Towards F-transform of a Higher Degree

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Abstract— The aim of this contribution is to show how the F-transform technique can be generalized from the case of constant components to the case of polynomial components. For this purpose, we choose complete functional spaces with inner products. After a general presentation of F^m-transform where $m \geq 0$, a detailed characterization of the F^1-transform is given. We applied a technique of numeric integration in order to simplify computation of F^1-transform components. The inverse F^m-transform, $m \geq 0$, is defined in the same way as the ordinary F-transform.

Keywords— F-transform, F^m-transform, fuzzy partition, orthogonal basis, Gaussian quadrature

1 Introduction

The goal of this paper is to provide a deeper analysis of fuzzy modeling and its contribution to general mathematics. In [1], we introduced the notion of a fuzzy transform (F-transform, for short) which explains modeling with fuzzy IF-THEN rules as a specific transformation. This enabled us to compare the success of fuzzy modeling with the success of classical transforms (Fourier, Laplace, integral, wavelet etc.). From this point of view, fuzzy transforms contribute to the theory of linear and, in particular, integral transforms. Moreover, they initiated a theory of semi-linear spaces (see [2]).

In [1], the approximation property of fuzzy transform has been described and then, in [3], it has been shown how shapes of basic functions influence the approximation quality. The F-transform has many other useful properties and great potential for various applications, such as special numerical methods, solution of ordinary and partial differential equations with fuzzy initial condition [4, 5], mining dependencies from numerical data [6], applications to signal processing, compression and decompression of images [7, 8], and fusion of images [9].

We aim at showing how the F-transform technique can be generalized in the sense that constant components considered as polynomials of 0 degree are replaced by polynomials of degree $m \geq 1$. For this purpose, we choose complete functional spaces with inner products. After presentation of the generalized approach, a detailed characterization of the F^1-transform is given. We applied a technique of numeric integration in order to simplify computation of F^1-transform components. The inverse F^m-transform, $m \geq 1$, is defined in the same way as the inverse F-transform.

The paper is organized as follows: in Section 2, we introduce the concept of F^m-transform of finite degree $m \geq 0$, and show some of its properties and its relation to the ordinary F-transform. In Section 3, F^1-transform is introduced in details. Moreover, a simplification of F^1-transform components computation is discussed. The inverse F^m-transform is discussed in Section 4.

2 F-transform of an Arbitrary Finite Degree

Let us recall that the (direct) F-transform of an integrable function is a certain vector with real components. In [1], we proposed various formulas which represent components of the F-transform and showed a relationship between a given function and its F-transform. In this contribution, we propose to generalize our view on the F-transform and consider it as a vector of components that are orthogonal projections of an original function onto a linear subspace of certain functions that have polynomial representation. In the particular, if the degree of polynomials is zero, we obtain the originally proposed F-transform which will be referred to as F^0-transform in the sequel.

2.1 Fuzzy partition

Let $[a, b]$ be an interval on the real line $\mathbb{R}$. Fuzzy sets on $[a, b]$ will be identified with their membership functions, i.e. mappings from $[a, b]$ into $[0, 1]$. The notion of fuzzy partition is a principle one for our construction so that we will repeat it from [1] and slightly change below.

Definition 1
Let $[a, b]$ be an interval on $\mathbb{R}$, $n \geq 2$ and $x_1, \dots, x_n$ nodes such that $a = x_1 < \dots < x_n = b$. We say that fuzzy sets $A_1, \dots, A_n$, identified with their membership functions, constitute a fuzzy partition *of $[a, b]$ if for $k = 1, \dots, n$, they fulfill the following conditions :*

1. *$A_k : [a, b] \longrightarrow [0, 1]$, $A_k(x_k) = 1$;*
2. *for $k = 2, \dots, n$, $A_k(x) = 0$ if $x \in [a, x_{k-1}]$ and for $k = 1, \dots, n-1$, $A_k(x) = 0$ if $x \in [x_{k+1}, b]$,*
3. *$A_k(x)$ is continuous;*
4. *for $k = 2, \dots, n$, $A_k(x)$ strictly increases on $[x_{k-1}, x_k]$ and for $k = 1, \dots, n-1$, $A_k(x)$ strictly decreases on $[x_k, x_{k+1}]$;*
5. *for all $x \in [a, b]$*

$$\sum_{k=1}^{n} A_k(x) = 1. \tag{1}$$

The membership functions $A_1, \dots, A_n$ are called basic functions.

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Let us extend $[a, b]$ by two extra nodes $x_0 = 2a - x_2$ and $x_{n+1} = 2b - x_{n-1}$ and for $x \in [x_0, a)$ define $A_1(x) = A_1(2a - x)$, and for $(b, x_{n+1}]$ define $A_n(x) = A_n(2b - x)$. Further on we will always assume that $[a, b]$ and membership functions A_1, A_n are extended.

We say that the fuzzy partition $A_1, \ldots, A_n$, $n \geq 3$, is h-*uniform* if nodes $x_0, \ldots, x_{n+1}$ are h-equidistant, i.e. for all $k = 1, \ldots, n$, $x_k = a + h(k-1)$ where $h = (b-a)/(n-1)$, and two additional properties are met:

6. $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0, h]$, $k = 1, \ldots, n$,
7. $A_k(x) = A_{k-1}(x - h)$, for all $k = 2, \ldots, n$ and $x \in [x_{k-1}, x_{k+1}]$.

2.2 F-transform in a space of functions with scalar product

Let us fix $[a, b]$, $n \geq 3$, nodes $x_0 < \ldots < x_{n+1}$ and fuzzy partition $A_1, \ldots, A_n$ of $[a, b]$. For every $k = 1, \ldots, n$, let us denote $L_2(A_k)$ a set of functions $f : [x_{k-1}, x_{k+1}] \longrightarrow \mathbb{R}$ for which the following integral

$$\int_{x_{k-1}}^{x_{k+1}} f(x)^2 A_k(x) dx$$

exists. Let

$$(f, g)_k = \int_{x_{k-1}}^{x_{k+1}} f(x) g(x) A_k(x) dx, \qquad (2)$$

be a weighted scalar product of functions f and g. Then $L_2(A_k)$ is a *linear space of functions with scalar product*.

For every integer $m \geq 0$, let $\varphi_0^k, \varphi_1^k, \ldots, \varphi_m^k \in L_2(A_k)$, be an orthogonal system of polynomials where $\varphi_0^k = 1$ and orthogonality is considered with respect to (2). Denote $L_2^m(A_k)$ a linear subspace of $L_2(A_k)$ with the basis $\varphi_0^k, \varphi_1^k, \ldots, \varphi_m^k$.

The following trick allows us to extend arbitrary function $f : [a, b] \longrightarrow \mathbb{R}$ to function $f^{ex} : [x_0, x_{n+1}] \longrightarrow \mathbb{R}$:

$$f^{ex}(x) = \begin{cases} f(a-x) = 2f(a) - f(a+x), & \text{if } x \in [0, x_2 - a], \\ f(x), & \text{if } x \in [a, b], \\ f(b+x) = 2f(b) - f(b-x), & \text{if } x \in [0, b - x_{n-1}]. \end{cases}$$

Definition 2
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a given function and f^{ex} its extension to $[x_0, x_{n+1}]$. Assume that for all $k = 1, \ldots, n$, $f^{ex}|_{[x_{k-1}, x_{k+1}]} \in L_2(A_k)$. Let F_k^m be the k-th orthogonal projection of f^{ex} on $L_2^m(A_k)$, $k = 1, \ldots, n$. Then the n-tuple $[F_1^m, \ldots, F_n^m]$ of functions is the F^m-transform of f with respect to $A_1, \ldots, A_n$. Every function F_k^m, $k = 1, \ldots, n$, is called the F^m-transform component.

Remark 1
Definition 2 does not provide us with a formal representation of components $F_1^m, \ldots, F_n^m$. Let us show how they can be obtained. According to the definition above, every component F_k^m minimizes the scalar product $(f^{ex} - g, f^{ex} - g)_k$ where g is an arbitrary function from $L_2^m(A_k)$. Taking into account that $\varphi_0^k, \varphi_1^k, \ldots, \varphi_m^k$ is the basis of $L_2^m(A_k)$, we can represent g by a linear combination $x_0\varphi_k^0 + x_1\varphi_k^1 + \ldots + x_m\varphi_k^m$ of basis functions. Assume that $c_0, c_1 \ldots, c_m$ are coefficients that minimize the following integral

$$\int_{x_{k-1}}^{x_{k+1}} (f^{ex}(x) - (x_0\varphi_k^0(x) + x_1\varphi_k^1(x) + \ldots + x_m\varphi_k^m(x)))^2 A_k(x) dx. \qquad (3)$$

Hence, $F_k^m = c_0\varphi_k^0 + c_1\varphi_k^1 + \ldots + c_m\varphi_k^m$.

Below, we will prove the following fact: the original F-transform (see, e.g. [1]) is actually (up to the first and the last components) the F^0-transform. This requires to show that every F-transform component F_k, $k = 2, \ldots, n-1$, that has been originally introduced by

$$F_k = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x) A_k(x) dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx}, \qquad (4)$$

is the k-th orthogonal projection of f on $L_2^0(A_k)$. In order to prove this fact, we recall that the basis of $L_2^0(A_k)$ consists of the constant function φ_k^0 that has the value 1. Then the assertion above immediately follows from the proposition given below which has been proved in [1]:

Theorem 1
Let f be a continuous function on $[a, b]$ and $A_1, \ldots, A_n$ be basic functions which constitute a fuzzy partition of $[a, b]$. Then the k-th component F_k (4) of the F-transform gives minimum to the function

$$\Phi(y) = \int_a^b (f(x) - y)^2 A_k(x) dx$$

defined on $[f(a), f(b)]$.

3 F^1-transform

On the example of F^1-transform, we will show how an arbitrary component of F^m, $m \geq 1$, can be computed. Actually, one possibility of computation directly follows from minimization of integral (3) (see Remark 1). We will use another approach which corresponds to the definition of F^1-transform.

Throughout this section, we fix an h-uniform partition $A_1, \ldots, A_n$ of $[a, b]$ where $n \geq 3$, and assume that $f : [a, b] \longrightarrow \mathbb{R}$ is a function such that for all $k = 1, \ldots, n$, $f^{ex}|_{[x_{k-1}, x_{k+1}]} \in L_2(A_k)$. The explanation below will be given for an arbitrary k, $k = 1, \ldots, n$.

3.1 F^1-transform components

Since component F_k^1 of the F^1-transform is the k-th orthogonal projection of f^{ex} on $L_2^m(A_k)$, and $L_2^m(A_k)$ is a linear subspace of $L_2(A_k)$ with orthogonal basis φ_k^0, φ_k^1, we will begin this subsection with a characterization of orthogonal polynomials φ_k^0, φ_k^1 of degrees 0 and 1.

Lemma 1
Polynomials $\varphi_k^0, \varphi_k^1 : [x_{k-1}, x_{k+1}] \longrightarrow \mathbb{R}$ such that for all $x \in [x_{k-1}, x_{k+1}]$, $\varphi_k^0(x) = 1$, $\varphi_k^1(x) = x - x_k$ are orthogonal with weight A_k, $k = 1, \ldots, n$.

PROOF: The proof is technical and follows from the following assertions:

(i) $\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx = h$,

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(ii) $\int_{x_{k-1}}^{x_{k+1}} xA_k(x)dx = hx_k$,

and properties of the uniform partition. □

Remark 2
Orthogonal polynomials of degrees 0 *and* 1*, described in Lemma 1, are invariant to shapes of* A_k*.*

Theorem 2
Under the assumptions above, the F^1*-transform of* f *with respect to* $A_1, \ldots, A_n$ *is the vector* $[F_1^1, \ldots, F_n^1]$ *of linear functions such that an arbitrary component* F_k^1*,* $k = 1, \ldots, n$*, is represented as follows:*

$$F_k^1(x) = c_k^0 + c_k^1(x - x_k), \quad x \in [x_{k-1}, x_{k+1}]$$

where

$$c_k^0 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{h}, \tag{5}$$

$$c_k^1 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x)dx}. \tag{6}$$

PROOF: The proof will be given for one fixed component F_k^1 where $k = 1, \ldots, n$. By Definition 2, F_k^1 is the k-th orthogonal projection of f^{ex} on $L_2^1(A_k)$ where the orthogonality is determined by (2). Therefore,

$$f^{ex}|_{[x_{k-1},x_{k+1}]} = c_k^0 + c_k^1(x - x_k) + R_k(x)$$

where R_k is orthogonal to each basis functions φ_k^0 and φ_k^1. Therefore,

$$c_k^0 = \frac{(f^{ex}, \varphi_k^0)_k}{(\varphi_k^0, \varphi_k^0)_k},$$

$$c_k^1 = \frac{(f^{ex}, \varphi_k^1)_k}{(\varphi_k^1, \varphi_k^1)_k}$$

which after substitution gives the required expressions for c_k^0 and c_k^1. □

Corollary 1
Under the assumptions above, $c_k^0 = F_k$ *where* F_k *is the* k*-th component (4) of the ordinary* F*-transform.*

PROOF: The proof follows from expressions (4), (6) and equality $\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx = h$. □

3.2 *Simplification of* F^1*-transform components computation*

In this subsection, we will show how Gaussian quadratures and the properties of orthogonal polynomials can be used for replacing integral $\int_{x_{k-1}}^{x_{k+1}} (x-x_k)^2 A_k(x)dx$ in the denominator of c_k^1 by its precise value.

At first, let us recall the notion of Gaussian quadratures (see e.g.[10]). The approximate equality

$$\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx \approx \frac{h}{2}\sum_{i=1}^{N} d_i f(t_i) \tag{7}$$

which is precise for all polynomials of the highest possible degree is a Gaussian quadrature. We will put $N = 2$ and characterize arguments t_1, t_2 and the highest possible degree of polynomials which turn (7) into a precise equality. The following statement (see e.g.[10]) gives the required characterization: if

(i) t_1, t_2 are roots of polynomial φ_k^2 which is orthogonal to φ_k^0 and φ_k^1, and

(ii) equality

$$\int_{x_{k-1}}^{x_{k+1}} P_l(x)A_k(x)dx = h(d_1 P_l(t_1) + d_2 P_l(t_2)), \tag{8}$$

holds true for some coefficients d_1, d_2 and all polynomials P_l of degrees $l = 0, 1$,

then (8) holds true for all polynomials P_l of degrees $0 \leq l \leq 3$.

Thus, our next purpose is to find a polynomial of the degree 2, orthogonal to φ_k^0 and φ_k^1, as well as to find its roots. Due to positivity and symmetry of A_k, two lemmas below hold true.

Lemma 2
If t_1*,* t_2 *are symmetrical with respect to* x_k *then equalities*

$$\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx = h(\frac{1}{2} + \frac{1}{2}) = h,$$

$$\int_{x_{k-1}}^{x_{k+1}} (x - x_k)A_k(x)dx = \frac{h}{2}((t_1 - x_k) + (t_2 - x_k)) = 0,$$

hold true for $l = 0, 1$*.*

Lemma 3
If we denote

$$I_2 = \int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x)dx$$

then polynomial φ_k^2*, represented by*

$$\varphi_k^2(x) = (x - x_k)^2 - \frac{I_2}{h} \tag{9}$$

is orthogonal to φ_k^0 *and* φ_k^1 *on* $[x_{k-1}, x_{k+1}]$*.*

The roots t_1, t_2 *of* φ_k^2 *belong to* (x_{k-1}, x_{k+1}) *and are symmetrical with respect to* x_k*, i.e. for some* δ*,* $t_1 = x_k - \delta$ *and* $t_2 = x_k + \delta$*.*

The statement above together with Lemmas 2, 3 leads to the equality

$$\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x)dx = h\delta^2.$$

Thus, the desired simplified representation of c_k^1 is as follows:

$$c_k^1 = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k)A_k(x)dx}{h\delta^2}.$$

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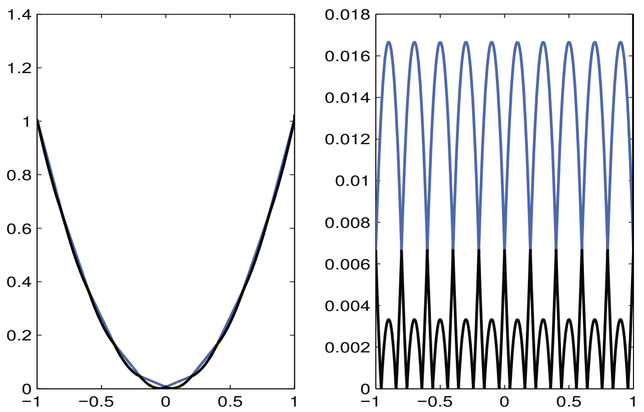

Figure 1: *Left.* The function x^2 and its inverse F^0 (gray line) and F^1 (black line) transforms. *Right.* Graphs of the error functions. Maximal errors of approximation are: 0.017 (the inverse F^0-transform) and 0.062 (the inverse F^1-transform).

4 Inverse F^m-transform

Similarly to the ordinary F-transform, the inverse F^m-transform is defined as a linear combination of the basic functions with "coefficients" given by the F^m-transform components.

Definition 3
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a given function and f^{ex} its extension to $[x_0, x_{n+1}]$ such that for all $k = 1, \ldots, n$, $f^{ex}|_{[x_{k-1},x_{k+1}]} \in L_2(A_k)$. For a certain $m \geq 0$, let $[F_1^m, \ldots, F_n^m]$ be the F^m-transform of f with respect to $A_1, \ldots, A_n$. Then the function

$$f_{F,m,n}(x) = \sum_{k=1}^{n} F_k^m A_k(x) \qquad (10)$$

is called the inverse F^m-transform.

The following recurrent formula easily follows from Definition 3 and the whole structure of the F^m-transform, $m \geq 1$:

$$f_{F,m,n}(x) = f_{F,m-1,n}(x) + \sum_{k=1}^{n} c_m \varphi_k^m(x) A_k(x). \qquad (11)$$

By Remark 1, the components F_k^m, $k = 1, \ldots, n$, $m \geq 0$, are the best approximation of f among all elements represented by linear combinations $a_0\varphi_k^0 + a_1\varphi_k^1 + \ldots + a_m\varphi_k^m$. Therefore, for each $k = 1, \ldots, n$, F_k^m is a better approximation of f than F_k^{m-1}. This observation together with (11) implicitly demonstrates that the quality of approximation of f by $f_{F,m,n}$ is better than that of approximation by $f_{F,m-1,n}$. The same conclusion follows also from our experiments (see Figure 1). The proof of this assertion is a matter of future investigation.

5 Conclusion

We have generalized the F-transform technique to the case where its components are polynomials. A detailed characterization of the F^1-transform with linear components was given. We have shown how a computation of F^1-transform components can be simplified if the technique of Gaussian quadratures is used. The inverse F^m-transform, $m \geq 1$, is defined in the same way as the inverse F-transform.

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