* and—with some abuse of notation—we denote it again by I_2.

The standard interval algebra is often considered with additional operations. These are often derived from the *standard MV-algebra* $([0,1], 0, \oplus, ')$, where $'$ is the standard fuzzy negation and $\oplus$ is the Łukasiewicz t-conorm $a \oplus b = \min(1, a+b)$. Its dual operation is the Łukasiewicz t-norm $a \odot b = (a' \oplus b')' = \max(0, a+b-1)$. These operations are applied to elements of I_2 coordinatewise,

$$(x_1,y_1) \odot (x_2,y_2) = (x_1 \odot x_2, y_1 \odot y_2) , \tag{7}$$

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1 \oplus x_2, y_1 \oplus y_2) . \tag{8}$$

Then $D \subset I_2$ becomes isomorphic to the standard MV-algebra.

Remark 1. *Other t-norms and t-conorms on* I_2*, not originated this way, are studied, e.g., in [3]. We do not deal with them here.*

Another line of research is due to S. Weber [10, 11]. He extends the MV-algebra D to a Girard algebra with operations

$$(x_1,y_1) \sqcap (x_2,y_2) = (x_1 \odot x_2, \max((x_1 \odot y_2), (y_1 \odot x_2))) , \tag{9}$$

$$(x_1,y_1) \sqcup (x_2,y_2) = (\min((x_1 \oplus y_2), (y_1 \oplus x_2)), y_1 \oplus y_2) . \tag{10}$$

Although rather strange, these operations arise as a unique solution of well-motivated algebraic conditions [11]. On D, they

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coincide with the Łukasiewicz operations and with (7), (8). The lattice reduct $(I_2, \vee, \wedge)$ of the Girard algebra is the same as in the former approach.

As we shall deal only with the lattice operations, the (standard) negation, and with the Łukasiewicz operations on D, we need not deal with the differences between the above two approaches; we study features which are common to both of them.

3 Measures on the interval algebra

In order to study probability of fuzzy events, we need to introduce (probability) measures on fuzzy sets. First of all, we need measures on the algebras of membership degrees; in our case, on the standard interval algebra I_2. Different conditions are imposed in order to extend the notion of probability from classical sets to fuzzy sets and interval-valued fuzzy sets. Here we follow one of the approaches suggested by S. Weber in [11], but analogous conditions are studied in many other papers. We refer to [12] for a comparison of many approaches to measures on collections of fuzzy sets.

Without referring to a specific definition, let us accept that the standard MV-algebra admits one very natural candidate for a measure—the identity. It commutes with the standard negation and satisfies all forms of additivity conditions (see [9] for details). Thus here we assume that the only measure on the standard MV-algebra is the identity. Also S. Weber in [11] studies measures m which are extensions of this measure on D, i.e.,

$$\forall x \in I : m(x,x) = x\,. \tag{M1+}$$

(For simplicity, we write $m(x,x)$ instead of $m((x,x))$.) For all elements $(x,y) \in I_2$, he imposes the following conditions as minimal requirements on so-called *uncertainty measures*:

$$m(\mathbf{0}) = 0\,, \qquad m(\mathbf{1}) = 1\,, \tag{M1}$$

$$A \le B \implies m(A) \le m(B)\,, \tag{M2}$$

$$m(A') = 1 - m(A)\,. \tag{M3}$$

(Condition (M1) follows from (M1+).) One of the additional conditions studied in [11] is the *valuation property*:

$$m(A \wedge B) + m(A \vee B) = m(A) + m(B)\,. \tag{V}$$

In this paper, we study mappings $m\colon I_2 \to [0,1]$ satisfying (M1+), (M2), (M3), and (V); we call them *valuation measures*.

3.1 *Characterization of valuation measures on the standard interval algebra*

Our principal contribution is the following (μ is a $\frac{1}{2}$*-Lipschitz* function if $|\mu(u) - \mu(v)| \le \frac{1}{2}\,|u - v|$ for all u, v):

Theorem 2. *A mapping $m\colon I_2 \to [0,1]$ is a valuation measure if and only if there is a $\frac{1}{2}$-Lipschitz function μ on $\left[0, \frac{1}{2}\right]$ such that m is of the form*

$$m(x,y) = \frac{x+y}{2} + \mu\left(\left|x - \tfrac{1}{2}\right|\right) - \mu\left(\left|y - \tfrac{1}{2}\right|\right)\,. \tag{12}$$

Proof: First, let us suppose that μ is a $\frac{1}{2}$-Lipschitz function and m is given by (12). If $x = y$, we obtain (M1+). Condition (M3) is verified as follows:

$$\begin{aligned} &m(1-y, 1-x) \\ &= \frac{1-y+1-x}{2} + \mu\left(\left|(1-y) - \tfrac{1}{2}\right|\right) - \mu\left(\left|(1-x) - \tfrac{1}{2}\right|\right) \\ &= 1 - \frac{x+y}{2} - \mu\left(\left|x - \tfrac{1}{2}\right|\right) + \mu\left(\left|y - \tfrac{1}{2}\right|\right) \\ &= 1 - m(x,y)\,. \end{aligned} \tag{13}$$

Monotonicity (M2) can be verified in each coordinate separately and follows from the $\frac{1}{2}$-Lipschitz condition. For $x_1 < x_2$, we obtain

$$\begin{aligned} &m(x_2, y) - m(x_1, y) \\ &= \frac{x_2 - x_1}{2} + \mu\left(\left|x_2 - \tfrac{1}{2}\right|\right) - \mu\left(\left|x_1 - \tfrac{1}{2}\right|\right) \\ &\ge \frac{x_2 - x_1}{2} - \frac{1}{2}\left|\left|x_2 - \tfrac{1}{2}\right| - \left|x_1 - \tfrac{1}{2}\right|\right| \\ &\ge \frac{x_2 - x_1}{2} - \frac{|x_2 - x_1|}{2} = 0\,. \end{aligned} \tag{14}$$

For $y_1 < y_2$, we obtain

$$\begin{aligned} &m(x, y_2) - m(x, y_1) \\ &= \frac{y_2 - y_1}{2} + \mu\left(\left|y_2 - \tfrac{1}{2}\right|\right) - \mu\left(\left|y_1 - \tfrac{1}{2}\right|\right) \\ &\ge \frac{y_2 - y_1}{2} - \frac{1}{2}\left|\left|y_2 - \tfrac{1}{2}\right| - \left|y_1 - \tfrac{1}{2}\right|\right| \\ &\ge \frac{y_2 - y_1}{2} - \frac{|y_2 - y_1|}{2} = 0\,. \end{aligned} \tag{15}$$

To check the valuation property (V), take $A = (x_1, y_1)$, $B = (x_2, y_2)$. Notice that, independently of all possible orderings of x_1, x_2 (resp. y_1, y_2), $m(A \wedge B) + m(A \vee B)$ expressed by (12) contains the same summands as $m(A) + m(B)$:

$$\begin{aligned} &m(A \wedge B) + m(A \vee B) \\ &= \frac{x_1 + x_2 + y_1 + y_2}{2} \\ &+ \mu\left(\left|x_1 - \tfrac{1}{2}\right|\right) + \mu\left(\left|x_2 - \tfrac{1}{2}\right|\right) \\ &- \mu\left(\left|y_1 - \tfrac{1}{2}\right|\right) - \mu\left(\left|y_2 - \tfrac{1}{2}\right|\right) \\ &= m(A) + m(B)\,. \end{aligned} \tag{16}$$

Thus m is a valuation measure.

To prove the reverse implication, let us assume that m is a valuation measure. Whenever $x + y = 1$, the element (x,y) is invariant to the negation and, due to (M3), $m(x,y) = \frac{1}{2}$.

We may define

$$\mu(z) = \tfrac{z}{2} + m\left(\tfrac{1}{2} - z, \tfrac{1}{2}\right)\,, \tag{17}$$

in particular,

$$\mu(0) = m\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{2}\,. \tag{18}$$

For $x \le y = \frac{1}{2}$, the right-hand side of (12) is

$$\begin{aligned} \tfrac{x}{2} + \tfrac{1}{4} + \mu\left(\tfrac{1}{2} - x\right) - \mu(0) &= \tfrac{x}{2} - \tfrac{1}{4} + \mu\left(\tfrac{1}{2} - x\right) \\ &= m\left(x, \tfrac{1}{2}\right) \end{aligned} \tag{19}$$

as desired.

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For $x \leq y < \frac{1}{2}$, we take $A = \left(x, \frac{1}{2}\right)$, $B = (y, y)$. Then $A \wedge B = (x, y)$, $A \vee B = \left(y, \frac{1}{2}\right)$. Using the valuation property, (19), and (M1+), we obtain

$$\begin{aligned} m\,(x, y) &= m(A \wedge B) = m(A) + m(B) - m(A \vee B) \\ &= m\left(x, \tfrac{1}{2}\right) + m\,(y, y) - m\left(y, \tfrac{1}{2}\right) \\ &= \tfrac{x}{2} + \mu\left(\tfrac{1}{2} - x\right) + y - \tfrac{y}{2} - \mu\left(\tfrac{1}{2} - y\right) \\ &= \tfrac{x}{2} + \mu\left(\tfrac{1}{2} - x\right) + \tfrac{y}{2} - \mu\left(\tfrac{1}{2} - y\right) \end{aligned} \tag{20}$$

in accordance with (12).

The case $\frac{1}{2} < x \leq y$ is obtained by duality.

In the remaining case $x \leq \frac{1}{2} < y$, without loss of generality we may suppose that $x + y \leq 1$ (the other case is dual). We take $A = (x, y)$, $B = \left(1 - y, \frac{1}{2}\right)$. Then $A \wedge B = \left(x, \frac{1}{2}\right)$, $A \vee B = (1 - y, y)$. Using the valuation property and (19), we obtain

$$\begin{aligned} m\,(x, y) &= m(A) = m(A \wedge B) + m(A \vee B) - m(B) \\ &= m\left(x, \tfrac{1}{2}\right) + m\,(1 - y, y) - m\left(1 - y, \tfrac{1}{2}\right) \\ &= \tfrac{x}{2} + \mu\left(\tfrac{1}{2} - x\right) + \tfrac{1}{2} - \tfrac{1-y}{2} + \mu\left(\tfrac{1}{2} - (1 - y)\right) \\ &= \tfrac{x}{2} + \mu\left(\tfrac{1}{2} - x\right) + \tfrac{y}{2} - \mu\left(y - \tfrac{1}{2}\right) . \end{aligned} \tag{21}$$

It remains to prove that μ is $\frac{1}{2}$-Lipschitz. Suppose that $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$. One inequality follows from the monotonicity of m:

$$\begin{aligned} \mu(x_2) - \mu(x_1) &= \tfrac{x_2}{2} + m\left(\tfrac{1}{2} - x_2, \tfrac{1}{2}\right) - \tfrac{x_1}{2} - m\left(\tfrac{1}{2} - x_1, \tfrac{1}{2}\right) \\ &\leq \tfrac{x_2 - x_1}{2} . \end{aligned} \tag{22}$$

To prove the second inequality, we take $A = \left(\frac{1}{2} - x_2, \frac{1}{2}\right)$, $B = \left(\frac{1}{2} - x_1, \frac{1}{2} - x_1\right)$. Then $A \wedge B = \left(\frac{1}{2} - x_2, \frac{1}{2} - x_1\right)$, $A \vee B = \left(\frac{1}{2} - x_1, \frac{1}{2}\right)$. We obtain

$$m(A \vee B) - m(A) = m(B) - m(A \wedge B)\,, \tag{23}$$

$$\begin{aligned} & m\left(\tfrac{1}{2} - x_1, \tfrac{1}{2}\right) - m\left(\tfrac{1}{2} - x_2, \tfrac{1}{2}\right) \\ & = m(\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_1) - m\left(\tfrac{1}{2} - x_2, \tfrac{1}{2} - x_1\right) \\ & \leq m(\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_1) - m\left(\tfrac{1}{2} - x_2, \tfrac{1}{2} - x_2\right) = x_2 - x_1 \,, \end{aligned} \tag{24}$$

$$\begin{aligned} & \mu(x_1) - \mu(x_2) \\ & = \tfrac{x_1}{2} + m\left(\tfrac{1}{2} - x_1, \tfrac{1}{2}\right) - \tfrac{x_2}{2} - m\left(\tfrac{1}{2} - x_2, \tfrac{1}{2}\right) \\ & \leq \tfrac{x_1 - x_2}{2} + x_2 - x_1 = \tfrac{x_2 - x_1}{2} . \end{aligned} \tag{25}$$

Corollary 3. *Every valuation measure on I_2 is continuous, even 1-Lipschitz.*

Function μ in (12) is determined by m uniquely up to an additive constant. Thus we may choose, e.g., $\mu(0) = 0$, then μ attains values from $\left[-\frac{1}{4}, \frac{1}{4}\right]$.

Example 4. *For $\mu_0 = 0$, we obtain the valuation measure*

$$m_0(x, y) = \frac{x + y}{2} \tag{26}$$

which is additive with respect to $\oplus, \odot$.

The extreme cases are $\mu_1(z) = \frac{z}{2}$, $\mu_{-1}(z) = -\frac{z}{2}$, which lead to

$$m_1(x, y) = \begin{cases} y & \text{if } x \leq y \leq \frac{1}{2}\,, \\ \frac{1}{2} & \text{if } x \leq \frac{1}{2} < y\,, \\ x & \text{if } \frac{1}{2} < x \leq y\,, \end{cases} \tag{27}$$

$$m_{-1}(x, y) = \begin{cases} x & \text{if } x \leq y \leq \frac{1}{2}\,, \\ x + y - \frac{1}{2} & \text{if } x \leq \frac{1}{2} < y\,, \\ y & \text{if } \frac{1}{2} < x \leq y\,. \end{cases} \tag{28}$$

The choice of function μ is not limited to linear functions in (12).

3.2 Convex structure of the space of valuation measures

In this section, we consider the set $\mathcal{M}$ of all valuation measures on I_2. We consider $\mathcal{M}$ equipped with the product topology inherited from $[0, 1]^{I_2} \supseteq \mathcal{M}$. As the defining conditions of valuation measures can be written in the form of equations using only continuous operations, the set $\mathcal{M}$ is closed. It can be easily checked that a convex combination of valuation measures is a valuation measure; thus $\mathcal{M}$ is also convex. Valuation measures μ_1, μ_{-1} of Example 4 are (some of) its extreme points.

Moreover, convex combinations are preserved by (12) in the following sense:

Proposition 5. *Let m_a, resp. m_b, be a valuation measure on I_2 and μ_a, resp. μ_b, the corresponding $\frac{1}{2}$-Lipschitz function satisfying (12). Let $\lambda \in [0, 1]$. Then the convex combination $\lambda\, m_a + (1 - \lambda)\, m_b$ is a valuation measure which satisfies (12) for $\mu = \lambda\, \mu_a + (1 - \lambda)\, \mu_b$.*

4 Generalizations

Theorem 2 described valuation measures on the standard interval algebra. This case is important, but very special. It describes all interval-valued fuzzy subsets of a singleton. Here we extend Theorem 2 to more general cases.

4.1 Valuation measures on the set of all interval-valued fuzzy sets

Now we shall extend Theorem 2 to finite products of algebras isomorphic to the standard interval algebra. These products are isomorphic to the collections of all interval-valued fuzzy subset of a finite set.

Let $n \in \mathbb{N}$. We denote by I_2^n the product of n standard interval algebras, i.e., the cartesian product $\prod_{i=1}^n I_2$ with the lattice operations and negation applied to each coordinate separately.

Theorem 6. *Let $n \in \mathbb{N}$. A mapping $m\colon I_2^n \to [0, 1]$ is a valuation measure if and only if there are valuation measures $m_1, \ldots, m_n$ on I_2 and real coefficients $c_1, \ldots, c_n \in [0, 1]$ such that $\sum_{i=1}^n c_i = 1$ and m is of the form*

$$m(A_1, \ldots, A_n) = \sum_{i=1}^{n} c_i\, m_i(A_i)\,. \tag{29}$$

Proof: Suppose first that m is of the form (29), i.e., a convex combination of valuation measures $m_1, \ldots, m_n$ on I_2. For each $i = 1, \ldots, n$, we denote by π_i the projection on the ith coordinate,

$$\pi_i(A_1, \ldots, A_n) = A_i\,. \tag{30}$$

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It is a lattice homomorphism of I_2^n onto I_2. Thus the composition $m_i \circ \pi_i : I_2^n \to [0,1]$,

$$(m_i \circ \pi_i)(A_1, \ldots, A_n) = m_i(\pi_i(A_1, \ldots, A_n)) = m_i(A_i), \tag{31}$$

is a valuation measure on I_2^n. As a convex combination of valuation measures, m is also a valuation measure.

Second, suppose that m is a valuation measure on I_2^n. Let $\mathcal{B}$ be the collection of all sharp (=crisp) elements of I_2^n, i.e., all n-tuplets $(A_1, \ldots, A_n)$ such that $A_i \in \{\mathbf{0}, \mathbf{1}\}$ for all $i = 1, \ldots, n$. Then $\mathcal{B}$ is a finite Boolean algebra with n atoms $(\delta_{i1}, \ldots, \delta_{in})$, $i = 1, \ldots, n$, where

$$\delta_{ik} = \begin{cases} \mathbf{1} & \text{if } i = k, \\ \mathbf{0} & \text{if } i \neq k. \end{cases} \tag{32}$$

This means that $(\delta_{i1}, \ldots, \delta_{in}) = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) \in I_2^n$, where the unit is in the ith coordinate. A valuation measure on a Boolean algebra is a classical probability measure. Thus m it is of the form

$$m(A_1, \ldots, A_n) = \sum_{i=1}^{n} c_i \, s_i(A_1, \ldots, A_n), \tag{33}$$

where $c_i = m(\delta_{i1}, \ldots, \delta_{in})$, $i = 1, \ldots, n$, and $s_1, \ldots, s_n$ are valuation measures on I_2^n such that

$$s_i(\delta_{i1}, \ldots, \delta_{in}) = 1. \tag{34}$$

As

$$s_i((\delta_{i1}, \ldots, \delta_{in})') = s_i(\delta'_{i1}, \ldots, \delta'_{in}) = 0, \tag{35}$$

we obtain

$$s_i(B_1, \ldots, B_n) = 0 \tag{36}$$

whenever

$$(B_1, \ldots, B_n) \leq (\delta_{i1}, \ldots, \delta_{in})'. \tag{37}$$

Thus $s_i(A_1, \ldots, A_n)$ depends only on the ith coordinate, $A_i = \pi_i(A_1, \ldots, A_n)$. We may express s_i in the form $s_i = m_i \circ \pi_i$ for some mapping $m_i : I_2 \to [0,1]$. The image of I_2^n under π_i is isomorphic to I_2. As π_i is a homomorphism, m_i must be a valuation measure on I_2. The proof is finished.

The characterization can be extended also to infinite products of the standard interval algebra, thus to collections of interval-valued fuzzy subsets of an infinite universe. This requires valuation measures concentrated not only in coordinates, but also in the points of the underlying Stone space of the Boolean algebra in question. This way, we may also admit subalgebras of products, so that the underlying Boolean algebra is not the whole power set. We leave these extensions for future research.

Theorems 2 and 6 can be combined. We obtain a characterization of valuation measures on the product I_2^n using $\frac{1}{2}$-Lipschitz functions μ_i, $i = 1, \ldots, n$. These functions can be different in different coordinates.

4.2 *Extension to subalgebras*

S. Weber in [11] investigates valuation measures on finite Girard algebras, mainly on finite subalgebras of I_2 of the form

$$\tilde{\mathbb{C}}_n = \left\{ \left(\tfrac{j}{n}, \tfrac{k}{n}\right) \middle| j, k \in \mathbb{Z},\ 0 \leq j \leq k \leq n \right\}, \tag{38}$$

where $n \geq 1$ is a fixed integer. Valuation measures on I_2, when restricted to $\tilde{\mathbb{C}}_n$, become valuation measures on $\tilde{\mathbb{C}}_n$. Conversely, each valuation measure on $\tilde{\mathbb{C}}_n$ admits an extension to a valuation measure on I_2. This holds also for other sublattices of I_2:

Theorem 7. *Let L be a sublattice of I_2. Suppose that $(0,0), (1,1) \in L$ and $(x,y) \in L$ implies (x,x), (y,y), $(y',x') \in L$. A mapping $m : L \to [0,1]$ is a valuation measure if and only if it satisfies the condition of Theorem 2.*

The proof follows the pattern of Theorem 2. Analogously, we can also generalize Theorem 6 and obtain a characterization of valuation measures on subalgebras of I_2^n. These subalgebras are isomorphic to general collections of (not all possible) interval-valued fuzzy sets. For this, it is necessary to add assumptions on closedness of these collections under the lattice operations. We leave further details to future research.

5 Conclusions

We studied measures on finite products of the algebra of subintervals of $[0,1]$. We gave a characterization of measures which (besides the obvious properties of monotonicity and boundary conditions)

1. are compatible with the negation,
2. extend the identity on elements of the form (x,x), $x \in [0,1]$,
3. are valuations.

This is aimed as the first step towards the characterization of such measures on collections of interval-valued fuzzy sets.

Our conditions do not represent the only choice, rather a necessary minimum. Other conditions could be added, depending on the application domain. Our general characterization forms a basis for description of measures restricted by additional conditions, too.

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