Fuzzy Implications and the Weak Law of Importation

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Abstract— *Some open problems on fuzzy implications dealing with the so-called importation law are studied and totally or partially solved in this work. In particular, it is proved that such property (in fact a weaker version than the law of importation) is stronger than the exchange principle. Along this study, new characterizations of (S, N)-implications and R-implications involving the law of importation are showed.*

Keywords— Implication function, exchange principle, law of importation, (S, N)-implication, R-implication.

1 Introduction

Fuzzy implications are an essential tool in fuzzy control and approximate reasoning, as well as in many fields where these theories apply. This is because they are used not only to model fuzzy conditionals, but also to make inferences in any fuzzy rule based system (see for instance [1] or [2]) through the modus ponens and modus tollens. Moreover, fuzzy implications are also useful in many other fields like fuzzy relational equations and fuzzy mathematical morphology ([3]), fuzzy DI-subsethood measures and image processing ([4] and [5]), and data mining ([6]). Due to this great quantity of applications many authors have focused their interest in the theoretical study of fuzzy implications. See for instance the recent book [7], exclusively devoted to fuzzy implications, and the references therein.

From this theoretical study some open problems on fuzzy implications have been recently posed in some works about this topic. In this paper we want to deal with some of them involving the so-called law of importation, that is derived from the tautology in classical logic

$$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r) \tag{1}$$

which, translated to fuzzy logic, becomes the functional equation:

$$I(T(x, y), z) = I(x, I(y, z)) \quad \text{for all } x, y, z \in [0, 1] \tag{2}$$

where T is a t-norm and I is a fuzzy implication. This property has been extensively studied in [8] and [9] for many kinds of implications derived from t-norms and t-conorms. Moreover, an extension of (2) involving uninorms instead of t-norms has been also studied in [10]. From this study some open problems for fuzzy implications involving the law of importation have been posed (see [11]) that we will try to solve in next sections. During the process, new characterizations of (S, N)-implications and R-implications involving the law of importation are showed.

2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms, t-conorms and fuzzy negations (all necessary results and notations can be found in [12]). We recall here only some facts on implications.

Definition 1 *A binary operator $I : [0, 1]^2 \rightarrow [0, 1]$ is said to be an* implication function*, or an* implication*, if it satisfies:*

(I1) $I(x, z) \geq I(y, z)$ *when* $x \leq y$*, for all* $z \in [0, 1]$.

(I2) $I(x, y) \leq I(x, z)$ *when* $y \leq z$*, for all* $x \in [0, 1]$.

(I3) $I(0, 0) = I(1, 1) = 1$ *and* $I(1, 0) = 0$.

Note that, from the definition, it follows that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$ whereas the symmetrical values $I(x, 0)$ and $I(1, x)$ are not derived from the definition.

Special interesting properties for implication functions are:

- The *law of importation* with a t-norm T,
$$I(T(x, y), z) = I(x, I(y, z)), \quad \text{for all} \quad x, y, z \in [0, 1]. \tag{LI}$$
- The *exchange principle*,
$$I(x, I(y, z)) = I(y, I(x, z)), \quad \text{for all} \quad x, y, z \in [0, 1]. \tag{EP}$$
- The *ordering property*,
$$x \leq y \Longleftrightarrow I(x, y) = 1, \quad \text{for all} \quad x, y \in [0, 1]. \tag{OP}$$

Among other models (see [3] and [7]), the most used fuzzy implications are derived from t-norms and t-conorms and they are:

- R-implications derived from a left-continuous t-norm T,
$$I_T(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leq y\} \tag{3}$$
for all $x, y \in [0, 1]$.
- (S, N)-implications derived from a t-conorm S and a strong negation[1] N,
$$I_{S,N}(x, y) = S(N(x), y) \qquad \text{for all} \quad x, y \in [0, 1]. \tag{4}$$

[1] This kind of implications has been characterized also when N is strict or continuous, see [13]. However, (LI) has been studied only in the cases when N is strict or strong, but not when N is only continuous.

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But also QL-implications and D-implications respectively given by

$$I_{QL}(x,y) = S(N(x), T(x,y)) \tag{5}$$

for all $x, y \in [0,1]$, and

$$I_D(x,y) = S(T(N(x), N(y)), y) \tag{6}$$

for all $x, y \in [0,1]$, where T is a t-norm, S a t-conorm and N a strong negation. For all these kinds of implications the (LI) is equivalent to the (EP) and moreover, there is one and only one t-norm for which the (LI) holds (see [8] and [9]).

Finally let us recall the characterizations of (S,N) and R-implications. Several of these characterizations can be found in the literature:

Theorem 1 *([13]) For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i). I is an (S,N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N

(ii). I satisfies (I1), (EP) and $I(x,0) = N(x)$ is a continuous (strict, strong) fuzzy negation.

Theorem 2 *([14], [15] or [7]) Let $I : [0,1]^2 \to [0,1]$ be a function. Then I is an R-implication derived from a left-continuous t-norm, if and only if, I satisfies (I2), (OP), (EP) and I is right-continuous with respect to the second variable.*

With the assumption of continuity we have a characterization of the following subclass of R-implications, that are also (S,N)-implications, known as the Smets-Magrez Theorem, see [16] and see also [17] for the current version.

Theorem 3 *Let $I : [0,1]^2 \to [0,1]$ be a function. Then I is a continuous function satisfying (OP), (EP), if and only if, I is conjugate with the Łukasiewicz implication, that is, there exists a unique increasing bijection $\varphi : [0,1] \to [0,1]$ such that*

$$I(x,y) = \varphi^{-1}(\min\{1, 1 - \varphi(x) + \varphi(y)\}) \tag{7}$$

for all $x, y \in [0,1]$.

3 Main results

Due to the commutativity of the t-norm T (or the uninorm), it is clear that the law of importation implies the exchange principle, but in all known cases the converse also holds and the t-norm for which (LI) holds is uniquely determined. The first two open problems (posed in [11], Problem 8.1) that we want to study are just the following:

1. **(OP1)** Does the exchange principle imply that there exists a t-norm such that the law of importation holds? If not, give an example and characterize all fuzzy implications for which the answer is positive.

2. **(OP2)** Is the t-norm in the law of importation uniquely determined?

Note that for R, (S,N), QL and D-implications derived from t-norms and t-conorms the answer of both problems is positive (see [8] and [9]). With respect to (OP1) a negative answer has been done in [10] (see also [7], Section 7.3) by using implications derived from uninorms. The examples given in these works are implications satisfying (EP) such that they do not satisfy (LI) for any t-norm. However, in all these cases there exists a conjunctive uninorm for which the (LI) holds and thus, the problems above are just translated to the following more general versions:

1. **(OP1')** Does the exchange principle imply that there exists a uninorm (or a more general binary operation) such that the law of importation holds? If not, give an example and characterize all fuzzy implications for which the answer is positive.

2. **(OP2')** Is the uninorm in the law of importation uniquely determined?

Moreover, the answer of both problems is again positive for R, S, QL and D-implications derived from uninorms (see [10]).

Along the paper, we will prove that (OP1') has a negative answer in general and we will characterize some particular cases for which the answer is positive. On the other hand, with respect to (OP2) some counterexamples have been proved in [7] for implications I for which $I(x,0)$ equals the greatest or the least fuzzy negation (the same examples work for (OP2')). We will discuss however the case when $I(x,0)$ is at least a continuous fuzzy negation.

3.1 The Weak Law of Importation (WLI)

To deal with these problems let us introduce a weaker law of importation than (LI), by reducing the requirements on the function T. To maintain its relation with the exchange principle we need to take T commutative, and due to the monotonicity of implication functions it seems also adequate to take T nondecreasing, but no more conditions are necessary. Thus we introduce the following

Definition 2 *Let $I : [0,1]^2 \to [0,1]$ be a function. We say that I satisfies the* weak law of importation *if there exists a commutative and nondecreasing function $F : [0,1]^2 \to [0,1]$ such that*

$$I(F(x,y), z) = I(x, I(y,z)) \quad \textit{for all } x,y,z \in [0,1] \tag{\textbf{WLI}}$$

and then we say that I satisfies (WLI) with the function F.

First of all, note that (WLI) clearly implies (EP).

On the other hand, when I is an (S,N), a QL or a D-implication it satisfies the following boundary property

$$I(x,0) = N(x), \quad \text{for all} \quad x \in [0,1] \tag{\textbf{BP}}$$

where N is a (continuous, strict, strong) fuzzy negation.

We want to study the properties of those functions $I : [0,1]^2 \to [0,1]$ that satisfy (WLI) and (BP). These properties will be useful in order to find a counterexample to the equivalence of (EP) and (LI).

Proposition 1 *Let $I : [0,1]^2 \to [0,1]$ be a function which satisfies (WLI) with a commutative and nondecreasing function F and (BP) with a fuzzy negation N. Then*

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- *I satisfies (I3).*
- *I satisfies right contrapositive symmetry, that is,*

$$I(x, N(y)) = I(y, N(x)) \text{ for all } x, y \in [0, 1]. \quad (8)$$

Moreover, if N is continuous, then

- *I satisfies (I1) and (I2), and consequently I is a fuzzy implication.*
- *I satisfies left neutrality principle, that is,*

$$I(1, y) = y \text{ for all } y \in [0, 1]. \quad (9)$$

Finally, if N is strong then

- *I satisfies contrapositive symmetry, that is,*

$$I(N(y), N(x)) = I(x, y) \text{ for all } x, y \in [0, 1]. \quad (10)$$

Proof: I satisfies (EP) due to the commutativity of F and (WLI). Furthermore, I satisfies also (I3) because

$$\begin{aligned} I(0,0) &= N(0) = 1 \\ I(1,0) &= N(1) = 0 \\ I(1,1) &= I(1, I(0,0)) = I(0, I(1,0)) = I(0,0) = 1. \end{aligned}$$

In addition, I satisfies right contrapositive symmetry because

$$I(x, N(y)) = I(x, I(y, 0)) = I(y, I(x, 0)) = I(y, N(x)).$$

From now on, N will be a continuous negation. Let us prove (I1). If $x \leq y$, using the continuity of N, we only need to show that $I(x, N(z)) \geq I(y, N(z))$ for all $z \in [0, 1]$.

$$\begin{aligned} I(x, N(z)) &= I(x, I(z,0)) = I(F(x,z),0) = N(F(x,z)) \\ &\geq N(F(y,z)) = I(F(y,z),0) = I(y, I(z,0)) \\ &= I(y, N(z)) \end{aligned}$$

A similar argument shows (I2). If $y \leq z$, using the continuity of N, we will show that $I(x, N(y)) \geq I(x, N(z))$ for all $x, y, z \in [0, 1]$.

$$\begin{aligned} I(x, N(y)) &= I(x, I(y,0)) = I(F(x,y),0) = N(F(x,y)) \\ &\geq N(F(x,z)) = I(F(x,z),0) = I(x, I(z,0)) \\ &= I(x, N(z)) \end{aligned}$$

Furthermore, I satisfies the left neutrality principle

$$\begin{aligned} I(1, N(y)) &= I(1, I(y,0)) = I(y, I(1,0)) \\ &= I(y, 0) = N(y) \end{aligned}$$

Finally, if N is strong, we have

$$\begin{aligned} I(N(y), N(x)) &= I(N(y), I(x,0)) = I(x, I(N(y),0)) \\ &= I(x, N(N(y))) = I(x, y). \end{aligned}$$

∎

It is known that in general, (EP) and (BP) with a continuous (even strong) fuzzy negation N do not imply (I1). For example, the function

$$I(x, y) = \begin{cases} 1 - x & \text{if } y = 0 \\ y & \text{if } x = 1 \\ 0.5 & \text{otherwise} \end{cases} \quad (11)$$

satisfies (EP) and (BP) with $N(x) = 1 - x$ the classical strong negation, but it does not satisfy the nonincreasingness in the first variable, (I1) (see [13]). Consequently, I can not satisfy (WLI), by the previous proposition. So, we have just found a counterexample proving that, for functions $I : [0, 1]^2 \to [0, 1]$ in general, (EP) does not imply (WLI). In section 3.3 we will prove that a counterexample for fuzzy implications is also available.

3.2 (WLI) and (S, N)-implications

The main target of this section is the study of (WLI) on (S, N)-implications. We will show a new characterization of (S, N)-implications based on (WLI) and we will partially solve (OP2).

Proposition 2 *Let $I : [0, 1]^2 \to [0, 1]$ be a binary operator satisfying (WLI) with a commutative and nondecreasing function F and (BP) with a continuous (strict, strong) fuzzy negation N. Then I is an (S, N)-implication generated from a t-conorm S and the negation N.*

Proof: Due to proposition 1, it follows that I satisfies (I1) and (EP). So, by using theorem 1, I is an (S, N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N. ∎

After that let us take a look to the verification of (WLI) on (S, N)-implications.

Proposition 3 *An (S, N)-implication I generated from a t-conorm S and a strict (strong) negation N satisfies (WLI) with a function $F : [0, 1]^2 \to [0, 1]$ if and only if F is the N-dual t-norm of S, that is*

$$F(x, y) = N^{-1}(S(N(x), N(y))) \text{ for all } x, y \in [0, 1]. \quad (12)$$

Proof: Let I be an (S, N)-implication generated from a t-conorm S and a strict negation N. If I satisfies (WLI) with a function F, then for all $x, y, z \in [0, 1]$ we have

$$\begin{aligned} I(F(x,y), z) &= S(N(F(x,y)), z) \\ I(x, I(y,z)) &= S(N(x), I(y,z)) = S(N(x), S(N(y), z)) \end{aligned}$$

Taking $z = 0$ in both expressions and using (WLI), we obtain $N(F(x, y)) = S(N(x), N(y))$, and consequently, $F(x, y) = N^{-1}(S(N(x), N(y)))$ for all $x, y \in [0, 1]$. Conversely, if F is the N-dual t-norm of S, a straightforward computation proves that I satisfies (WLI) with F. ∎

Remark 1 *Proposition 3 is the counterpart for (WLI) of an analogous result for (LI) in [8] and [13]. Note that none property of F is used in the proof. Thus, for (S, N)-implications with N strict or strong (LI) and (WLI) are equivalent and in fact, the N-dual t-norm of S is the only function $F : [0, 1]^2 \to [0, 1]$ (without any further assumption) for which (WLI) is satisfied.*

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We will see that the equivalence in the previous remark also holds for (S,N)-implications with N continuous (not necessarily strict), but something different occurs in this case. We will see that they also satisfy (WLI), but the function F for which they satisfy (WLI) needs not to be unique. However, among the functions F for which the (S,N)-implication satisfies (WLI), there is at least one of them that is a t-norm.

To do this, let us consider the following definition.

Definition 3 *(see [13]) Given any continuous fuzzy negation N, we consider the function $\mathfrak{R}_N : [0,1] \to [0,1]$ defined by*

$$\mathfrak{R}_N(x) = \begin{cases} N^{(-1)}(x) & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases} \tag{13}$$

where $N^{(-1)}$ stands for the pseudo-inverse of N given by

$$N^{(-1)}(x) = \sup\{z \in [0,1] \mid N(z) > x\} \quad \text{for all } x \in [0,1]. \tag{14}$$

In [13], it is proved that $\mathfrak{R}_N$ is a strictly decreasing negation, $\mathfrak{R}_N^{(-1)} = N$, $N \circ \mathfrak{R}_N = id_{[0,1]}$ and

$$\mathfrak{R}_N \circ N|_{\text{Ran}(\mathfrak{R}_N)} = id|_{\text{Ran}(\mathfrak{R}_N)},$$

where Ran$(\mathfrak{R}_N)$ stands for the range of function $\mathfrak{R}_N$. This function $\mathfrak{R}_N$ plays the role of the "inverse" function of N and therefore, we have the following result:

Proposition 4 *An (S,N)-implication I generated from a t-conorm S and a continuous fuzzy negation N satisfies (WLI) with the function $F(x,y) = \mathfrak{R}_N(S(N(x),N(y)))$ for all $x,y \in [0,1]$.*

Proof: It is clear from its definition that F is commutative and nondecreasing. Now,

$$\begin{aligned} I(T(x,y),z) &= S(N(T(x,y)),z) \\ &= S(N(\mathfrak{R}_N(S(N(x),N(y)))),z) \\ (N \circ \mathfrak{R} = id_{[0,1]}) \quad &= S(S(N(x),N(y)),z) \\ &= S(N(x),S(N(y),z)) \\ &= S(N(x),I(y,z)) = I(x,I(y,z)) \end{aligned}$$

■

Contrary to what happens in the case when N is strict (or strong), the function F in the proposition above is not the only one for which the (S,N)-implication satisfies (WLI). Note that the key fact in the proof of the previous proposition is that $N \circ \mathfrak{R}_N = id_{[0,1]}$ (in fact it is only necessary that $N(F(x,y)) = S(N(x),N(y))$ for all $x,y \in [0,1]$). When N is continuous but non-strict there are other functions than $\mathfrak{R}_N$ satisfying this condition, and consequently different functions F for which I satisfies the (WLI). See for instance the following example.

Example 1 *(see [13]) Consider the continuous fuzzy negation N given by*

$$N(x) = \begin{cases} -2x+1 & \text{if } x \in [0,0.25] \\ 0.5 & \text{if } x \in (0.25,0.75) \\ -2x+2 & \text{if } x \in [0.75,1]. \end{cases} \tag{15}$$

An easy calculation shows that

$$\mathfrak{R}_N(x) = \begin{cases} -0.5x+1 & \text{if } x \in [0,0.5) \\ -0.5x+0.5 & \text{if } x \in [0.5,1], \end{cases} \tag{16}$$

but taking N_1 given by

$$N_1(x) = \begin{cases} -0.5x+1 & \text{if } x \in [0,0.5] \\ -0.5x+0.5 & \text{if } x \in (0.5,1], \end{cases} \tag{17}$$

we also obtain $N \circ N_1 = id_{[0,1]}$. Consequently, given I any (S,N)-implication, derived from a t-conorm S and the negation N, defining $F_1(x,y) = N_1(S(N(x),N(y)))$ we have that I satisfies (WLI) with both functions, F (obtained from Proposition 4) and F_1.

Proposition 5 *Let S be a t-conorm and N a continuous fuzzy negation. Then the function $F(x,y) = \mathfrak{R}_N(S(N(x),N(y)))$ for all $x,y \in [0,1]$ is a t-subnorm, i.e., it is nondecreasing, commutative, associative and such that $F(x,y) \le \min(x,y)$ for all $x,y \in [0,1]$.*

Proof: F is clearly nondecreasing and commutative. To see associativity, note that from the associativity of S we easily obtain

$$\begin{aligned} F(F(x,y),z) &= \mathfrak{R}_N(S(S(N(x),N(y)),N(z)) \\ &= F(x,F(y,z)). \end{aligned}$$

Finally, using the definition of $\mathfrak{R}_N$ we easily obtain

$$\mathfrak{R}_N(S(N(x),N(y))) \le x \text{ for all } x \in [0,1]$$

and so $F(x,y) \le \min(x,y)$ for all $x,y \in [0,1]$. ■

However, they are not t-norms in general because we have

$$F(1,y) = \mathfrak{R}_N(S(0,N(y))) = \mathfrak{R}_N(N(y))$$

and $\mathfrak{R}_N \circ N$ is the identity function only for values in Ran$(\mathfrak{R}_N)$. Nevertheless, one can always find a t-norm for which the (S,N)-implication satisfies (WLI) as follows.

Proposition 6 *Let I be a (S,N)-implication generated from a t-conorm S and a continuous fuzzy negation N. Then*

$$T(x,y) = \begin{cases} \mathfrak{R}_N(S(N(x),N(y))) & \text{if } \max\{x,y\} < 1 \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1 \end{cases}$$

is a t-norm such that I satisfies (LI) with T.

Proof: T is clearly a t-norm since it is constructed in the usual way to obtain a t-norm from a given t-subnorm (see for instance, [12], Proposition 1.6). Thus, we only need to prove that I satisfies (LI) with T. From Proposition 4 it is sufficient to verify (LI) when $x = 1$ or $y = 1$. But, for $x = 1$ we have

$$I(T(1,y),z) = I(y,z) = I(1,I(y,z))$$

and similarly for $y = 1$. ■

Corollary 1 *Let I be an implication satisfying (BP) with a continuous fuzzy negation N. Then*

$$I \text{ satisfies (LI)} \quad \Leftrightarrow \quad I \text{ satisfies (WLI)}$$

Remark 2 *Note that the construction given in the Propositions 5 and 6 works only for the function $\mathfrak{R}_N$. For the other functions N_1 satisfying $N \circ N_1 = id_{[0,1]}$ the monotonicity of T is not guaranteed (it is enough to take $S = \max$, N and N_1 as in Example 1 and $x = 0.5 > y$, then $N_1(\max(N(0.5),N(y))) = 0.75 > 0.5$). Thus, for (S,N)-implications with N continuous we have found many functions F for which (WLI) holds, but only one t-norm for which (LI) holds. However, we are convinced that there can be found t-conorms S such that even the t-norm is not unique.*

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At this point, we propose a new characterization of (S, N)-implications based on (WLI).

Theorem 4 *For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i). I is an (S, N)-implication generated from a t-conorm S and a continuous (strict, strong) fuzzy negation N

(ii). I satisfies (WLI) with a commutative and nondecreasing function F and (BP) with a continuous (strict, strong) fuzzy negation N.

In this case possible functions F are given by $F(x, y) = N_1(I(x, N(y)))$ for all $x, y \in [0,1]$, where N_1 is a fuzzy negation with $N \circ N_1 = id_{[0,1]}$. Moreover, when N is strict (strong) then such F is unique with $N_1 = N^{-1}$.

Proof: If I satisfies i), then $I(x, 0) = S(N(x), 0) = N(x)$ and I satisfies (BP) with N. Moreover, it also satisfies (WLI) from Proposition 4. The converse is guaranteed by Proposition 2. ∎

Finally and returning to the question about the equivalence of (EP) and (LI), we have the following result.

Proposition 7 *Let I be an implication satisfying (BP) with a continuous fuzzy negation N. Then*

I satisfies (EP) $\Leftrightarrow$ I satisfies (LI) $\Leftrightarrow$ I satisfies (WLI)

Proof: We already know that (LI) implies (EP). To see the converse, note that if I is an implication it satisfies in particular (I1) and then Theorem 1 implies that I is an (S, N)-implication generated from a t-conorm S and a continuous fuzzy negation N. Finally, Proposition 4 and Corollary 1 ensures that I satisfies (LI). The other equivalence has been already proved in Corollary 1. ∎

3.3 Counterexample for implications

To sum up, we have proved that (EP) and (LI) are equivalent for implications satisfying (BP) with a continuous fuzzy negation N. So, to find a counterexample we need to search among those implications I such that $I(x, 0) = N(x)$ is non-continuous (it is always a fuzzy negation). In next proposition we give such a counterexample which proves not only that (EP) does not imply (LI), but also that it does not imply (WLI).

Proposition 8 *Let S be a nilpotent t-conorm and N a strict negation. Let $I : [0,1]^2 \to [0,1]$ be the function given by*

$$I(x, y) = \begin{cases} 0 & \text{if } y = 0 \text{ and } x \neq 0 \\ S(N(x), y) & \text{otherwise.} \end{cases} \tag{18}$$

Then I is a fuzzy implication that satisfies (EP), but there is no function F for which I satisfies (WLI).

Proof: We divide the proof in two steps.

- I satisfies (EP). When $z \neq 0$ we have $I(x, z) \neq 0$ for all $x \in [0,1]$ and then I satisfies (EP) because it is given as an (S, N)-implication.

  When $z = 0$ and $x, y \neq 0$ then

  $$I(x, I(y, 0)) = I(y, I(x, 0)) = 0.$$

  Finally, when $z = 0$ and $\min\{x, y\} = 0$ then

  $$I(x, I(y, 0)) = I(y, I(x, 0)) = 1.$$

- I does not satisfy (WLI) with any F. Suppose on the contrary that I satisfies (WLI) with a function F. Then for all $x, y \in [0,1]$ and $z \neq 0$ (WLI) derives into

  $$S(N(F(x, y)), z) = S(S(N(x), N(y)), z). \tag{19}$$

  If we denote by N_S the strong negation associated to S (i.e., $N_S(x) = \varphi^{-1}(1 - \varphi(x))$ where φ is the normalized additive generator of S), the left hand side of the previous equation equal 1 if and only if $N(F(x, y)) \geq N_S(z)$, whereas the right hand side equals 1 if and only if $S(N(x), N(y)) \geq N_S(z)$. Consequently for all $x, y \in [0,1]$ and $z \neq 0$

  $$N(F(x, y)) \geq N_S(z) \Longleftrightarrow S(N(x), N(y)) \geq N_S(z).$$

  Let us now prove from this equivalence that

  $$F(x, y) = N^{-1}(S(N(x), N(y)) \quad \text{for all } x, y \in [0,1]. \tag{20}$$

  Note that when $F(x, y) = 0$, $N(F(x, y)) \geq N_S(z)$ for all $z \in [0,1]$ and, from the equivalence before we obtain $S(N(x), N(y)) = 1$ and equation (20) holds. On the other hand, when $F(x, y) > 0$ there exists $z \in [0,1]$ such that $N_S(z) > N(F(x, y))$ and then equation (19) can be written as

  $$\begin{aligned} &\varphi^{-1}(\varphi(N(F(x, y))) + \varphi(z)) \\ &= \varphi^{-1}(\varphi(S(N(x), N(y))) + \varphi(z)) \end{aligned}$$

  and consequently equation (20) also holds.

  Thus, F should be given by equation (20). However, I does not satisfy (WLI) with this F when $z = 0$. To see it just take $x, y \neq 0$ such that $S(N(x), N(y)) = 1$ since then $F(x, y) = 0$ and we obtain

  $$\begin{aligned} I(x, I(y, 0)) &= I(x, 0) = 0 \\ I(F(x, y), 0) &= I(0, 0) = 1 \end{aligned}$$

∎

3.4 R-implications derived from left-continuous t-norms

A characterization of those functions $I : [0,1]^2 \to [0,1]$ that are R-implications derived from a left-continuous t-norm is given in Theorem 2, and the case of continuous R-implications is characterized in Theorem 3. Note that in this last case condition (I2) is not necessary to derive the conclusion. It was claimed in [17] that the same happens in the general case. Specifically it was claimed that (EP) and (OP) imply (I2). This is not true and a counterexample was recently proved in [18].

In this section, we will prove that the situation is completely different if we change (EP) by (WLI). We begin by proving that (I2) can be derived from (WLI) and (OP).

Proposition 9 *Let $I : [0,1]^2 \to [0,1]$ be a function satisfying (OP) and (WLI) with a commutative and nondecreasing function F. Then I satisfies (I2), i.e, it is nondecreasing in the second variable.*

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Proof: Consider $x, y, z \in [0,1]$ with $y \leq z$ and let us divide our argument in two cases.

- If $x \leq z$, then $I(x,y) \leq I(x,z) = 1$.
- If $y \leq z < x$, then by (OP) and (WLI),

$$1 = I(I(x,y), I(x,y)) = I(F(I(x,y),x), y)$$

and consequently $F(I(x,y),x) \leq y \leq z$. Thus

$$1 = I(F(I(x,y),x), z) = I(I(x,y), I(x,z))$$

and therefore, $I(x,y) \leq I(x,z)$. ∎

Moreover, we have also the following properties.

Proposition 10 *Let $I : [0,1]^2 \to [0,1]$ be a function satisfying (OP) and (WLI) with a commutative and nondecreasing function F. Then*

i) I satisfies (I1), i.e, it is nonincreasing in the first variable.

ii) $I(x,-)$ is right-continuous for all $x \in [0,1]$.

Proof: To prove i) consider $x, y, z \in [0,1]$ with $x \leq y$. We divide our argument in two cases.

- If $x \leq z$, then $I(y,z) \leq I(x,z) = 1$.
- If $x > z$, then by (OP) and (WLI),

$$1 = I(I(y,z), I(y,z)) = I(F(I(y,z),y), z)$$

and consequently $F(I(y,z),y) \leq z$. Since F is nondecreasing we also have $F(I(y,z),x) \leq z$. Thus

$$1 = I(F(I(y,z),x), z) = I(I(y,z), I(x,z))$$

and therefore, $I(y,z) \leq I(x,z)$.

To prove ii), since I is nondecreasing in the second variable, it is enough to show that $\inf\{I(x,y_n)\} = I(x, \inf\{y_n\})$ for all $x \in [0,1]$. Say $y = \inf\{y_n\}$, it is clear that $I(x,y) \leq I(x,y_n)$ for all n and consequently $I(x,y) \leq \inf\{I(x,y_n)\}$. Moreover, for all n we have

$$1 = I(I(x,y_n), I(x,y_n)) = I(F(I(x,y_n),x), y_n)$$

which implies $F(I(x,y_n),x) \leq y_n$ for all n and thus $F(I(x,y_n),x) \leq y$. Finally, we have

$$1 = I(F(I(x,y_n),x), y)) = I(I(x,y_n), I(x,y)),$$

that is, $I(x,y_n) \leq I(x,y)$ and thus $\inf\{I(x,y_n)\} \leq I(x,y)$. ∎

Due to the previous results we can reformulate the characterization of R-implications given by theorem 2.

Theorem 5 *Let $I : [0,1]^2 \to [0,1]$ be a function. Then I is an R-implication derived from a left-continuous t-norm, if and only if, I satisfies (OP) and (WLI) with a commutative and nondecreasing function F.*

Moreover, in this case F must be the t-norm from which I is obtained by residuation.

Finally, it is well know that if a function $I : [0,1]^2 \to [0,1]$ satisfies (OP), (EP) and (BP) with a continuous fuzzy negation N, then N must be strong. However, note that from the previous results more properties are derived.

Corollary 2 *Let $I : [0,1]^2 \to [0,1]$ satisfy (OP), (EP) and (BP) with a continuous fuzzy negation N. Then I satisfies (I2), I is right-continuous in the second variable and so, I is an R-implication derived from a left-continuous t-norm.*

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References

[1] E.E. Kerre, C. Huang, and D. Ruan. *Fuzzy Set Theory and Approximate Reasoning*. Wu Han University Press, Wu Chang, 2004.

[2] S. Gottwald. *A Treatise on Many-Valued Logic*. Research Studies Press, Baldock, 2001.

[3] M. Mas, M. Monserrat, J. Torrens, and E. Trillas. A survey on fuzzy implication functions. *IEEE Transactions on Fuzzy Systems*, 15(6):1107–1121, 2007.

[4] H. Bustince, V. Mohedano, E. Barrenechea, and M. Pagola. Definition and construction of fuzzy DI-subsethood measures. *Information Sciences*, 176:3190–3231, 2006.

[5] H. Bustince, M. Pagola, and E. Barrenechea. Construction of fuzzy indices from fuzzy DI-subsethood measures: application to the global comparison of images. *Information Sciences*, 177:906–929, 2007.

[6] P. Yan and G. Chen. Discovering a cover set of ARsi with hierarchy from quantitative databases. *Information Sciences*, 173:319–336, 2005.

[7] M. Baczyński and B. Jayaram. *Fuzzy Implications*. Springer, Berlin Heidelberg, 2008.

[8] B. Jayaram. On the law of importation $(x \wedge y) \to z \equiv (x \to (y \to z))$ in fuzzy logic. *IEEE Transactions on Fuzzy Systems*, 16:130–144, 2008.

[9] M. Mas, M. Monserrat, and J. Torrens. The law of importation for QL and D-implications. *Submitted*, 2008.

[10] M. Mas, M. Monserrat, and J. Torrens. On the law of importation for some kinds of fuzzy implications derived from uninorms. *In Proceedings of IPMU'08*, Torremolinos (Málaga):1303–1310, 2008.

[11] E.P. Klement and R. Mesiar. Open problems posed at the eighth international conference in fuzzy set theory and applications (FSTA 2006, Liptovský Ján, Slovakia). *Kybernetika*, 42:225–235, 2006.

[12] E.P. Klement, R. Mesiar, and E. Pap. *Triangular norms*. Kluwer Academic Publishers, Dordrecht, 2000.

[13] M. Baczyński and B. Jayaram. On the characterization of (S,N)-implications. *Fuzzy Sets and Systems*, 158:1713–1727, 2007.

[14] J.C. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[15] M. Baczyński and B. Jayaram. (S,N)- and R-implications: A state-of-the-art survey. *Fuzzy Sets and Systems*, 159:1836–1859, 2008.

[16] P. Smets and P. Magrez. Implications in fuzzy logic. *International Journal of Approximate Reasoning*, 1:327–347, 1987.

[17] M. Baczyński. Residual implications revisited. notes on the smets-magrez theorem. *Fuzzy Sets and Systems*, 145:267–277, 2004.

[18] R. Smogura. A note on the article: Residual implications revisited. Notes on the Smets-Magrez theorem. *Fuzzy Sets and Systems*, 157:466–469, 2006.