# Fuzzy Transform and Smooth Functions

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***Abstract***— *We describe parametric fuzzy partitions based on parametric shape functions and we show that the Perfilieva's Fuzzy Transform defined on parametric fuzzy partitions has the properties of a smoother. We also illustrate some examples.*

***Keywords***— Parametric Fuzzy Partition, Fuzzy Transform, Smoothing Functions, Fuzzy Numbers

## 1 Introduction

The fuzzy transform has been recently introduced by I. Perfilieva and the theoretical investigations and several connections between theory and applications are covered e.g. in [3], [4], [5] and [9].

In a recent paper (see [6]), we suggested the use of a parametrized class of fuzzy numbers (see [7], [8]) to define general parametric partitions for the fuzzy transform. The proposed flexible family of fuzzy partitions allows the possibility of estimating the shapes of the fuzzy partition to obtain better approximation properties.

In this paper we suggest an extension of the fuzzy partitions (section 3) and we describe the corresponding direct and inverse fuzzy transforms (section 4). Then, in section 5, we show that the inverse fuzzy transform based on extended partitions has an important smoothing property and we illustrate it by numerical examples.

## 2 Fuzzy Transform and Parametric Fuzzy Partitions

We briefly recall the basic definitions and properties

**Definition:** (Perfilieva[3]) A *fuzzy partition* of a given real compact interval $[a,b]$ is constructed by a decomposition $\mathbb{P} = \{a = x_1 < x_2 < ... < x_n = b\}$ of $[a,b]$ into $n-1$ subintervals $[x_{k-1}, x_k]$, $k = 2,...,n$ and by a family $\mathbb{A} = \{A_1, A_2, ..., A_n\}$ of $n$ fuzzy numbers (the *basic functions*), identified by their membership functions $A_1(x), A_2(x), ..., A_n(x)$ for $x \in [a,b]$ and with the properties (to complete this notation, we set $x_0 = a$ and $x_{n+1} = b$):

1. $A_k : [a,b] \longrightarrow [0,1]$ is continuous with $A_k(x_k) = 1$, $A_k(x) = 0$ for $x \notin [x_{k-1}, x_{k+1}]$;
2. for $k = 2,3,...,n$, $A_k$ is increasing on $[x_{k-1}, x_k]$ and decreasing on $[x_k, x_{k+1}]$; $A_1$ is decreasing on $[a, x_2]$; $A_n$ is increasing on $[x_{n-1}, b]$;
3. for all $x \in [a,b]$ the following partition of unity condition holds

$$\sum_{k=1}^{n} A_k(x) = 1.$$

We denote a fuzzy partition by the pair $(\mathbb{P}, \mathbb{A})$. The fuzzy numbers $A_1, A_2, ..., A_n$ giving the family of basic functions can be defined by $n-1$ increasing functions $L_2(x), ..., L_n(x)$ as follows

$$A_1(x) = 1 - L_2(x) \text{ if } x \in [a, x_2],$$

$$\text{for } k = 2,...,n-1$$
$$A_k(x) = \begin{cases} L_k(x) & \text{if } x \in [x_{k-1}, x_k] \\ 1 - L_{k+1}(x) & \text{if } x \in [x_k, x_{k+1}] \end{cases} \quad (1)$$
$$A_n(x) = L_n(x) \text{ if } x \in [x_{n-1}, b]$$

where $L_k(x_{k-1}) = 0$ and $L_k(x_k) = 1$.

In the standard case, the support of each basic function $A_k(x)$ is the compact interval $[x_{k-1}, x_{k+1}]$ so that, on each subinterval $[x_{k-1}, x_k]$ of the decomposition $\mathbb{P}$ only two basic functions $A_{k-1}(x)$ and $A_k(x)$ are non zero for $k = 2,...,n$.

**Definition 1:** (Perfilieva[3]) Given a continuous function $f : [a,b] \longrightarrow R$ and a fuzzy partition $(P, A)$, the direct fuzzy transform (F-transform) of $f$ with respect to $(P, A)$ is the $n$-tuple of real numbers $F = (F_1, F_2, ..., F_n)^T$ given by (notation $(\cdot)^T$ means transposition)

$$\text{For } k = 1,2,...,n \quad (2)$$
$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx}$$

**Definition 2:** (Perfilieva[3]) Given a continuous function $f : [a,b] \longrightarrow R$, a fuzzy partition $(P, A)$ and the direct fuzzy transform $(F_1, F_2, ..., F_n)^T$ of $f$ with respect to $(P, A)$, the inverse F-transform is the continuous function $\widehat{f}_F : [a,b] \longrightarrow R$ given by

$$\widehat{f}_F(x) = \sum_{k=1}^{n} F_k A_k(x) \text{ for } x \in [a,b]. \quad (3)$$

The following property (Th. 2 in Perfilieva[3]) is one of the fundamentals of the F-transform setting.

**Property 1**: If $f : [a,b] \longrightarrow R$ is a continuous function then, for any positive real $\varepsilon$, there exists a fuzzy partition $(P_\varepsilon, A_\varepsilon)$ such that the corresponding F-transform $F_\varepsilon = (F_{1,\varepsilon}, F_{2,\varepsilon}, ..., F_{n_\varepsilon,\varepsilon})^T$ and inverse fuzzy transform $\widehat{f}_{F_\varepsilon} : [a,b] \longrightarrow R$ satisfy

$$\left| f(x) - \widehat{f}_{F_\varepsilon}(x) \right| < \varepsilon \text{ for all } x \in [a,b].$$

To define a fuzzy partition $(\mathbb{P}, \mathbb{A})$ we can use a general decomposition $\mathbb{P} = \{a = x_1 < x_2 < ... < x_n = b\}$ of $[a,b]$ into $n-1$ subintervals $[x_{k-1}, x_k]$, and a general set of basic functions $\mathbb{A} = \{A_1, A_2, ..., A_n\}$ of $n$ fuzzy numbers

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$A_1, A_2, ..., A_n$ on $[a,b]$; denote, in general, $h_k = x_k - x_{k-1}$, $k = 2, ..., n$.

If the decomposition $\mathbb{P}$ is uniform, we have
$x_k = a + (k-1)h$ where $h = \frac{b-a}{n-1}$.
We can have uniform $\mathbb{P}$ and general $\mathbb{A}$.

To obtain families of basic functions $A_1, A_2, ..., A_n$ in (1), defined by $n-1$ increasing functions $L_2, ..., L_n$, we suggested to use parametric monotonic functions $p(t;\beta_0,\beta_1)$, $t \in [0,1]$ such that, for any parameters $\beta_0, \beta_1 \geq 0$, the following Hermite-type interpolation conditions are satisfied:

$$\begin{aligned} p(0;\beta_0,\beta_1) &= 0,\ p(1;\beta_0,\beta_1) = 1, \\ p'(0;\beta_0,\beta_1) &= \beta_0,\ p'(1;\beta_0,\beta_1) = \beta_1 \end{aligned}$$

and $p'(t;\beta_0,\beta_1) \geq 0\ \forall t \in [0,1]$ (derivatives at $t=0$ and $t=1$ are, respectively, right and left derivatives).

Valid examples of function $p(t;\beta_0,\beta_1)$ are reported in see [7]; to exemplify, we will use here the (2,2)-rational spline:

$$p_{rat22}(t;\beta_0,\beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)}. \quad (4)$$

By using parametric monotonic functions $p(t;\beta_0,\beta_1)$, $t \in [0,1]$ we obtain increasing functions $L_2, ..., L_n$ that define the basic membership functions of the fuzzy partition $A_1, A_2, ..., A_n$ :

$$\begin{aligned} L_k(x) &= p(\frac{x - x_{k-1}}{x_k - x_{k-1}};\beta_{k,0},\beta_{k,1}) \quad (5) \\ \text{for } x &\in [x_{k-1}, x_k] \text{ and } k = 2, ..., n \end{aligned}$$

and each fuzzy partition $(\mathbb{P},\mathbb{A})$ contains $2(n-1)$ parameters $\beta_{k,0}, \beta_{k,1}$, $k = 2, ..., n$ giving the first derivatives $L'_k(x_{k-1}) = \beta_{k,0}/(x_k - x_{k-1})$ and $L'_k(x_k) = \beta_{k,1}/(x_k - x_{k-1})$ (to be intended as right and left derivatives, respectively).

We define a *uniform fuzzy partition* $(\mathbb{P},\mathbb{A})$ if $\mathbb{P}$ is uniform and $\beta_{k,0} = \beta_{j,0}$, $\beta_{k,1} = \beta_{j,1}$ for all $j, k$.

We define a *balanced fuzzy partition* $(\mathbb{P},\mathbb{A})$ if $\mathbb{P}$ is uniform and $\beta_{k,0} = \beta_{k+r,0}$, $\beta_{k,1} = \beta_{k+r,1}$ for all $k$..

We obtain a uniform and balanced partition $(\mathbb{P},\mathbb{A})$ by requiring that $\beta_{k,0} = \beta_{k,1} = \beta_{j,0} = \beta_{j,1}$ for all $j, k$.

## 3 Extended Fuzzy Partitions

On the interval $[a,b]$ with a decomposition $\mathbb{P} = \{a = x_1 < x_2 < ... < x_n = b\}$ into $n-1$ subintervals $[x_{k-1}, x_k]$, $k = 2, ..., n$ the basic functions of the fuzzy partition can be defined such that the support of each $A_k$ is more extended than two intervals, identified by three points $x_{k-1}, x_k, x_{k+1}$. Consider an integer $r \geq 1$ and $2r+1$ consecutive points of $\mathbb{P}$, $x_{k-r}, ..., x_k, ..., x_{k+r}$ for all $k = 1, 2, ..., n$ (to complete the notation, we extend the points to $x_{1-r} < ... < x_0 < a$ and $b < x_{n+1} < ... < x_{n+r}$).

**Definition 3**: we define a *fuzzy r-partition* of $[a,b]$ to be $(\mathbb{P},\mathbb{A}^{(r)})$ with the family of $n+2r-2$ continuous, normal, convex fuzzy numbers

$$\begin{aligned} \mathbb{A}^{(r)} &= \{A_k^{(r)} : [a,b] \longrightarrow [0,1] | \\ k &= -r+2, ..., 1, 2, ..., n, ..., n+r-1\} \end{aligned}$$

where $A_k^{(r)}$ has core $\{x_k\}$ and support $[x_{k-r}, x_{k+r}]$ i.e.

a. for $k = 1, 2, ..., n$, $A_k^{(r)}$ is a continuous fuzzy number with $A_k^{(r)}(x_k) = 1$ (i.e. the core is $\{x_k\}$) and $A_k^{(r)}(x) = 0$ for $x \notin [x_{k-r}, x_{k+r}]$ (i.e. the support is $[x_{k-r}, x_{k+r}]$);

b. for $k = 1, 2, ..., n$, $A_k^{(r)}$ is increasing on $[x_{k-r}, x_k]$ and decreasing on $[x_k, x_{k+r}]$;

c. for $k = -r+2, ..., 0$, $A_k^{(r)}$ is decreasing on $[x_k, x_{k+r}]$;

d. for $k = n+1, ...n+r-1$, $A_k^{(r)}$ is increasing on $[x_{k-r}, x_k]$;

e. for all $x \in [a,b]$ the following $r-$*partition of unity* condition holds

$$\frac{1}{r}\sum_{k=-r+2}^{n+r-1} A_k^{(r)}(x) = 1.$$

If $r = 1$ we have the standard partition of unity.

We can construct parametric *fuzzy r-partition* of $[a,b]$ by considering $n+r-2$ shape functions $p(t;\beta_0,\beta_1)$ of types (4); then by defining

$$\begin{aligned} \text{for } k &= 2, ..., n+r-1 \\ L_k^{(r)}(x) &= p(\frac{x - x_{k-r}}{x_k - x_{k-r}};\beta_{k,0},\beta_{k,1}) \end{aligned}$$

the basic functions are

$$\text{for } k = 2, ..., n-1$$
$$A_k^{(r)}(x) = \begin{cases} L_k^{(r)}(x) & \text{if } x \in [x_{k-r}, x_k] \\ 1 - L_{k+r}^{(r)}(x) & \text{if } x \in [x_k, x_{k+r}] \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$\text{for } k = -r+2, ..., 1$$
$$A_k^{(r)}(x) = \begin{cases} 1 - L_{k+r}^{(r)}(x) & \text{if } x \in [x_k, x_{k+r}] \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

$$\text{for } k = n, ..., n+r-1$$
$$A_k^{(r)}(x) = \begin{cases} L_{k-r}^{(r)}(x) & \text{if } x \in [x_{k-r}, x_k] \\ 0 & \text{otherwise} \end{cases}$$

It is a matter of simple algebraic manipulations to show that for all $x \in [a,b]$ the $r-$*partition of unity* condition holds; in fact for $k = 1, 2, ..., n$ and $x \in [x_{k-1}, x_k]$, only $A_{k-r}^{(r)}, ..., A_k^{(r)}, ..., A_{k+r-1}^{(r)}$ are different from zero and

$$\begin{aligned} \sum_{j=k-r}^{k+r-1} A_j^{(r)}(x) &= \sum_{j=k-r}^{k-1}(1 - L_{j+r}^{(r)}(x)) + \sum_{j=k}^{k+r-1} L_j^{(r)}(x) \\ &= \sum_{j=k}^{k+r-1}(1 - L_j^{(r)}(x)) + \sum_{j=k}^{k+r-1} L_j^{(r)}(x) \\ &= r. \end{aligned}$$

Consider for example $r = 2$; we need $n+r-2 = n$ shape functions and (we remove the superscript $^{(r)}$ for simplicity) define $L_k(x) = p(\frac{x - x_{k-2}}{x_k - x_{k-2}};\beta_{k,0},\beta_{k,1})$ for $k = 2, ..., n+1$. So (in the relevant subintervals, otherwise the functions are null)

$$\text{for } k = 2, ..., n-1$$
$$A_k(x) = \begin{cases} L_k(x) & \text{if } x \in [x_{k-2}, x_k] \\ 1 - L_{k+2}(x) & \text{if } x \in [x_k, x_{k+2}] \end{cases}$$

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$$\begin{aligned} \text{for } k &= 0, 1, n, n+1 \\ A_0(x) &= 1 - L_2(x) \text{ if } x \in [x_0, x_2] \\ A_1(x) &= 1 - L_3(x) \text{ if } x \in [x_1, x_3] \\ A_n(x) &= L_n(x) \text{ if } x \in [x_{n-2}, x_n] \\ A_{n+1}(x) &= L_{n+1}(x) \text{ if } x \in [x_{n-1}, x_{n+1}]. \end{aligned}$$

The figures below illustrate three *fuzzy r-partition* $(\mathbb{P}, \mathbb{A}^{(r)})$ for $r = 1$ (standard case), $r = 2$ (each $A_k(x)$ covers four intervals) and $r = 3$ (each $A_k(x)$ covers six intervals).

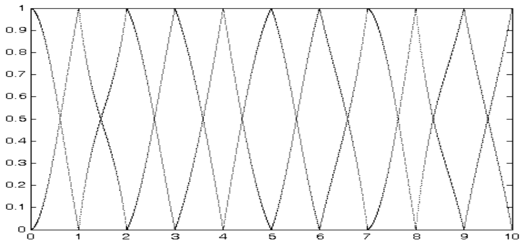

Figure 1a) *fuzzy r-partition* $(\mathbb{P}, \mathbb{A}^{(r)})$ with $r = 1$

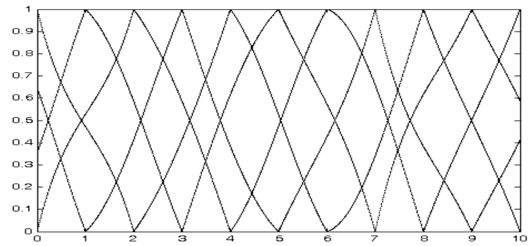

Figure 1b) *fuzzy r-partition* $(\mathbb{P}, \mathbb{A}^{(r)})$ with $r = 2$

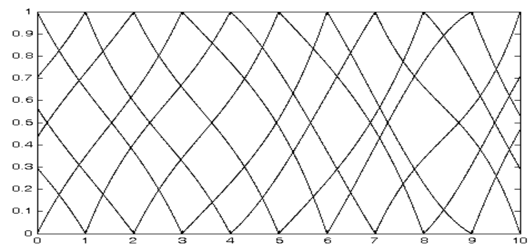

Figure 1c) *fuzzy r-partition* $(\mathbb{P}, \mathbb{A}^{(r)})$ with $r = 3$

## 4 Fuzzy Transform with Extended Partitions

The direct F$^{(r)}$-transform (of degree $r$) based on the given *fuzzy r-partition* $(\mathbb{P}, \mathbb{A}^{(r)})$ can be defined by the vector $F^{(r)} = (F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)})^T$

$$F_k^{(r)} = \frac{\int_a^b f(x) A_k^{(r)}(x) dx}{\int_a^b A_k^{(r)}(x) dx}, k = 1, 2, ..., n \quad (8)$$

and the inverse F$^{(r)}$-transform by

$$\widehat{f}^{(r)}(x) = \frac{1}{r} \sum_{k=1}^{n} F_k^{(r)} A_k^{(r)}(x). \quad (9)$$

We see that each $\widehat{f}^{(r)}(x_k)$ has the structure of a moving average of the values $\{F_j^{(r)}, j = 1, ..., n\}$; in fact, at the points $x = x_k$ we have (assume $F_k^{(r)} = 0$ if $k < 1$ or $k > n$)

$$\widehat{f}^{(r)}(x_k) = \frac{1}{r} \sum_{j=k-r}^{k+r} F_j^{(r)} A_j^{(r)}(x_k) \quad (10)$$

i.e. a weighted average of $F_{k-r}^{(r)}, ..., F_k^{(r)}, ..., F_{k+r}^{(r)}$ with weights $\frac{A_{k-r}^{(r)}(x_k)}{r}, ..., \frac{1}{r}, ..., \frac{A_{k+r}^{(r)}(x_k)}{r}$, respectively.

It is possible to see that the F$^{(r)}$-transform has analogues properties to theorem 1 and theorem 2 of Perfilieva ([3]) i.e.

1. the local (for any $k = 1, 2, ..., n$) "error" function $\frac{1}{r} \int_a^b [f(x) - y]^2 A_k^{(r)}(x) dx$ is minimized by $y = F_k^{(r)}$;

2. the minimizer of $\frac{1}{r} \sum_{k=1}^{n} \int_a^b [f(x) - y_k]^2 A_k^{(r)}(x) dx$ is $y = F^{(r)} = (F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)})^T$ (notation $(\cdot)^T$ means transposition);

3. for fixed $r \geq 1$ and for any $\varepsilon > 0$ there exists a *fuzzy r-partition* $(\mathbb{P}_\varepsilon, \mathbb{A}_\varepsilon^{(r)})$ with the corresponding inverse F$^{(r)}$-transform $\widehat{f}_\varepsilon^{(r)}$ such that $\sup_x \left| f(x) - \widehat{f}_\varepsilon^{(r)}(x) \right| < \varepsilon$.

For a given family of parametric basic functions $A_k^{(r)}$ consider the linear combination (with $\theta = (\theta_1, \theta_2, ..., \theta_n)^T$)

$$C_{\mathbb{A}^{(r)}}(x; \theta, \beta) = \frac{1}{r} \sum_{j=1}^{n} \theta_j A_j^{(r)}(x; \beta) \text{ for } x \in [a, b] \quad (11)$$

where $\beta$ is the vector of the non negative slope parameters $\beta_{j,0}, \beta_{j,1}$ giving $L_j(x; \beta_{j,0}, \beta_{j,1})$.

The following minimization defines $C_{\mathbb{A}^{(r)}}(x; \widehat{\theta}, \widehat{\beta})$ as the *least squares approximation* of $f(x)$ over interval $[a, b]$

$$\underset{\theta,\beta}{Minimize} \quad \int_a^b [f(x) - C_{\mathbb{A}^{(r)}}(x; \theta, \beta)]^2 dx \quad (12)$$

where $(\widehat{\theta}, \widehat{\beta})$ is the minimizing point.

For fixed $\beta$, the linear parameters $\widehat{\theta}$ are obtained by putting to zero the partial derivatives (for $i = 1, 2, ..., n$) and we obtain the equations

$$\int_a^b A_i^{(r)}(x; \beta) f(x) dx = \frac{1}{r} \sum_{j=1}^{n} \widehat{\theta}_j \int_a^b A_i^{(r)}(x; \beta) A_j^{(r)}(x; \beta) dx \quad (13)$$

From definition of the F$^{(r)}$-transform $F^{(r)}(\beta)$ as a function of $\beta$, we have

$$\int_a^b A_i^{(r)}(x; \beta) f(x) dx = F_i^{(r)}(\beta) \int_a^b A_i^{(r)}(x; \beta) dx \quad (14)$$

and we finally obtain the linear system of $n$ equations

$$\begin{aligned} C^{(r)}(\beta) \widehat{\theta} &= r F^{(r)}(\beta) \quad i.e. \\ \sum_{j=1}^{n} c_{i,j}^{(r)}(\beta) \widehat{\theta}_j &= r F_i^{(r)}(\beta), \quad i = 1, 2, ..., n \end{aligned} \quad (15)$$

where the matrix $C^{(r)}(\beta)$ has elements

$$c_{i,j}^{(r)}(\beta) = \frac{1}{r} \int_a^b A_i^{(r)}(x; \beta) A_j^{(r)}(x; \beta) dx. \quad (16)$$

For fixed parameters $\beta$, we can obtain a matrix form for $\theta$ in the definition of function (11).

Given $f : [a, b] \longrightarrow \mathbb{R}$ and a fuzzy $r$-partition $(\mathbb{P}, \mathbb{A}^{(r)}(\beta))$ with the associated direct F$^{(r)}$-transform $F^{(r)}(\beta) = (F_1^{(r)}, F_2^{(r)}, ..., F_n^{(r)})^T$ and inverse F$^{(r)}$-transform

$$\widehat{f}_F^{(r)}(x; \beta) = \frac{1}{r} \sum_{j=1}^{n} F_j^{(r)}(\beta) A_j^{(r)}(x; \beta). \quad (17)$$

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Define the following vectors (we omit reference to $\beta$ and $r$ for simplicity of notation), obtained by evaluating $f(x)$ and $\widehat{f}_F(x)$ at points $x_1, x_2, ..., x_n$

$$f = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \\ f(x_n) \end{bmatrix}, \quad \widehat{f} = \begin{bmatrix} \widehat{f}_F(x_1) \\ \widehat{f}_F(x_2) \\ \\ \widehat{f}_F(x_n) \end{bmatrix}, \quad F = \begin{bmatrix} F_1^{(r)} \\ F_2^{(r)} \\ \\ F_n^{(r)} \end{bmatrix} \tag{18}$$

and the following $n \times n$ matrix

$$\begin{aligned} A^{(r)} &= [a_{i,j}] \text{ with} \\ a_{i,j} &= A_j^{(r)}(x_i). \end{aligned} \tag{19}$$

Matrix $A^{(r)}$ is a $(2r-1)$-band matrix and its $i-th$ row has the form $(i = 1, 2, ..., n)$

$$a_{i,j} = \begin{cases} 0 & j = 1, ..., i-r \\ A_j^{(r)}(x_i) & j = i-r+1, ..., i-1 \\ 1 & j = i \\ A_j^{(r)}(x_i) & j = i+1, ..., i+r-1 \\ 0 & j = i+r, ..., n \end{cases} \tag{20}$$

It is immediate that

$$\widehat{f} = \frac{1}{r}(A^{(r)})^T F \tag{21}$$

and $\frac{1}{r}(A^{(r)})^T$ represents a *moving average operator* acting on $F$ to produce $\widehat{f}$.

### 4.1 Discrete $F^{(r)}$-Transform

Matrix $A$ in (19) can be defined also for the discrete direct and inverse F$^{(r)}$-transforms. For $m$ points $(t_i, y_i), i = 1, 2, ..., m$, where $y_i = f(t_i)$ and defining $A^{(r)}(\beta) = [a_{i,j}(\beta) = A_j^{(r)}(t_i, \beta)]$ for the family $\{A_j^{(r)}(t; \beta) | j = 1, 2, ..., n\}$ of basic functions, the discrete F$^{(r)}$-transform is

$$\begin{aligned} \text{For } j &= 1, 2, ..., n \\ F_j^{(r)}(\beta) &= \frac{g_j(\beta)}{s_j(\beta)} \text{ where} \\ g_j(\beta) &= \sum_{i=1}^{m} y_i A_j^{(r)}(t_i; \beta) \text{ and } s_j = \sum_{i=1}^{m} A_j^{(r)}(t_i; \beta) > 0 \end{aligned}$$

If the elements $s_j$ are organized into a diagonal $n-$matrix

$$S = diag(s_1, s_2, ..., s_n)$$

so that

$$S^{-1} = diag(\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n})$$

and denote

$$\begin{aligned} y &= (y_1, ..., y_m)^T, \\ F^{(r)}(\beta) &= (F_1^{(r)}(\beta), ..., F_n^{(r)}(\beta))^T, \\ g(\beta) &= (g_1(\beta), ..., g_n(\beta))^T, \end{aligned}$$

then we have $(A^{(r)}(\beta))^T y = g(\beta)$ so that

$$F^{(r)}(\beta) = S(\beta)^{-1}(A^{(r)}(\beta))^T y.$$

Denote also, for $i = 1, 2, ..., m$ (note that for $i = r, ..., n-r+1$ we have $d_i = r$),

$$\begin{aligned} d_i(\beta) &= \sum_{j=1}^{n} A_j(t_i; \beta) > 0 \\ D &= diag(d_1, d_2, ..., d_m). \end{aligned}$$

For the discrete inverse F$^{(r)}$-transform

$$f_{F,\beta}^{(r)}(x) = \frac{\sum_{j=1}^{n} F_j^{(r)} A_j^{(r)}(x; \beta)}{\sum_{j=1}^{n} A_j^{(r)}(x; \beta)}$$

and denoting $f_{F,\beta}^{(r)} = (f_{F,\beta}^{(r)}(t_1), f_{F,\beta}^{(r)}(t_2), ..., f_{F,\beta}^{(r)}(t_m))^T$, we get

$$\begin{aligned} f_{F,\beta}^{(r)} &= D(\beta)^{-1} A^{(r)}(\beta) F^{(r)} \\ &= D(\beta)^{-1} A^{(r)}(\beta) S(\beta)^{-1} (A^{(r)}(\beta))^T y. \end{aligned} \tag{22}$$

## 5 Smoothing functions from $F^{(r)}$-Transform

In this section we investigate the smoothing effect of the inverse $F^{(r)}$-Transform $f^{(r)}$ on a finite set of data points $(t_i, y_i)$, $i = 1, 2, ..., m$; the discrete direct and inverse F$^{(r)}$-transform are obtained with a partition $(\mathbb{P}, \mathbb{A}^{(r)}(\beta))$, where

$$\begin{aligned} \mathbb{P} &= \{a = x_1 < x_2 < ... < x_n = b\} \text{ and} \\ a &= \min\{t_i\}, \; b = \max\{t_i\} \\ x_i &= a + (i-1)/(n-1), i = 1, 2, ..., n. \end{aligned}$$

We first consider the parameters $\beta$ as fixed for the family of basic functions $\mathbb{A}^{(r)}(\beta)$.

For a given function $f : [a, b] \longrightarrow \mathbb{R}$ a measure of smoothness can be given by its total variation

$$V(f) = \sup\{\sum_{j=1}^{k-1} |f(\alpha_{j+1}) - f(\alpha_j)|; \forall k \geq 2, \forall \alpha_j \in [a, b]\}$$

or by the following *total* variation (see [1], [2] and the references therein)

$$\widetilde{V}(f) = \sum_{j=1}^{m-1} |f(t_{j+1}) - f(t_j)|. \tag{23}$$

By increasing the integer value of $r$, the total variation $\widetilde{V}(f^{(r)})$ of the inverse transform functions $f^{(r)}$ tend to decrease to zero, as illustrated by the graphical representation in Fig. 2. we can obtain inverse fuzzy transform functions $f^{(r)}$ with arbitrarily small $\widetilde{V}(f^{(r)})$. A graphical representation is in Fig. 2.

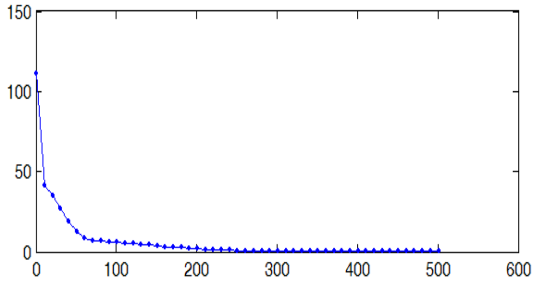


Figure 2. *Total variation $\widetilde{V}(f^{(r)})$ of inverse fuzzy transform function for increasing $r$.*

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It is interesting to see the combined effect of changing $n$ and $r$; in figures 3a)-3c), $m = 101$ data points are generated from function $f(t) = 1+0.2t+sin(t).cos(t)$ by adding a Gaussian random noise with zero mean and standard deviation $\sigma = 2$, i.e. $y_i = f(t_i) + z_i$ and $z_i \in N(0, \sigma^2)$ (the $t_i$ are uniform on $[-2\pi, 2\pi]$). In all the figures of this section, the continuos line is the original data, the dotted line is the inverse fuzzy transform $f^{(r)}$ and the circles are the direct $F^{(r)}$-transform.

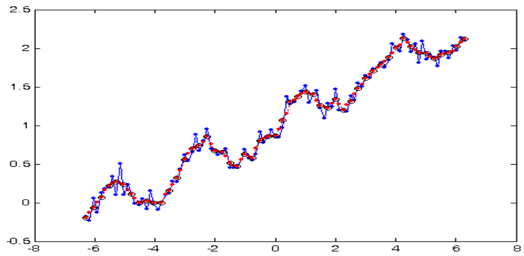

Figure 3a) Smoothing of m=101 data points, n=51, r=1.

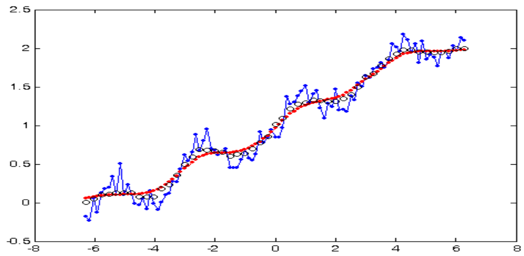

Figure 3b) Smoothing of m=101 data points, n=51, r=5.

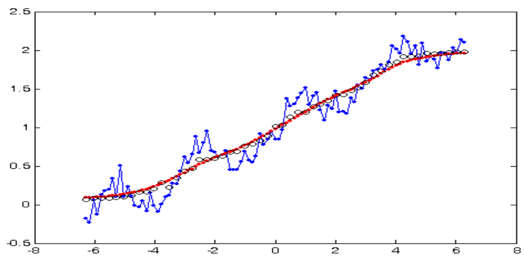

Figure 3c) Smoothing of m=101 data points, n=51, r=9.

In figures 4a)-4c) the data are generated from the same model but $n = 21$ is smaller then in previous case. When $n$ is reduced, the smoothing effect is increased but it is more visible at a global scale: compare figures 3a) with 4a), 3b) with 4b) and 3c) with 4c).

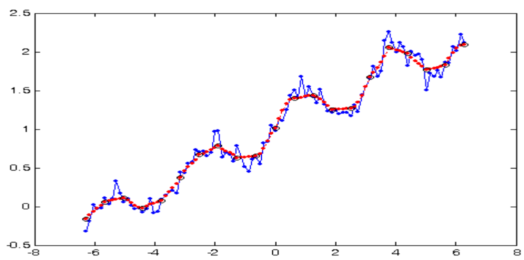

Figure 4a) Smoothing of m=101 data points, n=21, r=1.

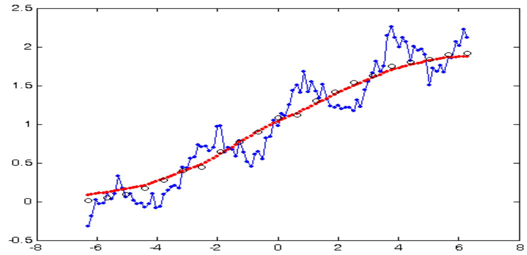

Figure 4b) Smoothing of m=101 data points, n=21, r=5.

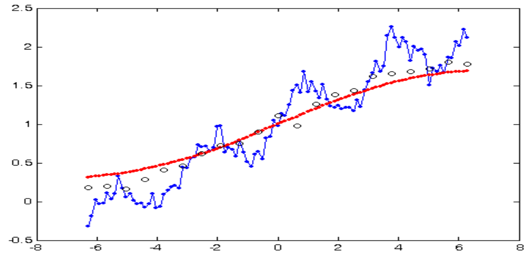

Figure 4c) Smoothing of m=101 data points, n=21, r=9.

If $n$ is small with respect to $m$ as in figures 5a) and 5b), the global smoothing is increased further.

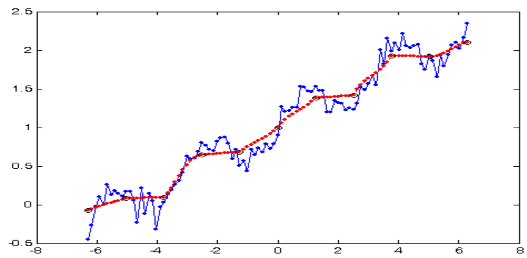

Figure 5a) Smoothing of m=101 data points, n=11, r=1.

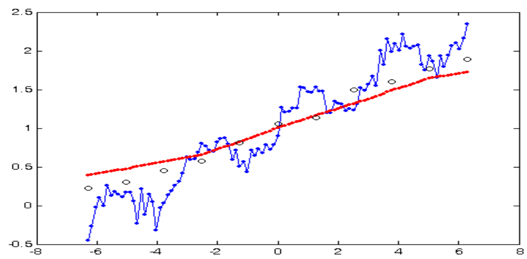

Figure 5b) Smoothing of m=101 data points, n=11, r=5.

So, according to the general properties of the F$^{(r)}$-transform, the discussion above illustrates that, in general, there are two possible sources for smoothness: for fixed m and r, reducing n will increase the smoothing effect; for fixed m and n, increasing r will increase the smoothness.

### *5.1 A Smoothing Procedure*

Let $(t_i, y_i)$, $i = 1, 2, ..., m$ be given observations and define $T_1 = \{i|x_1 \leq t_i \leq x_2\}$, $T_k = \{i|x_{k-1} < t_i \leq x_k\}$ for $k = 2, ..., n-1$.

The parameters $\beta$ (if not fixed) can be estimated from the data by solving the following criterion, a penalized nonlinear least squares problem where the objective function is penalized by the total variation and a positive tuning parameter $\lambda > 0$ is introduced

$$\underset{\beta}{\text{Minimize}} \quad \left\| f^{(r)}_{F,\beta} - y \right\|^2 + \lambda \widetilde{V}(f^{(r)}_{F,\beta}). \qquad (24)$$

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Suppose that each $T_k$ has a sufficient number of elements to allow the following approximation problem be well defined: determine $\beta = (\beta_{2,0}, ..., \beta_{n+r-1,0}\beta_{2,1}, ..., \beta_{n+r-1,1})$ to minimize the functional

$$\Phi^{(r)}(\beta) = \left\| L^{(r)}(\beta)F^{(r)}(\beta) - y \right\|^2 + \lambda \widetilde{V}(f^{(r)}_{F,\beta}) \quad (25)$$

where matrix $L^{(r)}(\beta)$ is

$$L^{(r)}(\beta) = D^{(r)}(\beta)^{-1}A^{(r)}(\beta) \text{ and}$$

$$F^{(r)}(\beta) = S^{(r)}(\beta)^{-1}(A^{(r)}(\beta))^T y$$

Alternatively, we can tune the total variation $\widetilde{V}(f^{(r)}_{F,\beta})$ by fixing a variation level $\overline{V} > 0$ and solving the following constrained problem

$$\Phi^{(r)}(\widehat{\beta}) = Minimize \left\| L^{(r)}(\beta)F^{(r)}(\beta) - y \right\|^2 \quad (26)$$

$$s.t. \quad \widetilde{V}(f^{(r)}_{F,\beta}) \le \overline{V}$$

We show the procedure with the following example and with the value of $\overline{V} = \gamma \widetilde{V}(y)$ (a fraction $\gamma$ of the total variation of the data); the generated data are $y_i = f(t_i) + 2z_i$ and $z_i \in N(0, 1)$ (the $m = 101$ points $t_i$ are uniform on $[0, 2]$) with the function $f(t) = 5e^{-0.5t} sin^2(\pi t)$. For the generated data, $\widetilde{V}(y) = 210.25$. Figs 6a)-6c) illustrate the best smoothing obtained by solving (26) for different values of $\gamma \in \{0.5, 0.2, 0.1\}$ and $n = 21$; Figs 7a)-7b) show the best smoothing for $\gamma \in \{0.2, 0.1\}$ and $n = 51$.

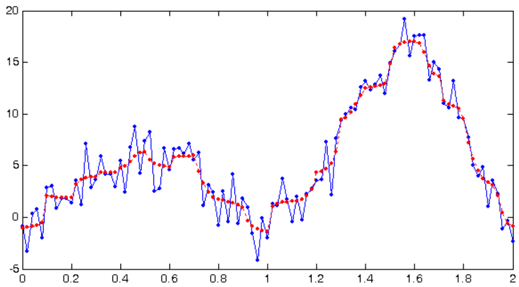

Figure 6a) Best smoothing fit with $n = 21, \gamma = 0.5$ $(r = 1)$

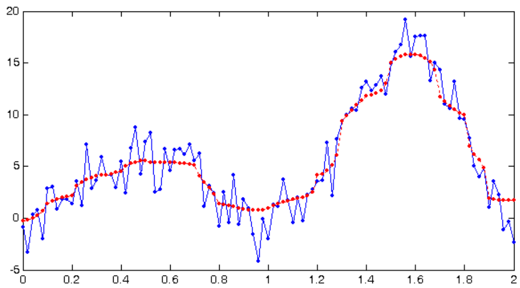

Figure 6b) Best smoothing fit with $n = 21, \gamma = 0.2$ $(r = 2)$

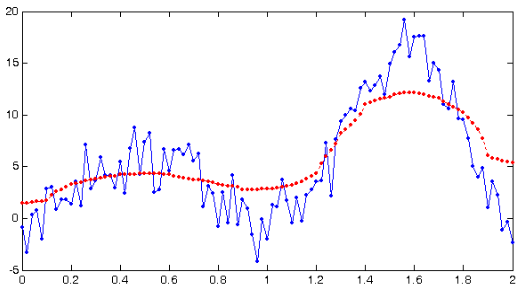

Figure 6c) Best smoothing fit with $n = 21, \gamma = 0.1$ $(r = 6)$

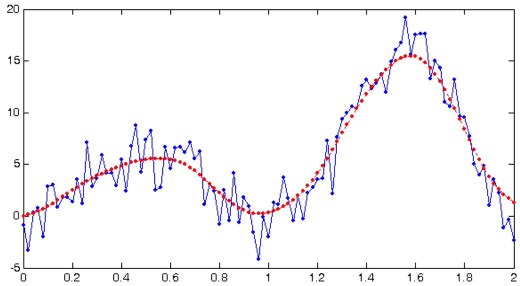

Figure 7a) Best smoothing fit with $n = 51, \gamma = 0.2$ $(r = 4)$

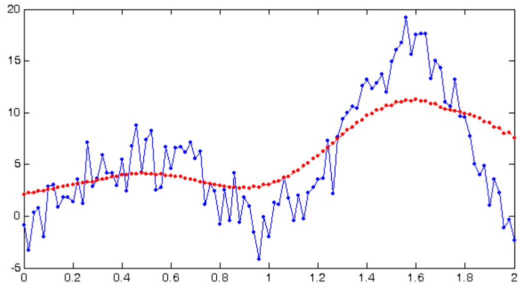

Figure 7b) Best smoothing fit with $n = 51, \gamma = 0.1$ $(r = 8)$

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