Some chaotic and mixing properties of Zadeh's extension

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Abstract— *Let X be a compact metric space, let φ be a continuous self-map on X, and let $\mathbb{F}(X)$ denote the space of fuzzy sets on X equipped with the levelwise topology. In this paper we study relations between various dynamical properties of a given (crisp) dynamical system (X, φ) and its Zadeh's extension Φ on $\mathbb{F}(X)$. Among other things we study various (weak, strong, mild etc.) mixing properties and also several kinds of chaotic behaviors (Li-Yorke chaos, ω-chaos, distributional chaos, topological chaos etc.).*

Keywords— Zadeh's extension, fuzzification, chaos, mixing, transitivity, topological entropy.

1 Introduction

Throughout this paper, let (X, d_X) be a compact metric space and let $C(X)$ denote the space of continuous maps $\varphi : X \to X$. A discrete dynamical system is a pair (X, φ). For other notions and notations mentioned in this section, we refer to Section **??**. It is well known ([**?**]) that the discrete dynamical system (X, φ) naturally induces a dynamical system $(\mathbb{F}(X), \Phi)$ on the space $\mathbb{F}(X)$ of all fuzzy compact subsets of X. The map Φ is called the *fuzzification* (or *Zadeh's extension*) (see (**??**)).

It is natural to ask the following **question:** *how is the dynamical complexity of the fuzzified (resp. crisp) dynamical system related to the dynamical properties of the original (resp. fuzzy) one*. There are only a few papers devoted to this question so far – for example, [**?**], [**?**] and [**?**], where different chaotic properties of fuzzy discrete dynamical systems were considered.

In this paper, we consider the space $\mathbb{F}(X)$ of upper semi-continuous fuzzy sets with compact supports. This space is equipped with the topology induced by the levelwise metric d_∞ (see (**??**)), since this topology is stronger than the other topologies commonly used in fuzzy topological dynamics (e.g. see [**?**]). We especially deal with the subspace $\mathbb{F}^1(X) \subseteq \mathbb{F}(X)$ of all normal fuzzy sets on X (see (**??**)). The reason for this is the following: no fuzzification $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ admits one of the simplest chaotic behaviors (namely the transitivity, see Proposition **??**) and, consequently, it does not admit more complex behavior.

This paper is a partial answer to the question mentioned above. Our results concerning the most commonly used chaotic and mixing properties can be summarized as follows:

- if P denotes either the distributional or Li-Yorke or topological or ω-chaos then

$$(\varphi \text{ has } P \Rightarrow \Phi \text{ has } P), \text{ but } (\Phi \text{ has } P \not\Rightarrow \varphi \text{ has } P), \quad (1)$$

- if P denotes either transitivity or total transitivity, then

$$(\Phi \text{ has } P \Rightarrow \varphi \text{ has } P), \text{ but } (\varphi \text{ has } P \not\Rightarrow \Phi \text{ has } P), \quad (2)$$

- if P denotes one of the following properties: exactness, sensitive dependence, weak mixing, mild mixing, or strong mixing, then

$$(\Phi \text{ has } P \Rightarrow \varphi \text{ has } P), \quad (3)$$

but the validity of the converse implication is unknown.

This paper is organized as follows: in Section **??**, we introduce notation and definitions used in this paper. Then, in Section **??**, some preliminary results are proven, showing also connections between the set-valued and fuzzified system induced by the same original system. Finally, the chaotic and mixing properties are studied in Section **??**.

2 Definitions and notation

Further we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of integers and real numbers, respectively. Now we define some classic notions from topological dynamics. For a given dynamical system (X, φ) and a given point $x \in X$, we define the n-th *iteration* of the point x inductively by $\varphi^0(x) = x$, $\varphi^{n+1}(x) = \varphi(\varphi^n(x))$ for any $n \in \mathbb{N}$. Then, the sequence $\{\varphi^n(x)\}_{n\in\mathbb{N}}$ of all iterations of x is called the *trajectory* of the point x. Any limit point of the trajectory of the point x is called an ω*-limit point* of the point x, and the union $\omega_\varphi(x)$ of all ω-limit points of the point x is the ω*-limit set* of the point x. The iterations of a given set $A \subseteq X$ are defined analogously. The point $x \in X$ is called *fixed* if $\varphi(x) = x$ or *periodic* if $\varphi^k(x) = x$ for some $k \in \mathbb{N}$. We denote by $\omega(\varphi)$, $P(\varphi)$ and $Fix(\varphi)$ the set of ω-limit, periodic and fixed points, respectively.

A map $\varphi \in C(X)$ is called *transitive* if for any non-empty open subsets $U, V \subseteq X$, there exists some $k \in \mathbb{N}$ such that $\varphi^k(U) \cap V \neq \emptyset$. The map φ is *totally transitive* if the n-th iteration of φ is transitive for any $n \in \mathbb{N}$. The map φ is *weakly mixing* if the product map $\varphi \times \varphi$ is transitive. The map φ is *strongly mixing* if for any non-empty open subsets $U, V \subseteq X$ there exists some $m \in \mathbb{N}$ such that $\varphi^k(U) \cap V \neq \emptyset$ for any $k \geq m$. The map φ is *topologically exact* (or simply *exact*) if for any non-empty subset $U \subseteq X$, there exists some $k \in \mathbb{N}$, such that $\varphi^k(U) = X$.

2.1 Chaotic properties

When defining chaotic properties we follow the notation introduced in [**?**]. The notion of distributional chaos was introduced in [**?**]. For any $x, y \in X$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$, set

$$\xi(x, y, t, n) = \#\{i,\ 0 \leq i < n \wedge d(\varphi^i(x), \varphi^i(y)) < t\}. \quad (4)$$

Set

$$F^\star_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \xi(x, y, t, n) \quad (5)$$

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and

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n). \qquad (6)$$

Obviously, both maps $F^\star_{xy}$ and F_{xy} are nondecreasing, $0 \le F_{xy}(t) \le F^\star_{xy}(t) \le 1$ for all $t \in \mathbb{R}$, $F^\star_{xy}(t) = 0$ if $t \le 0$ and $F^\star_{xy}(t) = 1$ if $t \ge diam(X)$. The map $F^\star_{xy}$ ($F_{xy}(t)$) is an upper (a lower) distribution function for $x, y \in X$.

The map φ is *distributionally chaotic* of type 1 (d_1C) if $F^\star_{xy} \equiv 1$ and $F_{xy}(t) = 0$ for some $t > 0$. The map φ is *distributionally chaotic* of type 2 (d_2C) if $F^\star_{xy} \equiv 1$ and $F^\star_{xy}(t) > F_{xy}(t)$ for some $t > 0$. Finally, the map φ is *distributionally chaotic* of type 3 (d_3C) if $F^\star_{xy}(t) > F_{xy}(t)$ for all $t \in J$, where J is a nondegenerate interval.

It follows from the definition that

$$d_1C \Rightarrow d_2C \Rightarrow d_3C. \qquad (7)$$

However, the converse implications are not valid (see, for instance, [**?**] and [**?**]).

Two points $x, y \in X$ form a *Li-Yorke pair* if

$$\limsup_{n\to\infty} d_X(\varphi^n(x), \varphi^n(y)) > 0 \qquad (8)$$

and

$$\liminf_{n\to\infty} d_X(\varphi^n(x), \varphi^n(y)) = 0. \qquad (9)$$

A set $S \subseteq X$ is a *LY-scrambled set* for the map φ if $\#S \ge 2$ and every pair from S is Li-Yorke. The map φ is *Li-Yorke chaotic* (shortly *LYC*) if there exists an uncountable LY-scrambled set.

A map $\varphi \in C(X)$ is ω*-chaotic* ([**?**]) (shortly ωC) if there exists an uncountable ω*-scrambled* set $S \subseteq X$, i.e. for any two points $x, y \in S$, the following conditions are satisfied: (i) $\omega_\varphi(x) \setminus \omega_\varphi(y)$ is uncountable, (ii) $\omega_\varphi(x) \setminus \omega_\varphi(y) \ne \emptyset$ and (iii) $\omega_\varphi(x) \cap P(\varphi) \ne \emptyset$.

If a map $\varphi : X \to X$ is transitive and $P(\varphi)$ is dense in X then φ is called *Devaney chaotic*. It should be mentioned that in the original definition of Devaney, φ *depends sensitively on initial conditions*,i.e. there exists $\delta > 0$ such that for any $x \in X$ and any open neighborhood U of x there is $y \in U$ satisfying $d_X(\varphi^k(x), \varphi^k(y)) > \delta$ for some $k \in \mathbb{N}$. But it was proved that this condition is implied by the transitivity and density of periodic points (see [**?**] and [**?**]).

The notion of positive topological entropy was firstly defined by Bowen ([**?**]). The topological entropy of a map φ is a number $h(\varphi) \in [0, \infty]$, defined by

$$h(\varphi) = \lim_{\varepsilon > 0} \limsup_{n\to\infty} \#E(n, \varphi, \varepsilon), \qquad (10)$$

where $E(n, \varphi, \varepsilon)$ is a $(n, \varphi, \varepsilon)$*-span* with a minimal possible number of points, i.e. a set such that for any $x \in X$ there exists a $y \in E(n, \varphi, \varepsilon)$ satisfying $d(\varphi^k(x), \varphi^k(y)) < \varepsilon$ for any j, $1 \le j \le n$. A map φ is *topologically chaotic* (shortly PTE) if $h(\varphi) > 0$. It is well-known that the topological entropy is monotone in the following way: for any $A, B \subseteq X$,

$$A \subseteq B \quad \Rightarrow \quad h(\varphi|_A) \le h(\varphi|_B). \qquad (11)$$

A map $\varphi \in C(X)$ has the *specification property* if for any $\varepsilon > 0$ there is a positive $M \in \mathbb{N}$ such that for any integer $k \ge 2$ and any k points $x_i \in X$, $i = 1, 2, \dots, k$ and any $2k$ integers $a_1 \le b_1 < a_2 \le b_2 < \dots < a_k \le b_k$ with $a_i - b_{i-1} \ge M$, there exists $z \in X$ for which

$$d(\varphi^n(z), \varphi^n(x_i)) < \varepsilon \qquad (12)$$

for any $n = a_i, \dots, b_i$ and any $i = 1, 2, \dots, k$.

The following implications are currently known among the chaotic and mixing properties mentioned above:

$$\begin{gathered}\text{specification property} \Rightarrow \text{strong mixing} \Rightarrow \\ \text{mild mixing} \Rightarrow \text{weak mixing} \Rightarrow \\ \text{total transitivity} \Rightarrow \text{transitivity}.\end{gathered} \qquad (13)$$

For further details and relations among the chaotic properties, we refer to [**?**] and to the references therein.

2.2 *Metric spaces of fuzzy sets*

Let (X, d) denote a compact metric space, and let A, B be non-empty closed subsets of X. The *Hausdorff metric* D_X between A and B is defined, as usual, by

$$D_X(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq U_\varepsilon(B) \wedge B \subseteq U_\varepsilon(A)\}, \quad (14)$$

where

$$U_\varepsilon(A) = \{x \in X \mid D(x, A) < \varepsilon\}, \qquad (15)$$

and

$$D(x, A) = \inf_{a\in A} d(x, a). \qquad (16)$$

By $\mathbb{K}(X)$ we denote the space of all nonempty compact subsets of X, equipped with the Hausdorff metric D_X. It is well known (c.f. [**?**]) that $(\mathbb{K}(X), D_X)$ is compact, complete and separable whenever X is compact, complete and separable.

A fuzzy set A on the space X is a function $A : X \to I$ where I denotes the closed unit interval $[0, 1]$. The α*-cuts* (or the α*-level sets*) $[A]_\alpha$ and the *support* $supp(A)$ of a given fuzzy set A are defined as usual by -

$$[A]_\alpha = \{x \in X \mid A(x) \ge \alpha\}, \ \alpha \in [0, 1], \qquad (17)$$

and

$$supp(A) = \overline{\{x \in X \mid A(x) > 0\}}. \qquad (18)$$

Further, we define $\mathbb{F}(X)$ as the system of all upper semi-continuous fuzzy sets $A : X \to I$ having compact supports. Moreover, let

$$\mathbb{F}^1(X) = \{A \in \mathbb{F}(X) \mid A(x) = 1 \text{ for some } x \in X\} \quad (19)$$

denote the system of all *normal* fuzzy sets on X. Finally, we define $\emptyset_X$ as the *empty fuzzy set* ($\emptyset_X(x) = 0$ for each $x \in X$) on the space X, and $\mathbb{F}_0(X)$ as the system of all nonempty fuzzy sets.

Let us define a *levelwise* metric d_∞ on $\mathbb{F}_0(X)$ by

$$d_\infty(A, B) = \sup_{\alpha\in(0,1]} D_X([A]_\alpha, [B]_\alpha). \qquad (20)$$

This equality defines the levelwise metric correctly only for non-empty fuzzy sets $A, B \in \mathbb{F}_0(X)$ whose maximal values are identical, since the Hausdorff distance D_X is only measured between two non-empty closed subsets of the space X.

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Thus, we consider the following extension of the Hausdorff metric D_X:

$$D_X(\emptyset, \emptyset) = 0 \text{ and } D_X(\emptyset, A) = diam\,(X) \quad (21)$$

for any $A \in \mathbb{K}(X)$. With this extension, (**??**) correctly defines the levelwise metric on $\mathbb{F}(X)$. It is obvious that

$$d_\infty(\emptyset_X, \emptyset_X) = 0 \text{ and } d_\infty(\emptyset_X, A) = diam\,(X) \quad (22)$$

for any $A \in \mathbb{F}_0(X)$.

It should be noted that the metric d_∞ is one of the three most commonly used metrics in fuzzy topological dynamics. We also recall that the metric space $(\mathbb{F}(X), d_\infty)$ is complete but is not separable and not compact and that the levelwise topology induced by d_∞ is stronger than the remaining (*sendograph* and *endograph*) ones. For more details we refer to [**?**] and to the references therein.

2.3 Zadeh's extension

Let X be a compact metric space and $\varphi \in C(X)$. Then a *fuzzification* (or *Zadeh's extension*) of the (crisp) dynamical system (X, φ) is a map $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ defined by

$$(\Phi(A))(x) = \sup_{y \in \varphi^{-1}(x)} \{A(y)\} \quad (23)$$

for any $A \in \mathbb{F}(X)$ and $x \in X$.

It is shown recently by [**?**] that, if X is a compact metric space, then the fuzzification $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ is continuous if and only if $\varphi : X \to X$ is continuous. The last statement was generalized about the case of locally compact metric spaces in [**?**] recently.

It is known that, for any $\alpha \in (0, 1]$ and any $A \in \mathbb{F}(X)$,

$$\varphi([A]_\alpha) = [\Phi(A)]_\alpha. \quad (24)$$

Similarly, $\varphi(supp(A)) = supp(\Phi(A))$ holds.

3 Preliminary results

Inspired by the results mentioned, for instance, in [**?**], we define some basic properties of generalized extensions. For any $U \subseteq X$, we define

$$e(U) = \{B \in \mathbb{F}(X) \mid supp(B) \subseteq U\} \quad (25)$$

It is obvious that $e(U) \neq \emptyset$ if and only if $U \neq \emptyset$. Moreover, we have the following assertion (Lemma **??**) that was partially proved in [**?**]:.

Lemma 1 *A subset U is a non-empty open subset of X if and only if $e(U)$ is a non-empty open subset of $\mathbb{F}(X)$.*

Proof. Since the implication "⇒" has been proven in [**?**], the converse remains to be proven. So let $e(U)$ be a non-empty open subset of $(\mathbb{F}(X), d_\infty)$. Assume by contradiction that U is not open. Take any $x \in U \setminus int(U)$ and consider a fuzzy set $\chi_{\{x\}}$. Then, for any $\varepsilon > 0$, an open ε-neighborhood $V \subseteq X$ of x intersects the exterior of U. Consequently, χ_V is ε-close to $\chi_{\{x\}}$ (by the definition of d_∞), but $\chi_V \notin e(U)$. Thus no ε-neighborhood V of $\chi_{\{x\}}$ is a subset of $e(U)$. This contradicts the fact that $e(U)$ is open in $(\mathbb{F}(X), d_\infty)$. □

Lemma 2 (Representation theorem of Negoita-Ralescu, e.g. [**?**]) *Consider a family $\{B_\alpha \mid \alpha \in [0, 1]\}$ of closed subsets of X satisfying the following two conditions:*

(a) $B_\beta \subseteq B_\alpha \subseteq B_0$ if $0 \leq \alpha \leq \beta$,

(b) if $\{\alpha_n\}$ is an increasing sequence in I converging to α_0 then $B_{\alpha_0} = \bigcap_{n \in \mathbb{N}} B_{\alpha_n}$.

Then there exists $B \in \mathbb{F}(X)$ such that $[B]_\alpha = B_\alpha$.

Conversely, if B is a fuzzy set on X then the system $\{B_\beta\}_{\beta \in I}$ defined by $B_\beta = [B]_\beta$ for any $\beta \in (0, 1]$ and $B_0 = supp(B)$ satisfies conditions (a) and (b).

Lemma 3 *Let U, V be two subsets of X and $\varphi \in C(X)$. Then*

(i) $e(U \cap V) = e(U) \cap e(V)$,

(ii) $\Phi(e(U)) \subseteq e(\varphi(U))$,

(iii) $\Phi(e(U)) = e(\varphi(U))$ whenever U is closed.

Proof. The statements (i) and (ii) were already proved in [**?**].

The statement (iii) still remains to be proven. The inclusion $\Phi(e(U)) \subseteq e(\varphi(U))$ in (iii) follows from (ii). Let us prove $e(\varphi(U)) \subseteq \Phi(e(U))$ if U is closed. Take any $A \in e(\varphi(U))$. We want to show that there exists $B \in e(U)$ such that $\Phi(B) = A$. Since $A \neq \emptyset_X$ and upper semi-continuous, there exists $\alpha_0 = \max_{x \in X}\{A(x) \mid A(x) > 0\}$. Moreover, for any $\alpha \in (0, \alpha_0]$, $[A]_\alpha$ is nonempty, closed and, consequently by the continuity of φ, $\varphi^{-1}([A]_\alpha) \cap U$ is also nonempty and closed.

By the definition of the fuzzification Φ, $\max(A) = \max(B)$ whenever $B \in \mathbb{F}(X)$ is any preimage of A. So a fuzzy set $B \in e(U)$ can be defined as follows. For any $\beta \in (0, \alpha_0]$ we put

$$[B]_\beta = \varphi^{-1}([A]_\beta) \cap U. \quad (26)$$

We also put $[B]_0 = U$ and $[B]_\beta = \emptyset$ for any $\beta \in (\alpha_0, 1]$. Obviously the system $\{[B]_\beta\}_{\beta \in I}$ satisfies the condition (a) of Lemma **??**. We shall now show that the condition (b) of Lemma **??** is satisfied, i.e. $B \in \mathbb{F}(X)$. Then, by the definition of B, we obtain that $B \in e(U)$.

Assume that $\{\beta_n\} \subseteq I$ is an increasing sequence that converges to $\beta_0 \leq \alpha_0$. Suppose by contradiction that $[B]_{\beta_0} \neq \bigcap_{n \in \mathbb{N}} [B]_{\beta_n}$. By the monotonicity of $\{\beta_n\}$ and Lemma **??** (a), the only possibility is that $[B]_{\beta_0}$ is a proper subset of $\bigcap_{n \in \mathbb{N}} [B]_{\beta_n}$, i.e.

$$\Big(\bigcap_{n \in \mathbb{N}} [B]_{\beta_n}\Big) \setminus [B]_{\beta_0} \neq \emptyset. \quad (27)$$

Take any $x_0 \in \bigcap [B]_{\beta_n} \setminus [B]_{\beta_0}$ and then take $x_n \in [B]_{\beta_n} \setminus [B]_{\alpha_0}$ such that $\{x_n\}$ converges to x_0. Obviously, since $x_0 \notin [B]_{\beta_0}$ we obtain from (**??**) that

$$\varphi(x_0) \notin [A]_{\beta_0}. \quad (28)$$

On the other hand, $x_n \in [B]_{\beta_n}$ for any $n \in \mathbb{N}$, i.e.,

$$\varphi(x_n) \in [A]_{\beta_n}, \text{ for any } n \in \mathbb{N}. \quad (29)$$

Now the continuity of φ implies that $\{\varphi(x_n)\}$ converges to $\varphi(x_0)$. Hence (**??**) and (**??**) imply that $[A]_{\beta_0}$ is a proper subset of $\bigcap_{n \in \mathbb{N}} [A]_{\beta_n}$, i.e. $A \notin \mathbb{F}(X)$ by Lemma **??** – a contradiction. Thus, we have shown that $B \in \mathbb{F}(X)$.

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Finally, by the construction of B, we have $\Phi(B) = A$. $\square$

We will need some further notation. For any $\alpha \in (0, 1]$ and $U \subseteq X$, set

$$e_\alpha(U) = \{A \in \mathbb{F}(X) \mid [A]_\alpha \neq \emptyset \text{ and } [A]_\alpha \subseteq U\} \quad (30)$$

and

$$\vartheta(U) = e_1(U) \cap e(U). \quad (31)$$

Lemma 4 *For any $\alpha \in (0, 1]$, $e_\alpha(U)$ is open in $(\mathbb{F}(X), \tau_\infty)$ if and only if $U \subseteq X$ is open in X.*

Proof. Let $\alpha \in (0, 1]$ be fixed. We shall show that $e_\alpha(U)$ is open in $(\mathbb{F}(X), \tau_\infty)$ if U is open. Take any $A \in e_\alpha(U)$. Since $[A]_\alpha \subseteq U$ is closed and U is open, there exists an open ε-neighborhood V of $[A]_\alpha$ lying in U for any $\varepsilon > 0$. Consequently, by the definition of d_∞, if we consider an open ε-neighborhood $V' \subseteq (\mathbb{F}(X), d_\infty)$ of A with $\varepsilon < diam(X)$, then for any $B \in V'$,

$$[B]_\alpha \subseteq V \subseteq U. \quad (32)$$

Thus, there exists an open neighborhood V' of A in $e_\alpha(U)$, i.e., $e_\alpha(U)$ is open in $(\mathbb{F}(X), \tau_\infty)$.

Let us prove the converse implication. Assume by contradiction that U is not open and take any $x \in U \setminus int(U)$. Then, by the definition of $e_\alpha(U)$, $\chi_{\{x\}}$ belongs to $e_\alpha(U)$ but no ε-neighborhood of $\chi_{\{x\}}$ is a subset of $e_\alpha(U)$, i.e. $e_\alpha(U)$ is not open - a contradiction. $\square$

Corollary 1 *A subset $U \subseteq X$ is open in X if and only if $\vartheta(U)$ is open in $(\mathbb{F}(X), \tau_\infty)$ (and therefore also in $(\mathbb{F}^1(X), d_\infty)$).*

Lemma 5 *Let X be a compact metric space and $\varphi \in C(X)$ Then, for any $\alpha \in (0, 1]$ and $U, V \subseteq X$,*
(i) $e_\alpha(U \cap V) = e_\alpha(U) \cap e_\alpha(V)$,
(ii) $\Phi(e_\alpha(U)) \subseteq e_\alpha(\varphi(U))$.

Proof. Clearly, for any $\alpha \in (0, 1]$, $A \in e_\alpha(U \cap V)$ if and only if $[A]_\alpha \subseteq U \cap V$, if and only if $[A]_\alpha \subseteq U$ and $[A]_\alpha \subseteq V$, if and only if $A \in e_\alpha(U)$ and $A \in e_\alpha(V)$, if and only if $A \in e_\alpha(U) \cap e_\alpha(V)$. Thus, (i) holds.

Let us prove (ii). Consider any $A \in \Phi(e_\alpha(U))$. Then there exists $B \in e_\alpha(U)$ for which $\Phi(B) = A$. Since $[B]_\alpha \subseteq U$ it follows from (**??**) and from the continuity of φ that

$$\varphi([B]_\alpha) = [\Phi(B)]_\alpha = [A]_\alpha \subseteq \varphi(U), \quad (33)$$

i.e. $A \in e_\alpha(\varphi(U))$, and the inclusion is valid for any $\alpha \in (0, 1]$. $\square$

We are ready to modify Lemma **??** and to prove the next lemma, which is used for the further study of dynamics in the space of normal fuzzy sets on X. For completeness, we note the obvious fact that $\vartheta(U) \neq \emptyset$ if and only if $U \neq \emptyset$.

Lemma 6 *Let U, V be two subsets of X and $\varphi \in C(X)$. Then*
(i) $\vartheta(U \cap V) = \vartheta(U) \cap \vartheta(V)$,
(ii) $\Phi(\vartheta(U)) \subseteq \vartheta(\varphi(U))$,
(iii) $\Phi(\vartheta(U)) = \vartheta(\varphi(U))$ whenever U is closed.

Proof. The statements (i) and (ii) are easy consequences of Lemmas **??** and **??**. Moreover, the proof of the statement (iii) is a slight modification of the proof of Lemma **??** ($\max_{x \in X} A(x) = \alpha_0 = 1$). $\square$

At the end of this section we mention two simple properties of the usual fuzzification. By χ_A we denote the characteristic function of a given set $A \subseteq X$.

Lemma 7 *Let X be a compact metric space, $\varphi \in C(X)$ let Φ be a fuzzification of φ. Then*

$$\Phi(\alpha\chi_A) = \alpha\chi_{\varphi(A)} \quad (34)$$

for any $\alpha \in (0, 1]$.

Proof. Obvious. $\square$

Lemma 8 *Let X be a compact metric space. Then the map $i : (\mathbb{K}(X), D_X) \to (\mathbb{F}(X), d_\infty)$ defined by $i(A) = \chi_A$ for any $A \in \mathbb{K}(X)$ is an isometrical embedding.*

Proof. It is obvious that the map i is correctly defined. It follows directly from the definition of the metric d_∞ on the space of fuzzy sets that

$$d_X(x, y) = d_\infty(\chi_{\{x\}}, \chi_{\{y\}}) \quad (35)$$

for any $x, y \in X$, and also

$$D_X(U, V) = d_\infty(\chi_U, \chi_V) \quad (36)$$

for any $U, V \in \mathbb{K}(X)$. Thus, by Lemma **??**, i is injective, continuous and isometric. $\square$

Remark 1 *The same assertion is true when we consider either the sendograph or endograph topology instead of the levelwise one, since the equalities used in the proof are valid also in those topologies. Consequently, some results (Propositions **??** and **??**) presented in the next section are valid also in all mentioned topologies. The property mentioned in Lemma **??** was firstly mentioned in Kloeden's pioneering paper [**?**] in a little bit different setting.*

4 Chaotic and mixing properties.

Firstly we study some chaotic properties. It follows from Lemma **??** that any fuzzified system is an extension of the set-valued dynamical system $(\mathbb{K}(X), \tilde{\varphi})$ induced by the original dynamical system (X, φ). Consequently, we can use results published in [**?**] to easily provide some answers (Propositions **??** and **??**) to the question mentioned at the beginning of this paper.

Lemma 9 [**?**] *Let X be a compact metric space and $\varphi \in C(X)$.*

(i) If there is a set $S \subseteq X$ that is d_jC-scrambled ($j = 1, 2, 3$) (ω-scrambled, LY-scrambled, resp.) for the map φ then there exists d_jC-scrambled ($j = 1, 2, 3$) (ω-scrambled, LY-scrambled, resp.) set for $\tilde{\varphi}$ with the same cardinality as S.

(ii) If φ is d_jC (LYC, ωC, resp.) then the same holds for $\tilde{\varphi}$.

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Lemma 10 [**?**] *There exists a compact metric space X with a zero topological entropy map φ for which there exist no LY pairs, d_3C-scrambled sets, or ω-scrambled sets, such that $\tilde{\varphi}$ is PTE, d_1C, ωC and LYC.*

Proposition 1 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be a fuzzification of φ. Then*

(i) If there is a set $S \subseteq X$ that is d_jC-scrambled ($j = 1,2,3$) (ω-scrambled, LY-scrambled, resp.) for the map φ, then there exists a d_jC-scrambled ($j = 1,2,3$) (resp. ω-scrambled, LY-scrambled) set S' for Φ, with the same cardinality as S.

(ii) If φ is d_jC (resp. LYC, ωC), then the same holds for Φ.

Proof. Let $S' = \{A \in \mathbb{F}(X) \mid A = \chi_{\{x\}} \text{ and } x \in S\}$. Then this proposition is a corollary of Lemmas **??**, **??** and **??**. □

Proposition 2 *Let Φ denote a fuzzification of φ. There exists a compact metric space X with a zero topological entropy map φ for which there exists no LY pairs, neither d_3C-scrambled set, nor ω-scrambled set such that Φ is PTE, d_1C, ωC and LYC.*

Proof. This proposition is an easy consequence of Lemmas **??**, **??** and **??**, and of the fact that the three versions of distributional chaos are comparable (**??**). □

The following proposition justifies why we study dynamical properties of fuzzified dynamical systems on the space of normal fuzzy sets in the rest of this paper.

Proposition 3 *Let X be a compact metric space and $\varphi \in C(X)$. Then no fuzzification $\Phi : (\mathbb{F}(X), d_\infty) \to (\mathbb{F}(X), d_\infty)$ of φ is transitive.*

Proof. Take any $A, B \in \mathbb{F}(X)$ such that $\max(A) \neq \max(B)$. By the definition of d_∞, there is an open ε-neighborhood U_ε of A (resp., an open ϑ-neighborhood V_ϑ of B) for some $\varepsilon, \vartheta > 0$ such that $\max(A') = \max(A)$ for any $A' \in U_\varepsilon$ (resp., $\max(B') = \max(B)$ for any $B' \in V_\vartheta$). Moreover, Φ preserves the maxima of fuzzy sets from U_ε and V_ϑ. Thus, by the choice of A and B, $\Phi^n(U_\varepsilon) \cap V_\vartheta = \emptyset$ (resp., $U_\varepsilon \cap \Phi^n(V_\vartheta) = \emptyset$) for any $n \in \mathbb{N}$, i.e., Φ is not transitive. □

Corollary 2 *Let X be a compact metric space and $\varphi \in C(X)$. Then a fuzzification $\Phi : (\mathbb{F}(X), d_\infty) \to (\mathbb{F}(X), d_\infty)$ of φ has none of the properties listed in (**??**) (specification property, strong mixing, mild mixing, weak mixing and total transitivity).*

Proof. The proof is an obvious consequence of Proposition **??** and (**??**). □

For completeness, it follows directly from Lemmas **??** and (**??**) that if the original dynamical system (X, φ) is topologically chaotic then also the fuzzified system $(\mathbb{F}^1(X), \Phi)$ (and hence $(\mathbb{F}(X), \Phi)$) is topologically chaotic. Now we would like to mention a recent paper [**?**] where the conditions defining Devaney chaos ([**?**]) were studied on the space of all fuzzy sets on X. We recall that we extended their results since no such fuzzification can be Devaney chaotic by Lemma **??**.

Moreover, dynamical properties of fuzzifications on the space of normal fuzzy sets on X have been never studied before. So we study them in the rest of this section.

Proposition 4 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is transitive then φ is transitive, but the converse implication does not hold.*

Proof. Let the assumptions be fulfilled. Consider any open subsets $U, V \subseteq X$. We want to show that φ is transitive, i.e. there exists $n \in \mathbb{N}$ such that

$$\varphi^n(U) \cap V \neq \emptyset. \tag{37}$$

First we take open subsets $U', V' \subseteq X$ such that $U' \subseteq \overline{U'} \subseteq U$ and $V' \subseteq \overline{V'} \subseteq V$. Then, by Corollary **??**, $\vartheta(U')$ and $\vartheta(V')$ are open subsets of $(\mathbb{F}^1(X), \tau_\infty)$. Since Φ is transitive, there exists $n \in \mathbb{N}$ for which $\Phi^n(\vartheta(U')) \cap \vartheta(V') \neq \emptyset$. Then $\Phi^n(\vartheta(\overline{U'})) \cap \vartheta(\overline{V'}) \neq \emptyset$, and by using results (i) and (iii) of Lemma **??**, we obtain

$$\vartheta(\varphi^n(\overline{U'})) \cap \vartheta(\overline{V'}) \neq \emptyset \quad \Rightarrow \quad \vartheta(\varphi^n(\overline{U'}) \cap \overline{V'}) \neq \emptyset. \tag{38}$$

Therefore $\varphi^n(\overline{U'}) \cap \overline{V'} \neq \emptyset$, and consequently, (**??**) is proved.

A counterexample (an irrational rotation on the unit circle) showing that the converse implication does not hold was presented in [**?**] for the fuzzification Φ defined on $\mathbb{F}(X)$, but it can be applied also to $\Phi|_{\mathbb{F}^1(X)}$. □

Proposition 5 *Let X be a compact metric space, $\varphi \in C(X)$, and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is totally transitive, then φ is totally transitive, but the converse implication does not hold.*

Proof. This proposition is an easy corollary of Proposition **??**, since the transitivity of any iteration Φ^n of Φ on $\mathbb{F}^1(X)$ implies the transitivity of φ^n on X for any $n \in \mathbb{N}$.

As a counterexample to the converse, we again can use the example mentioned in [**?**]. Their example was used as the counterexample to

$$\varphi \text{ transitive } \not\Rightarrow \Phi \text{ transitive}, \tag{39}$$

but the map φ used in [**?**] is an irrational rotation on the unit circle, and it is obvious that any irrational rotation on the unit circle is totally transitive. □

Proposition 6 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is sensitively dependent, then φ is sensitively dependent.*

Proof. Let the assumptions be fulfilled. We shall show that φ is sensitively dependent for the same sensitivity constant δ as Φ. Take any $x \in X$ and any open neighborhood U of x. Then $\vartheta(U)$ is an open neighborhood of $\chi_{\{x\}}$ by Corollary **??**. Since Φ is sensitively dependent, there exist $A \in \vartheta(U)$ and $n \in \mathbb{N}$ such that

$$d_\infty(\Phi^n(\chi_{\{x\}}), \Phi^n(A)) > \delta. \tag{40}$$

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By Lemma **??**

$$d_\infty(\Phi^n(\chi_{\{x\}}), \Phi^n(A)) = d_\infty(\chi_{\{\varphi^n(x)\}}, \Phi^n(A)) > \delta. \quad (41)$$

Thus, by the definition of the metric d_∞, there exists $y_0 \in supp(A)$ such that $d_X(\varphi^n(x), \varphi^n(y_0)) > \delta$. Obviously, y_0 lies in U and this shows that φ is sensitively dependent. □

Proposition 7 *Let X be a compact metric space, $\varphi \in C(X)$, and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is exact, then φ is exact.*

Proof. Let the assumptions be fulfilled, and let $U \subseteq X$ be an open set. Then there exists a closed $U' \subseteq U$ with nonempty interior $int(U')$. By Corollary **??**, $\vartheta(int(U'))$ is an open subset of $\mathbb{F}^1(X)$. Since Φ is exact, there exists some $k \in \mathbb{N}$ for which $\Phi^k(\vartheta(int(U'))) = \mathbb{F}^1(X)$. Therefore, by the choice of U' and by Lemma **??**,

$$\mathbb{F}^1(X) = \Phi^k(\vartheta(int(U'))) \subseteq \Phi^k(\vartheta(U')) \quad (42)$$

and

$$\Phi^k(\vartheta(U')) = \vartheta(\varphi^k(U')) \subseteq \vartheta(\varphi^k(U)). \quad (43)$$

Thus, $\vartheta(\varphi^k(U))$ covers $\mathbb{F}^1(X)$ and hence, since $\mathbb{F}^1(X)$ also contains the characteristic functions of all singletons (see Lemma **??**), $\varphi^k(U) = X$. □

Remark 2 *The map Φ on $\mathbb{F}(X)$ cannot be exact since Φ cannot be transitive (see Lemma **??**). Moreover, the validity of the converse implication to Proposition **??** is still unknown.*

Proposition 8 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is strongly mixing then φ is strongly mixing.*

Proof. The proof of this proposition is a slight variation of the proof of Proposition **??**. □

Proposition 9 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is mildly mixing then φ is mildly mixing.*

Proof. Let the assumptions be fulfilled. Let Y be any compact metric space and (Y, ψ) be any transitive discrete dynamical system. According to the definition of the mild mixing property, we want to show that the product system $(X \times Y, \varphi \times \psi)$ is transitive, i.e. for any open sets $U, V \subseteq X \times Y$ there exists $k \in \mathbb{N}$ for which

$$(\varphi \times \psi)^k(U) \cap V \neq \emptyset. \quad (44)$$

So let $U, V \subseteq X \times Y$ be given. Then there are open subsets $U_1, V_1 \subseteq X$, $U_2, V_2 \subseteq Y$ for which $U_1 \times U_2 \subseteq U$ and $V_1 \times V_2 \subseteq V$. We also consider open subsets $U_1', V_1' \subseteq X$ such that $\overline{U_1'} \subseteq U_1$ and $\overline{V_1'} \subseteq V_1$. By Corollary **??**, $\vartheta(U_1')$ and $\vartheta(V_1')$ are open subsets of $\mathbb{F}^1(X)$. Thus, since Φ is mildly mixing, there exists $k \in \mathbb{N}$ for which

$$(\Phi \times \psi)^k(\vartheta(U_1') \times U_2) \cap (\vartheta(V_1') \times V_2) \neq \emptyset, \quad (45)$$

i.e.

$$\psi^k(U_2) \cap V_2 \neq \emptyset \quad (46)$$

and also

$$\Phi^k(\vartheta(U_1')) \cap \vartheta(V_1') \neq \emptyset. \quad (47)$$

As in the proof of Proposition **??** ($U_1' = U'$ and $V_1' = V'$), the last inequality implies that

$$\varphi^k(U_1) \cap V_1 \neq \emptyset. \quad (48)$$

Consequently, this together with (**??**) gives

$$(\varphi \times \psi)^k(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset. \quad (49)$$

Finally, by the choice of U_1, U_2, V_1, V_2, (**??**) is proven. □

Proposition 10 *Let X be a compact metric space, $\varphi \in C(X)$ and let Φ be the fuzzification of φ. If $\Phi : (\mathbb{F}^1(X), d_\infty) \to (\mathbb{F}^1(X), d_\infty)$ is weakly mixing, then φ is weakly mixing.*

Proof. The proof of this proposition is a slight variation of the proof of Proposition **??**. □

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