On the completion mechanism produced by the Choquet integral on some decision strategies

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Abstract— Preference modeling consists in constructing a preference relation from initial preferences given by a decision maker. We are interested in the preference relation obtained from the use of the Choquet integral. The necessity preference is constructed as the intersection of all preference relations corresponding to a Choquet integral which are compatible with the initial preferences of the decision maker. We study some properties of this necessity relation on some classes of initial preferences. This gives some properties on the completion or extension ability of the Choquet integral.

Keywords— Choquet integral, capacity, initial preferences.

1 Introduction

Multi-Criteria Decision Aid aims at representing the preferences of a Decision Maker (DM) over some options described by several points of view or attributes. We are interested in this paper in the case where the preference relation can be described by an overall utility in which the aggregation of the attributes is obtained with the Choquet integral. The preference model is then thoroughly constructed once the parameters of the Choquet integral - namely the fuzzy measure - are determined.

To this end, the facilitator asks the DM to provide some preferential information. The most broadly used type of preferential information is when the DM says that an option is preferred to another one. The major difficulty the facilitator faces is that there usually does not exist one single fuzzy measure that fulfils this preferential information. Most of the elicitation methods based on the Choquet integral consist in selecting a fuzzy measure (fulfilling the preferential information) that maximizes some functional, which may be an entropy for instance [1, 2, 3]. This is not quite satisfactory for the DM since he does not usually understand what maximizing the functional really means. The use of a maximization problem introduces some additional information that does not come from the DM.

The facilitator shall rather stick strictly to what the DM says and add no further information. One then looks for a robust way to recommend some comparisons among the options from the preferential information [4]. The concept of *necessity preference relation* has been recently introduced for robust quantitative multi-criteria decision models [5, 6]. It has been applied to the Choquet integral in [7]. An option is *necessarily* preferred to another option according to the necessity preference relation, if the first option is preferred to the second one according to all models that fulfill the preferential information provided by the DM. This necessity preference relation is usually incomplete, unless the model is completely specified from the preferential information of the DM. The necessity preference relation is cautious in the sense that its outcomes cannot be contradicted.

The goal of this paper is to study the necessity preference relation for several examples of typical preferential information, in the case where the underlying model is the Choquet integral. In particular, we want to analyze the strength of the completion mechanism obtained from the Choquet integral. This will give complementary insights (compare to the axiomatic characterizations) on the ability of the Choquet integral to represent DM preferences. For each preferential information, the layout is the following:

- Statement of the preferential information and precise description in terms of binary relations.
- Equivalence conditions on the model. We determine the necessary and sufficient conditions on the parameters of the model for this latter to satisfy the preferential information.
- Description of the necessity preference relation derived from the preferential information.

Section 2 presents a more formal description of the problem. The Choquet integral is defined in Section 3. Section 4 describes the process elicitation of the preferential information. Three decision strategies are then studied: the case of empty preference information (see Section 5), the relative importance (see Section 6) and the veto (see Section 7).

2 Description of the problem

Multi-Criteria Decision Aid aims at modeling the preferences of a Decision Maker (DM) over alternatives described by several points of view $N = \{1, \ldots, n\}$. The points of view to be taken into account in the decision making process are denoted by $X_1, \ldots, X_n$. An alternative is characterized by a value w.r.t. each point of view

ISBN: 978-989-95079-6-8

and is thus identified to a point in the Cartesian product X of the points of view: $X = X_1 \times \cdots \times X_n$.

Preference modeling aims at helping a DM to compare the options of a set $\Xi \subseteq X$ containing the alternatives of interest for him. We denote by $\succsim$ the preference relation over X that we wish to define. We are interested in the case where $\succsim$ results from an algebraic utility model. According to this model, the preference relation $\succsim$ can be represented by an overall utility $U : X \to \mathbb{R}$. Hence for all $x, y \in X$, we have

$$x \succsim y \iff U(x) \geq U(y).$$

A standard representation of U is the so-called transitive decomposable model $U(x) = F(u(x))$, where $u_i : X_i \to \mathbb{R}$ is the utility function on attribute $i \in N$, $u(x) = (u_1(x_1), \ldots, u_n(x_n))$, and $F : \mathbb{R}^N \to \mathbb{R}$ is the aggregation function. We assume here that u is known so that the options can be described by a vector in $\mathbb{R}^n$. Hence $X = \mathbb{R}^n$ and the utility functions are the identity function. The aggregation function is characterized by some parameter vector w that need to be determined. Let $\mathcal{W}$ the set of values of the parameters. The preference relation obtained from the parameter vector w is denoted by $\succsim_w$. We denote by $\succ_w$ and $\sim_w$ the asymmetric and symmetric parts of $\succsim_w$.

Any binary relation on $\mathbb{R}^n$ can be identified to a subset of $(\mathbb{R}^n)^2$ composed of the pairs (a, b) such that a is in relation with b.

The fact that the DM needs some help to compare the options in Ξ, means that the comparison of at least some of the options in Ξ is complex. Some preferential information, containing the preferences of the DM over several options in $X \backslash \Xi$, is asked to the DM. This preferential information is encoded by three partial orders $\trianglerighteq, \triangleright, \equiv$ over X. For $x, y \in X$, relation $x \trianglerighteq y$ means that the DM finds x at least as good as y, $x \triangleright y$ means that the DM finds x strictly better than y, and $x \equiv y$ means that the DM finds x similar to y. We set $\Box = (\trianglerighteq, \triangleright, \equiv)$. In order to deduce a comparison of the elements of Ξ from $\Box$, one necessarily needs some completion mechanism. We want to construct three binary relations $\succsim$, $\succ$ and $\sim$ on X that extend $\Box$. The strength of the completion is measured through the differences $\succsim \backslash \trianglerighteq$, $\succ \backslash \triangleright$ and $\sim \backslash \equiv$ compare to $\trianglerighteq, \triangleright, \equiv$ respectively. For a very weak completion, we have $\succsim \backslash \trianglerighteq = \emptyset$, $\succ \backslash \triangleright = \emptyset$ and $\sim \backslash \equiv = \emptyset$.

Let $\mathcal{W}(\Box)$ be the set of parameter vectors w for which $x \succsim_w y$ whenever $x \trianglerighteq y$, $x \succ_w y$ whenever $x \triangleright y$, and $x \sim_w y$ whenever $x \equiv y$. The completion is necessarily based on the family $\{\succsim_w\}_{w \in \mathcal{W}(\Box)}$. A classical completion is the preference relation $\succsim_w$ corresponding to a particular $w \in \mathcal{W}(\Box)$ solution to an optimization problem [1, 2, 3]

$$w = \text{argmax}_{w \in \mathcal{W}(\Box)} G_\Box(w) \tag{1}$$

where $G_\Box$ is the function depending on $\Box$ to be maximized. As said in the introduction, we are interested in the *necessity* preference relation $\succsim^{\mathcal{W}}_{\mathrm{N},\Box}$ that is constructed from $\Box$ as follows [7, 5, 6]:

$$x \succsim^{\mathcal{W}}_{\mathrm{N},\Box} y \iff [\forall w \in \mathcal{W}(\Box)\ \ x \succsim_w y]\,. \tag{2}$$

One may also define $\succ^{\mathcal{W}}_{\mathrm{N},\Box}$ and $\sim^{\mathcal{W}}_{\mathrm{N},\Box}$ by

$$x \succ^{\mathcal{W}}_{\mathrm{N},\Box} y \iff [\forall w \in \mathcal{W}(\Box)\ \ x \succ_w y]$$
$$x \sim^{\mathcal{W}}_{\mathrm{N},\Box} y \iff [\forall w \in \mathcal{W}(\Box)\ \ x \sim_w y]\,.$$

Note that one may have $\succsim^{\mathcal{W}}_{\mathrm{N},\Box} \neq \succ^{\mathcal{W}}_{\mathrm{N},\trianglerighteq} \cup \sim^{\mathcal{W}}_{\mathrm{N},\Box}$.

Another interesting order relation derived from $\Box$ is the so-called *possibility preference relation* defined as follows [7, 5, 6]:

$$x \succsim^{\mathcal{W}}_{\Pi,\Box} y \iff [\exists w \in \mathcal{W}(\Box)\ \ x \succsim_w y]\,.$$

Let us mention that these concepts of necessary and possible preference representation are originated from earlier works on preference modelling. In the context of decision rules, the necessary and possibility preference relations are the lower and upper approximations of a rough set [8]. Moreover, in qualitative decision models, these concepts correspond to the dominance and ordering queries for the Ceterus Paribus nets [9].

The choice of the representation of $\succsim_w$ (and thus of the family of aggregation functions F) is very important since it characterizes directly the completion mechanism. The choice of the family F results from a compromise: if the completion is too strong, we deduce much more than what the DM expresses and the DM may not agree with the outcome. If the completion is too weak, we basically do not deduce really more than what the DM says. Moreover, the stronger the completion mechanism, the less information $\Box$ needs to be given.

As an example, the completion resulting from a simple model such a weighted sum is stronger than that obtained from a more general model such as the Choquet integral. We will illustrate this in the main part of the paper. Consider the case where F is the usual weighted sum: $F(x) = \sum_{i \in N} w_i\, x_i$ where w_i is the weight of criterion i. Thanks to the additivity properties of this model, one can deduce that $(1, \frac{1}{2}, 0) \succsim^{\mathcal{W}}_{\mathrm{N},\Box} (0, 1, \frac{1}{2})$, from the simplier comparisons $(\frac{1}{2}, \frac{1}{2}, 0) \trianglerighteq (0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, 0) \trianglerighteq (0, \frac{1}{2}, 0)$ provided by the DM. The previous deduction is wrong for the Choquet integral. The weighted sum, which is a very simple aggregation model, implies a very strong completion mechanism.

The necessity preference relation $\succsim^{\mathcal{W}}_{\mathrm{N},\Box}$ corresponds to the comparisons among the options in X that can be made for sure. If the initial preferences $\Box$ of the DM are fixed, the recommendation given by the facilitator shall be based only on $\succsim^{\mathcal{W}}_{\mathrm{N},\Box}$.

Now, if $\succsim^{\mathcal{W}}_{\mathrm{N},\Box}$ is not enough complete, the facilitator will probably try to enrich $\Box$ with new comparisons provided by the DM. The candidate comparisons must be picked up from the possibility preference relation. Assume indeed that $\Box$ is enriched with a new information "x is at least as good as y" such that $x \not\succsim^{\mathcal{W}}_{\Pi,\Box} y$. The enriched preferential information is $\Box' = (\trianglerighteq', \triangleright', \equiv')$ with $\trianglerighteq' = \trianglerighteq \cup \{(x, y)\}$, $\triangleright' = \triangleright$ and $\equiv' = \equiv$. Since $x \not\succsim^{\mathcal{W}}_{\Pi,\Box} y$, we obtain $\mathcal{W}(\Box') = \emptyset$. Hence the new preferential information to add to $\Box$ must belong to $\succsim^{\mathcal{W}}_{\Pi,\Box}$, $\succ^{\mathcal{W}}_{\Pi,\Box}$ or $\sim^{\mathcal{W}}_{\Pi,\Box}$. One clearly has [5, Proposition 4.1]

$$\succsim^{\mathcal{W}}_{\mathrm{N},\Box} \subseteq \succsim^{\mathcal{W}}_{\Pi,\Box}\,.$$

ISBN: 978-989-95079-6-8

The comparisons contained in the necessity preference relation are automatically deduced from $\square$, and do not therefore enrich $\square$. In other words, if $(x,y) \in \succsim^{\mathcal{W}}_{\mathrm{N},\square}$, then $\mathcal{W}(\square') = \mathcal{W}(\square)$, where $\square' = (\trianglerighteq', \triangleright', \equiv')$ with $\trianglerighteq' = \trianglerighteq \cup \{(x,y)\}$, $\triangleright' = \triangleright$ and $\equiv'=\equiv$. This proves that the new relevant preferential information to add to $\square$ actually belongs to

$$\succsim^{\mathcal{W}}_{\Pi,\square} \setminus \succsim^{\mathcal{W}}_{\mathrm{N},\square}, \ \succ^{\mathcal{W}}_{\Pi,\square} \setminus \succ^{\mathcal{W}}_{\mathrm{N},\square} \quad \text{or} \quad \sim^{\mathcal{W}}_{\Pi,\square} \setminus \sim^{\mathcal{W}}_{\mathrm{N},\square}.$$

In this paper, we will not study specifically the possibility preference relation. We will focus only on the necessity preference relation. The reason is that $\succsim^{\mathcal{W}}_{\mathrm{N},\square}$ and $\succsim^{\mathcal{W}}_{\Pi,\square}$ are strongly linked together. More precisely, since $\succsim_w$ is a complete preorder, we have

$$x \succsim^{\mathcal{W}}_{\Pi,\square} y \iff y \not\succ^{\mathcal{W}}_{\mathrm{N},\square} x.$$

In other words, $\succsim^{\mathcal{W}}_{\Pi,\square} = X^2 \setminus (\succ^{\mathcal{W}}_{\mathrm{N},\square})^{-1}$.

3 Notation and the Choquet integral

We hereafter restrict to the case where F is a Choquet integral.

3.1 Choquet integral

A fuzzy measure (also called *capacity*) on a set N of criteria is a set function $\mu : 2^N \to [0,1]$ such that [10]

- $\mu(\emptyset) = 0$, $\mu(N) = 1$,
- $\forall A \subseteq B \subseteq N$, $\mu(A) \le \mu(B)$

Let $\mathcal{M}$ be the set of all fuzzy measures. A fuzzy measure is said to be additive if $\mu(A \cap B) = \mu(A) + \mu(B)$ for every pair (A,B) of disjoint coalitions.

The Choquet integral of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ defined w.r.t. a capacity μ has the following expression [11] :

$$C_\mu(x_1,\ldots,x_n) = \sum_{i=1}^{n} \left(x_{\tau(i)} - x_{\tau(i-1)}\right) \times \mu\left(\{\tau(i),\cdots,\tau(n)\}\right), \quad (3)$$

where τ is a permutation on N such that $x_{\tau(1)} \le x_{\tau(2)} \le \cdots \le x_{\tau(n)}$, and $x_{\tau(0)} := 0$. The Choquet integral has been proved to be able to model both the importance of criteria and the interaction between criteria. One fundamental property of the Choquet integral is that

$$C_\mu(1_A, 0_{N\setminus A}) = \mu(A). \quad (4)$$

3.2 k-additive capacities

We introduce a useful linear transformation of fuzzy measures. The *Möbius transform* m of a fuzzy measure μ is the unique solution of the equation

$$\forall A \subseteq N \quad \mu(A) = \sum_{B \subseteq A} m(B), \quad (5)$$

and is given by:

$$m^\mu(A) := \sum_{B \subseteq A} (-1)^{|A|-|B|} \mu(B). \quad (6)$$

A fuzzy measure is defined by 2^n coefficients, which is much more than a weighted sum. The concept of k-additive fuzzy measure has a complexity in-between a fuzzy measure and a weighted sum. More precisely, a fuzzy measure μ is said to be k-additive [12] if $m^\mu(A) = 0$ whenever $|A| > k$ and there exists A with $|A| = k$ such that $m^\mu(A) \neq 0$. We denote by $\mathcal{M}^k$ the set of fuzzy measures that are at most k-additive (i.e. 1, or 2, or ..., or k additive). Note that $\mathcal{M} = \mathcal{M}^n$ and that $\mathcal{M}^1$ are the additive fuzzy measures and corresponds to the usual weighted sum. An interesting particular case is when $k = 2$.

3.3 Interpretation

A capacity contains 2^n terms, which makes its interpretation complex for a DM. In order to ease its interpretation, several global indices have been defined. The first index called *importance index* aims at measuring the degree to which a criterion is important for the decision. The importance index of criterion $i \in N$ is defined by [13]:

$$v_i := \sum_{A \subset N\setminus\{i\}} \frac{(n-|A|-1)!|A|!}{n!} [\mu(A \cup \{i\}) - \mu(A)].$$

Note that the definition is originated from Cooperative Game Theory in which it corresponds to a fair share of a common wealth among several players.

The interaction index I_{ij} of the couple of criteria $\{i,j\}$ is defined by [14]

$$I_{ij} := \sum_{A \subset N\setminus\{i,j\}} \frac{(n-|A|-2)!|A|!}{(n-1)!} [\mu(A \cup \{i,j\}) - \mu(A \cup \{i\}) - \mu(A \cup \{j\}) + \mu(A)].$$

The Shapley index v_i can be interpreted as a kind of average value of the contribution of criterion i alone in all coalitions. The interaction index I_{ij} can be interpreted as a kind of average value of the *added value* obtained by putting criterion i and j together. When I_{ij} is positive (resp. negative), the interaction is said to be positive (resp. negative). These two concepts correspond to a mean behavior and are anyhow not so easy to understand for a DM.

3.4 Initial preferential information

We denote by $\mathcal{M}^k(\square)$ the set of capacities in $\mathcal{M}^k$ that satisfy the preferential information $\square = (\trianglerighteq, \triangleright, \equiv)$:

$$\mathcal{M}^k(\square) = \Big\{ \mu \in \mathcal{M}^k , \forall a,b \in \mathbb{R}^n \quad a \trianglerighteq b \Rightarrow C_\mu(a) \ge C_\mu(b) \quad a \triangleright b \Rightarrow C_\mu(a) > C_\mu(b) \quad a \equiv b \Rightarrow C_\mu(a) = C_\mu(b) \Big\}$$

The necessity preference relation on $\mathbb{R}^n$ is a cautious way to take into account the fact that the aggregation model is not uniquely determined from a knowledge of $\square$, and

ISBN: 978-989-95079-6-8

is defined by:

$$\forall a,b \in \mathbb{R}^n \quad , \quad a \succsim_{\mathrm{N},\square}^{\mathcal{M}^k} b \iff [\forall \mu \in \mathcal{M}^k(\square) \quad C_\mu(a) \geq C_\mu(a)] \; .$$

It is easy to see that $\succsim_{\mathrm{N},\square}^{\mathcal{M}^k}$ is usually incomplete, but is transitive and reflexive. One can define $\succ_{\mathrm{N},\square}^{\mathcal{M}^k}$ and $\sim_{\mathrm{N},\square}^{\mathcal{M}^k}$ as in Section 2. One has

$$\sim_{\mathrm{N},\square}^{\mathcal{M}^k} = \succsim_{\mathrm{N},\square}^{\mathcal{M}^k} \cap (\succsim_{\mathrm{N},\square}^{\mathcal{M}^k})^{-1} \quad \text{and} \quad \succ_{\mathrm{N},\square}^{\mathcal{M}^k} \subseteq \succsim_{\mathrm{N},\square}^{\mathcal{M}^k} \setminus \sim_{\mathrm{N},\square}^{\mathcal{M}^k} \; .$$

Let us define finally the incomparability as $I_{\mathrm{N},\square}^{\mathcal{M}^k} = (\mathbb{R}^n)^2 \setminus (\succsim_{\mathrm{N},\square}^{\mathcal{M}^k} \cup (\succsim_{\mathrm{N},\square}^{\mathcal{M}^k})^{-1})$.

4 Elicitation process

The preference model is usually constructed iteratively through a decision aiding process [15]. This process is centered around the interaction between the DM and the decision aiding system. The information that the DM expresses regarding his preferences can be of two types:

- The first type of information can be easily produced and interpreted by the DM. It is usually admitted that the DM can naturally compare some options. In this paper, the only information of this type corresponds to the three binary relations $\square = (\trianglerighteq, \triangleright, \equiv)$. It is called the *Preferential information*.

- The second type of information (called *Decision strategies*) corresponds to abstract information concerning the preference model. It can typically concern concepts such as the mean importance, veto, relative importance,... The DM can understand them but only to some extend, but it is unexpected that he will naturally provide information of this type. If the decision analyst guesses that the DM fulfills to a decision strategy, he needs to convert it to preferential information in order to check the validity of his guess.

At the end, the only learning data that is considered from the DM corresponds to some preferential information $\square$. It is transformed in the *internal representation* of the model. This is the set of parameter vectors fulfilling the preferential information, that is $\mathcal{M}^k(\square)$. It is not understood by the DM. Consistency and completeness is checked in this representation. The necessity preference relation can then be constructed. This relation can be presented to the DM in different ways. It can be directly applied to the set Ξ of options of interest for the DM. It is also possible to interpret the necessity preference relation in terms of decision strategies. The interpretation may concern some indices regarding the mean importance, veto, relative importance, if all the admissible capacities $\mathcal{M}^k(\square)$ share some common properties on one of these concepts.

The elicitation process can thus be seen as a flow diagram between the three concepts and is organized on the steps *a* to *f* in Figure 1:

- *a*: The DM can express some abstract information regarding his decision strategies. He may express for instance that a criterion is more important than another one.

- *b*: The DM can express information in the format of the order relations $\square$. He may say for instance that an option is preferred to another one. If the DM has already given some preferential information (denoted by $\square'$) in a previous step of the elicitation process and he wishes to enrich $\square'$, the new preferential information shall lie in the sets $\succsim_{\Pi,\square'}^{\mathcal{M}^k} \setminus \succsim_{\mathrm{N},\square'}^{\mathcal{M}^k}$, $\succ_{\Pi,\square'}^{\mathcal{M}^k} \setminus \succ_{\mathrm{N},\square'}^{\mathcal{M}^k}$, or $\sim_{\Pi,\square'}^{\mathcal{M}^k} \setminus \sim_{\mathrm{N},\square'}^{\mathcal{M}^k}$.

- *c*: When the DM expresses abstract information (decision strategies), they shall be converted into preferential information through $\square$ in order to be precisely set and taken into account by the elicitation method.

- *d*: The preferential information is transformed into constraints in the parameter set. This corresponds to the set $\mathcal{M}^k(\square)$. This step is sometimes called disaggregation step.

- *e*: An interpretation of the set $\mathcal{M}^k(\square)$ is presented in terms of, for instance, veto, importance, interaction indices. It can also provide the necessity and possibility preference relations. This step is sometimes called aggregation step.

- *f*: An interpretation of the results of the elicitation is displayed to the DM in some understandable way.

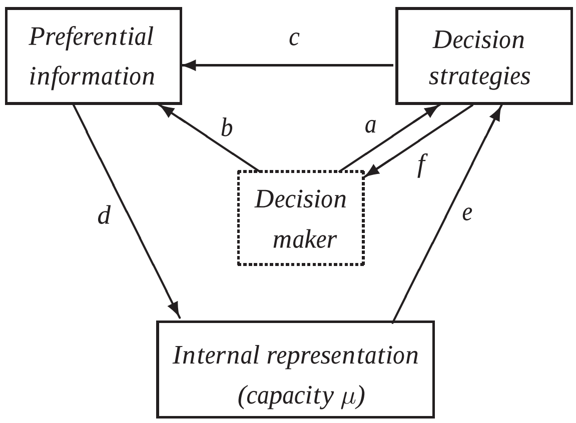

Figure 1: Eliciation process.

In the remaining of this paper, we are interested in some abstract decision strategies that the DM often wishes to express in practice. For each type of preferential information, we determine how it is converted into order relations $\square$ (step *c*). Then, we construct the necessity preference relation corresponding to this preferential information (steps *d* and *e*).

5 Case of empty preference information

We denote by $\square_\emptyset$ the empty preference information. When the DM says nothing, there is of course no new

ISBN: 978-989-95079-6-8

constraint on the fuzzy measure:

$$\mathcal{M}^k(\square_\emptyset) = \mathcal{M}^k.$$

Moreover, it is easy to check that for all $k \in \{1,\dots,n\}$

$$\succsim^{\mathcal{M}^k}_{\mathrm{N},\square_\emptyset} = \mathcal{P}_{\mathrm{a}} := \{(a,b) \in \mathbb{R}^n \ , \ \forall i \in N \ \ a_i \geq b_i\} \, .$$

This relation extends [5, Remark 4.1] to the case of the Choquet integral. This is the standard Pareto dominance order. When no preferential information is given, the necessary preferential information is similar to the Pareto dominance order, which is what one expects.

6 Relative importance between criteria

6.1 Statement of the preferential information

We want to express a preferential information regarding the relative importance between two criteria – namely that criterion p is more important than criterion q. The first idea is to interpret this as the relation $v_p > v_q$ based on the Shapley value. This relation is considered in [1, 7]. However, as explained in Section 3.3, the Shapley value is not a simple concept for a DM. It is not clear at all whether the statement *"criterion p is more important than criterion q"* expressed by a DM can be represented by the relation $v_p > v_q$. We wish only to express the previous statement only as binary relations denoted by $\square^{p,q}_{\mathrm{RelImp}} = (\trianglerighteq^{p,q}_{\mathrm{RelImp}}, \triangleright^{p,q}_{\mathrm{RelImp}}, \equiv^{p,q}_{\mathrm{RelImp}})$. The following definition seems to express quite accurately the previous statement.

Definition 1 ([16]) *A criterion $p \in N$ is said to be* more important *than criterion $q \in N \setminus \{p\}$ iff*

$$\left(\gamma_p, \beta_q, a_{N\setminus\{p,q\}}\right) \triangleright^{p,q}_{\mathrm{RelImp}} \left(\beta_p, \gamma_q, a_{N\setminus\{p,q\}}\right) \qquad (7)$$

for all $\gamma, \beta \in \mathbb{R}$ with $\gamma > \beta$, and all $a_{N\setminus\{p,q\}} \in \mathbb{R}^{n-2}$, where $\left(\alpha_p, \beta_q, a_{N\setminus\{p,q\}}\right)$ denotes the alternative having value α on criterion p, value β on criterion q, and value $a_{N\setminus\{p,q\}}$ on the remaining criteria.

Note that (7) makes sense only if the scales of the criteria are commensurate.

6.2 Conditions on the fuzzy measure

Let $\delta_{p,q}\mu(A) := \mu(A \cup \{p\}) - \mu(A \cup \{q\})$.

Lemma 1 *For all $k \in \{2,\dots,n\}$, criterion p is more important than criterion q with the Choquet integral with a capacity in $\mathcal{M}^k$ iff*

$$\forall A \subseteq N \setminus \{p,q\} \quad \delta_{p,q}\mu(A) > 0. \qquad (8)$$

6.3 Completion

We have the following result.

Proposition 1 *Let $p, q \in N$ with $p \neq q$. Then for all $k \in \{2,\dots,n\}$*

$$\begin{aligned}
&\succsim^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \\
&= \Big\{ \Big((\gamma_p, \beta_q, a_{N\setminus\{p,q\}}), (\beta'_p, \gamma'_q, a'_{N\setminus\{p,q\}}) \Big) \in (\mathbb{R}^n)^2 \ , \\
&\quad \gamma > \beta \ , \ \gamma' > \beta' \ , \gamma' \leq \gamma \ , \ \beta' \leq \beta \text{ and} \\
&\quad \text{for all } i \in N \setminus \{p,q\} \ \ a'_i \leq a_i \Big\} \bigcup \mathcal{P}_{\mathrm{a}}
\end{aligned}$$

$$\begin{aligned}
&\succ^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \\
&= \Big\{ \Big((\gamma_p, \beta_q, a_{N\setminus\{p,q\}}), (\beta'_p, \gamma'_q, a'_{N\setminus\{p,q\}}) \Big) \in (\mathbb{R}^n)^2 \ , \\
&\quad \gamma > \beta \ , \ \gamma' > \beta' \ , \gamma' \leq \gamma \ , \ \beta' \leq \beta \text{ and} \\
&\quad \text{for all } i \in N \setminus \{p,q\} \ \ a'_i \leq a_i \Big\} \bigcup \mathcal{P}^p_{\mathrm{a}\,>}
\end{aligned}$$

$$\sim^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} = \Big\{ (a,a) \ , \ \ a \in \mathbb{R}^n \Big\}$$

where $\mathcal{P}^p_{\mathrm{a}\,>} = \{(a,b) \in \mathcal{P}_{\mathrm{a}} \ , \ a_p > b_p\}$.

When $\gamma > \beta$, $(\gamma_p, \beta_q, a_{N\setminus\{p,q\}})$ is at least as good as $(\beta_p, \gamma_q, a_{N\setminus\{p,q\}})$ from $\square^{p,q}_{\mathrm{RelImp}}$. From the Pareto relation $\mathcal{P}_{\mathrm{a}}$, $(\beta_p, \gamma_q, a_{N\setminus\{p,q\}})$ is at least as good as $(\beta'_p, \gamma'_q, a'_{N\setminus\{p,q\}})$ whenever $\gamma' \leq \gamma$, $\beta' \leq \beta$ and $a'_k \leq a_k$ for all $k \in N \setminus \{p,q\}$. The completion of $\square^{p,q}_{\mathrm{RelImp}}$ by a k-additive Choquet integral (with $k \geq 2$) is just the combination of the definition of $\square^{p,q}_{\mathrm{RelImp}}$ and the Pareto dominance order. Therefore, there is no side effect coming from the Choquet model.

Lemma 2 *Let $p, q \in N$ with $p \neq q$. Then*

$$\begin{aligned}
&\succsim^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \\
&= \Big\{ \Big((\gamma_p, \beta_q, a_{N\setminus\{p,q\}}), (\beta'_p, \gamma'_q, a'_{N\setminus\{p,q\}}) \Big) \in (\mathbb{R}^n)^2 \\
&\quad \gamma \geq \beta' \ , \ \gamma - \beta + \beta' - \gamma' \geq 0 \\
&\quad \text{and for all } i \in N \setminus \{p,q\} \ \ a'_i \leq a_i \Big\} \bigcup \mathcal{P}_{\mathrm{a}}
\end{aligned}$$

$$\begin{aligned}
&\succ^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \\
&= \Big\{ \Big((\gamma_p, \beta_q, a_{N\setminus\{p,q\}}), (\beta'_p, \gamma'_q, a'_{N\setminus\{p,q\}}) \Big) \in (\mathbb{R}^n)^2 \\
&\quad \gamma > \beta' \ , \ \gamma - \beta + \beta' - \gamma' \geq 0 \ , \ \gamma' \leq \gamma \ , \ \beta' \leq \beta \\
&\quad \text{and for all } i \in N \setminus \{p,q\} \ \ a'_i \leq a_i \Big\} \cup \mathcal{P}^p_{\mathrm{a}\,>}
\end{aligned}$$

$$\sim^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} = \Big\{ (a,a) \ , \ \ a \in \mathbb{R}^n \Big\}$$

There is a strange condition $\gamma - \beta + \beta' - \gamma' \geq 0$ in the necessity preference relation of the model $\mathcal{M}^1$.

One has for all $k \in \{2,\dots,n\}$

$$\succsim^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \subset \succsim^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}}$$
$$\succ^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \subset \succ^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}}$$
$$\sim^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \subset \sim^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}}$$

The information $\succsim^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \setminus \succsim^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}}$ and $\succ^{\mathcal{M}^1}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}} \setminus \succ^{\mathcal{M}^k}_{\mathrm{N},\square^{p,q}_{\mathrm{RelImp}}}$ is contained neither in Definition

ISBN: 978-989-95079-6-8

1 nor in the Pareto set $\mathcal{P}_{\rm a}$. This proves that the completion resulting from the Choquet integral fits exactly with the preferential information provided by the DM, whereas the completion resulting from the weighted sum is strong. The Choquet is thus able to represent the relative importance without adding new information not provided by the DM.

7 Veto criteria

7.1 Statement of the preferential information

As in group decision, one can define a veto criterion. The preferential information corresponding to a veto criterion p is denoted by $\square^p_{\rm Veto} = (\trianglerighteq^p_{\rm Veto}, \triangleright^p_{\rm Veto}, \equiv^p_{\rm Veto})$.

Definition 2 *Criterion p is a* veto *criterion if for all $x, y \in \mathbb{R}^n$*

$$(x_{N\setminus\{p\}}, 0_p) \equiv^p_{\rm Veto} (y_{N\setminus\{p\}}, 0_p).$$

It is straightforward to see that the evaluation of the option $(x_{N\setminus\{p\}}, 0_p)$ is zero. This means that a bad score on a veto criterion cannot be saved by good scores on the remaining criteria.

7.2 Conditions on the fuzzy measure

Lemma 3 *Criterion $p \in N$ is a veto iff*

$$\forall S \subset N \setminus \{p\}\ , \quad \mu(S) = 0\ . \qquad (9)$$

7.3 Completion

There is a unique additive fuzzy measure for which p is a veto. It is defined by $\mu(A) = 1$ if $p \in A$ and $= 0$ otherwise. Criterion p is then interpreted as a dictator. All the other criteria are merely discarded. Hence $\succsim^{\mathcal{M}^1}_{{\rm N},\square^p_{\rm Veto}}$ is a complete order.

The following result shows that the necessity preference relation does not depend on k for $k \geq 2$.

Proposition 2 *Let P be the preferential information stating that criterion p is a veto. Then for all $k \in \{2, \ldots, n\}$*

$$\succsim^{\mathcal{M}^k}_{{\rm N},\square^p_{\rm Veto}} = \Big\{(a,b) \in (\mathbb{R}^n)^2\ ,\ a_p \geq b_p \textit{ and } \forall i \in N \setminus \{p\} \textit{ either } a_i \geq b_i \textit{ or } [a_i < b_i \textit{ and } a_i \geq b_p]\Big\}$$

$$\succ^{\mathcal{M}^k}_{{\rm N},\square^p_{\rm Veto}} = \Big\{(a,b) \in (\mathbb{R}^n)^2\ ,\ a_p > b_p \textit{ and } \forall i \in N \setminus \{p\} \textit{ either } a_i > b_i \textit{ or } [a_i \leq b_i \textit{ and } a_i > b_p]\Big\}$$

$$\sim^{\mathcal{M}^k}_{{\rm N},\square^p_{\rm Veto}} = \Big\{(a,b) \in (\mathbb{R}^n)^2\ ,\ a_p = b_p \textit{ and } \forall i \in N \setminus \{p\} \textit{ either } a_i = b_i \textit{ or } [a_i \geq a_p \textit{ and } b_i \geq a_p]\Big\}$$

The condition $a_i < b_i$ and $a_i \geq b_p$ means that the relatively bad score of a on criterion i is hidden by the score of b on the veto criterion p. Hence the necessity preference relation is completely natural. The completion mechanism of the Choquet integral is once more easily interpretable.

8 Conclusion

We have studied the completion mechanism produced by the Choquet integral on three types of preferential information: empty prior information, relative importance and veto. The result is trivial on the empty prior information since one recovers the Pareto ordering. Concerning the two other types of preferential information, we have shown that the Choquet integral does not introduce weird phenomena in the robust completion corresponding to the necessity preference relation. Moreover, the completion does not depend on order k for k-additive fuzzy measures, provided that $k \geq 2$. On the contrary, the weighted sum produced a strong completion.

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ISBN: 978-989-95079-6-8