Associativity of triangular norms in light of web geometry

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Abstract— The aim of this paper is to promote web geometry and, especially, the Reidemeister closure condition as a powerful and intuitive tool characterizing associativity of the Archimedean triangular norms. In order to demonstrate its possible applications, we provide the full solution to the problem of convex combinations of nilpotent triangular norms.

Keywords— Archimedean triangular norm, web geometry, Reidemeister closure condition.

1 Triangular norms

The notion of *triangular norm* was originally introduced within the framework of probabilistic metric spaces [12]. Since then, triangular norms have found diverse applications in the theory of fuzzy sets, fuzzy decision making, in models of certain many-valued logics or in multivariate statistical analysis; for a reference see the books by Alsina, Frank, and Schweizer [5] and by Klement, Mesiar, and Pap [8].

A conjunctor is a function $K\colon\ [0,1]^2 \to [0,1]$ which is nondecreasing in both arguments, commutative, and which satisfies the *boundary condition* $T(x,1) = x$ for all $x \in [0,1]$. A triangular norm (shortly a *t-norm*, usually denoted by T) is a conjunctor which satisfies the *associativity equation* $T(T(x,y),z) = T(x,T(y,z))$ for all $x,y,z \in [0,1]$. This paper deals mainly with *Archimedean t-norms*. Let us recall that for every t-norm T, a number $n \in \mathbb{N} \cup \{0\}$, and $x \in [0,1]$ a *natural power* of x, denoted $(x)_T^{(n)}$, is defined by:

$$(x)_T^{(n)} = \begin{cases} 1 & \text{if} \quad n = 0\,, \\ T\left(x, (x)_T^{(n-1)}\right) & \text{if} \quad n > 0\,. \end{cases} \tag{1}$$

A t-norm T is said to be *Archimedean* if and only if for every pair $x, y \in\ ]0,1[$, $x < y$, there exists a natural number $n \in \mathbb{N}$ such that $(y)_T^{(n)} < x$. A t-norm which is continuous and strictly increasing on the half-open square $]0,1]^2$ is said to be *strict*. A continuous t-norm T is called *nilpotent* if and only if for every $x \in\ ]0,1[$ there exists a natural number $n \in \mathbb{N}$ such that $(x)_T^{(n)} = 0$. A prototypical example of a strict and a nilpotent t-norm is the product t-norm, $T_{\mathbf{P}}(x,y) = x \cdot y$, and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max\{x+y-1, 0\}$, respectively. It is possible to show that a continuous Archimedean t-norm is either strict or nilpotent. A t-norm T is said to be *cancellative* if it satisfies $T(a,b) = T(a,c) \Rightarrow b = c$ for every $a,b,c \in [0,1]$, $a \neq 0$. A t-norm T is said to be *weakly cancellative* [9] if it satisfies $T(a,b) = T(a,c) \Rightarrow b = c$ for every $a,b,c \in [0,1]$ with $T(a,b) \neq 0$ and $T(a,c) \neq 0$. Every cancellative t-norm is weakly cancellative. Under the assumption of continuity, the set of cancelative t-norms and the set of strict t-norms coincide. Under the same assumption, the set of weakly cancelative t-norms coincides with the set of Archimedean t-norms.

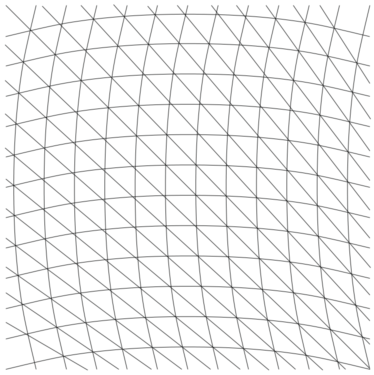

Figure 1: Example of a complete 3-web.

2 Web geometry and local loops

In this section, *web geometry* is explained as a tool allowing to visualize algebraic identities. In particular, it is shown that it visualizes associativity of Archimedean t-norms. A detailed introduction to the subject is given in the monograph by Blashke and Bol [6]. Also the collection of papers by Aczél, Akivis, and Goldberg [1, 2, 3] can serve as an (English) introductory text.

Definition 2.1 *A* groupoid *(or a* magma*) is an algebraic structure $\mathcal{G} = (G, \circ)$ on a set G where $\circ\colon G \times G \to G$ is a binary operation. A* quasigroup *is a groupoid in which the*

ISBN: 978-989-95079-6-8

equations $a \circ x = b$ *and* $y \circ a = b$ *have unique solutions for every* a *and* b *in* G*. Finally, a* loop *is an algebraic structure* $\mathcal{L} = (G, \circ, e)$ *where* $(G, \circ)$ *is a quasigroup and* e *is an identity element, i.e.* $x \circ e = x = e \circ x$ *for every* $x \in G$*.*

The definition of a quasigroup, $\mathcal{G} = (G, \circ)$, allows to define the left and the right inverse. For this purpose we introduce a prefix notation $g(x, y) = x \circ y$.The left inverse is then defined for every $u \in G$ as ${}^{-1}g(u, y) = x$ such that $g(x, y) = u$. Similarly, the right inverse is, for every $v \in G$, $g^{-1}(x, v) = y$ such that $g(x, y) = v$. We say that g is *invertible* in x, resp. y. Defining a point $A = (a, b) \in G \times G$ we may introduce a new operation, $\bullet\colon G \times G \to G$, as

$$u \bullet v = g\left({}^{-1}g(u, b), g^{-1}(a, v)\right) . \tag{2}$$

It is easy to show that $(G, \bullet)$ is a loop with a unit element $e = g(a, b)$; this loop is called a *local loop* of the quasigroup $(G, \circ)$ at the point $A = (a, b)$.

Definition 2.2 *Let* M *be a non-empty set and let* $\lambda_1, \lambda_2, \lambda_3$ *be three families of subsets of* M*. To elements of* M *we refer as to points and to elements of* λ_α*,* $\alpha \in \{1, 2, 3\}$*, as to lines. We say that the system* $(M, \lambda_1, \lambda_2, \lambda_3)$ *is a* complete 3-web *if and only if the following conditions hold:*

1. *Any point* $p \in M$ *is incident to just one line of each family* λ_α*,* $\alpha \in \{1, 2, 3\}$*.*
2. *Any two lines of different families are incident to exactly one point of* M*.*
3. *Two distinct lines from the same family* λ_α *are disjoint.*

Note that Item 3 of the definition is redundant and can be derived from Item 1. We kept the definition in this redundant form in order to keep some of the furher considerations simpler.

Usually M is equipped with a topology which turns the set into a manifold. In such a case the families λ_α are often required to be foliations. Figure 1 shows such an example of 3-web where M is a two dimmensional domain in plane and λ_α are foliations of co-dimension 1.

Notice that all the examples of 3-webs, shown in figures of this paper, are simplified. That means that only some particular lines from uncountable sets λ_α, $\alpha \in \{1, 2, 3\}$, are drawn.

Complete 3-webs are closely connected to quasigroups and, especially, loops. Every quasigroup defines a 3-web in the following way. Let $\mathcal{G} = (G, \circ)$ be a quasigroup defined on a manifold $M = G \times G$ and let $x, y \in G$. Let the lines of sets λ_1, λ_2, and λ_3 be given by the equations $x = a$, $y = b$, and $x \circ y = c$ respectively for fixed $a, b, c \in G$. The pair (M, λ_α), $\alpha \in \{1, 2, 3\}$, is then a 3-web.

Conversely, a 3-web (M, λ_α), $\alpha \in \{1, 2, 3\}$, on a manifold M, defines a binary operation which is invertible with respect to both operands; yet this time the result is not unique. Denote one set of the lines of the 3-web as λ_x, second as λ_y, and third as λ_z (the lines in the set λ_z are called *contour lines*), let $A \in M$ and let $g_x \in \lambda_x$ and $g_y \in \lambda_y$ be the lines passing through

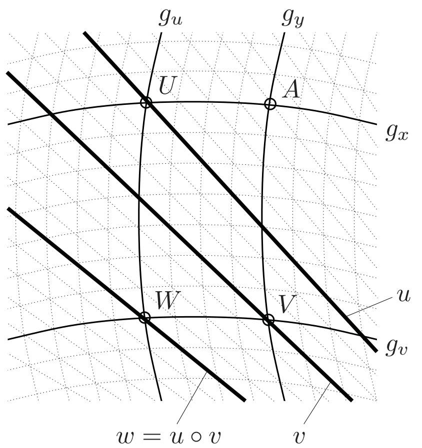

Figure 2: Operation defined on a complete 3-web.

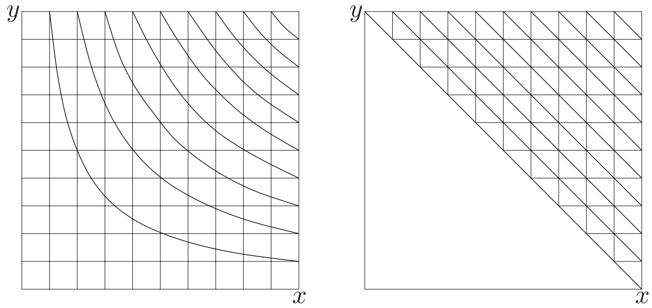

Figure 3: 3-webs given by the product t-norm, $T_{\mathbf{P}}$, and by the Łukasiewicz t-norm, $T_{\mathbf{L}}$.

A; cf. Figure 2. Now we define an operation $*\colon \lambda_z \times \lambda_z \to \lambda_z$. Take $u, v \in \lambda_z$, define points $U, V \in M$ as $U = u \cap g_x$ and $V = v \cap g_y$, respectively. Let $g_u \in \lambda_y$ be the line passing through U and let $g_v \in \lambda_x$ be the line passing through V. Denote the intersection of these two lines as $W = g_u \cap g_v$; the line w passing through W is the result of the operation $u * v$.

It can be shown easily that the operation $*$ is invertible in both variables. Moreover, the line $e \in \lambda_z$, passing through the point A, behaves, with respect to the oparation $*$, as the unit element. Thus $(\lambda_z, *)$ is a loop. Moreover, this loop coincides, up to an isomorphism, with the local loop of $(G, \circ)$ at the point A.

T-norms, defined on the unit interval $[0, 1]$, form neither groupoids nor loops. Nevertheless, the 3-web given by a continuous Archimedean (i.e. strict or nilpotent) t-norm T satisfies all the requirements given by Definition 2.2 if the manifold M is defined as the subset of $[0, 1]^2$ where the t-norm attains non-zero values. A manifold, induced this way by a binary operation $T\colon\ [0, 1]^2 \to [0, 1]$, will be denoted as $\operatorname{Man} T$. Thus

$$M = \operatorname{Man} T = \left\{(x, y) \in [0, 1]^2 \mid T(x, y) > 0\right\} . \tag{3}$$

Figure 3 shows 3-webs given by $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ as examples of a strict and a nilpotent t-norm respectively.

ISBN: 978-989-95079-6-8

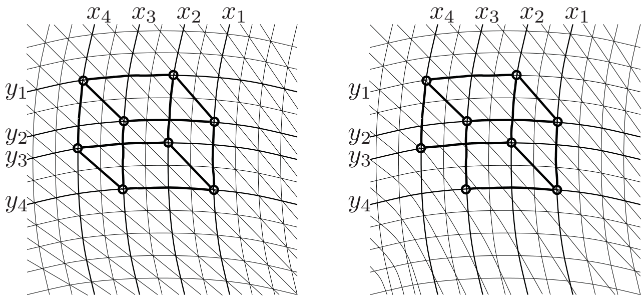

Figure 4: Example of a closed Reidemeister figure and a non-closed Reidemeister figure.

3 Reidemeister closure condition

Different types of 3-webs are characterized by closure conditions. These closure conditions have their counterparts in the related loops as algebraic properties of the loop operations. In this text we are interested in the associativity of t-norms since the associativity is the only property of a t-norm which cannot be intuitively interpreted from its graph. The 3-web counterpart of the associativity is the *Reidemeister closure condition*.

A 3-web satisfies the Reidemeister closure condition if and only if every Reidemeister figure in this web is closed; see Figure 4. Described in the terminology of contour lines, the Reidemeister closure condition is as follows. Let $x_1, x_2, x_3, x_4 \in \lambda_x$ and $y_1, y_2, y_3, y_4 \in \lambda_y$. If the points $(x_1 \cap y_2)$ and $(x_2 \cap y_1)$ lie on the same contour line (i.e. a line from the set λ_z), and if so do the pair of points $(x_1 \cap y_4)$ and $(x_2 \cap y_3)$, and the pair of points $(x_3 \cap y_2)$ and $(x_4 \cap y_1)$, then the points $(x_3 \cap y_4)$ and $(x_4 \cap y_3)$ also lie on the same contour line. The left part of Figure 4 shows a 3-web which satisfies the Reidemeister closure condition whereas the right part shows a 3-web where the condition is violated.

Let us now define a relation $\sim \subseteq M \times M$. We say that two points $A, B \in M$ are in the relation $\sim$, and we write $A \sim B$, if and only if they are both elements of the same line from the set λ_z. Using the language of $\sim$ the Reidemeister condition reads as:

$$\Big(\big((x_1 \cap y_2) \sim (x_2 \cap y_1)\big) \& \big((x_1 \cap y_4) \sim (x_2 \cap y_3)\big) \\ \& \big((x_3 \cap y_2) \sim (x_4 \cap y_1)\big)\Big) \\ \Rightarrow \big((x_3 \cap y_4) \sim (x_4 \cap y_3)\big) . \qquad (4)$$

4 Reidemeister closure condition and continuous Archimedean t-norms

We are going to show that the level set system of a continuous Archimedean t-norm always satisfies the Reidemeister closure condition. Moreover, the Reidemeister closure condition characterizes the associativity of a continuous Archimedean t-norm. For this purpose we recall that if we relax the requirement of the associativity in the definition of a t-norm, we obtain the definition of a conjunctor.

Let K be a continuous Archimedean conjunctor; for the sake of compactness we will use the infix notation: $K(x, y) =$

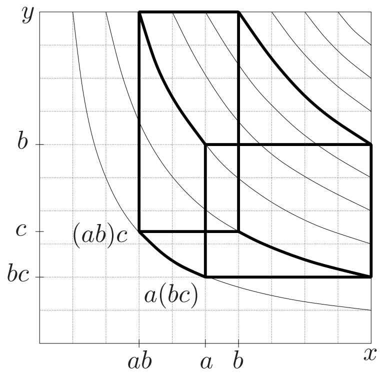

Figure 5: Reidemeister figure on the 3-web given by a continuous Archimedean t-norm.

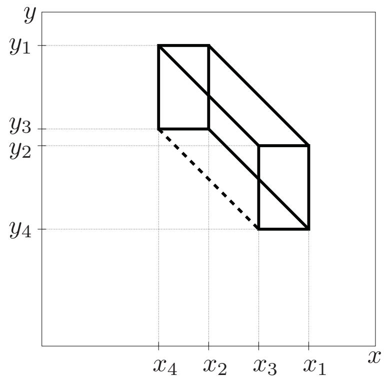

Figure 6: Illustration for the proof: an arbitrary Reidemeister figure.

$x \circ y$. Let (M, λ_α) be the corresponding 3-web defined on the manifold $M = \operatorname{Man} K = \{(x, y) \in [0, 1]^2 \mid x \circ y > 0\}$. Such a 3-web, as in the case of the continuous Archimedean t-norms, satisfies all the requirements given by Definition 2.2. In view of the previous section, the 3-web given by $\circ$ satisfies the Reidemeister closure condition if and only if the following condition holds for any $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in M$:

$$\Big(\big(x_1 \circ y_2 = x_2 \circ y_1\big) \& \big(x_1 \circ y_4 = x_2 \circ y_3\big) \\ \& \big(x_3 \circ y_2 = x_4 \circ y_1\big)\Big) \\ \Rightarrow \big(x_3 \circ y_4 = x_4 \circ y_3\big) . \qquad (5)$$

In the sequel, this condition will be denoted by $\mathbf{R}$.

Figure 5 shows that $\mathbf{R}$ implies the associativity of $\circ$ in a rather intuitive way. For any $a, b, c \in [0, 1]^2$, such that $(a \circ b) \circ c \in \operatorname{Man} \circ$ and $a \circ (b \circ c) \in \operatorname{Man} \circ$, there can be constructed a Reidemeister figure which, as can be seen in Figure 5, is closed if and only if $(a \circ b) \circ c = a \circ (b \circ c)$.

Figure 5 shows also that, in the other way round, if $\circ$ is associative then all the Reidemeister figures that "touch" the lines given by $x = 1$ and $y = 1$ are closed. In other words, $\mathbf{R}$ has to be satisfied for any $x_2, x_3, x_4, y_2, y_3, y_4 \in [0, 1]$ and, at least, for $x_1 = y_1 = 1$. Let us denote this weaker condition by $\mathbf{R_1}$. It is now clear that K is associative if and only if it

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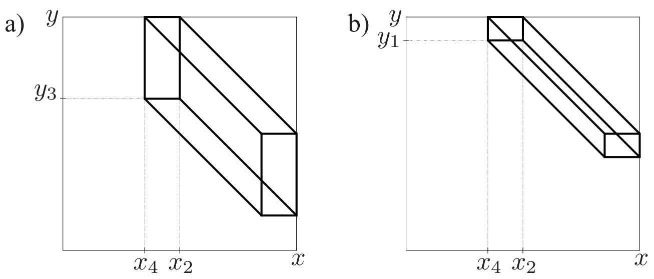

Figure 7: Illustration for the proof.

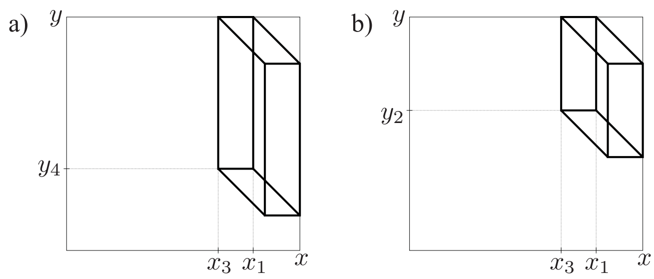

Figure 9: Illustration for the proof.

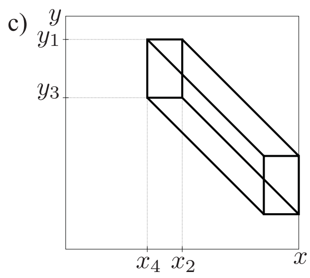

Figure 8: Illustration for the proof.

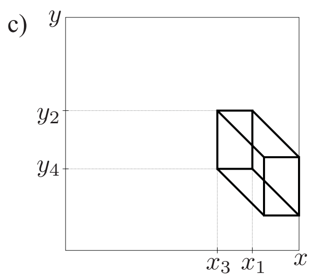

Figure 10: Illustration for the proof.

satisfies $\mathbf{R_1}$. In order to show that $\circ$ is associative if and only if it satisfies $\mathbf{R}$, we need to show that $\mathbf{R_1} \Leftrightarrow \mathbf{R}$. Obviously, $\mathbf{R} \Rightarrow \mathbf{R_1}$. The inverse implication, $\mathbf{R_1} \Rightarrow \mathbf{R}$, is given the following way.

Let us have a continuous Archimedean conjunctor K which satisfies $\mathbf{R_1}$. Let us have a Reidemeister figure drawn on its 3-web for arbitrary $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in [0, 1]$, see Figure 6. We are going to show that this figure shall be always closed. Thanks to $\mathbf{R_1}$, the Reidemeister figures, shown in Figure 7-a and Figure 7-b, are closed. Combining these two figures together it can be concluded that the Reidemeister figure in Figure 8-c is closed as well. By the same deduction, from the closedness of the Reidemeister figures in Figure 9-a and Figure 9-b it can be concluded that the Reidemeister figure in Figure 10-c is closed. Now, combining the closed Reidemeister figures in Figure 8-c and Figure 10-c, the closedness of the Reidemeister figure in Figure 6 is proven.

Corollary 4.1 *A continuous Archimedean conjunctor is associative (and, thus, a t-norm) if and only if it satisfies* $\mathbf{R}$.

5 Applications

5.1 Problem of convex combinations of t-norms

The question was introduced by Alsina, Frank, and Schweizer [4]. The approach of web geometry allows to give the following, recently published [11], answers:

Theorem 5.1 *Let T_1 and T_2 be two continuous Archimedean t-norms such that $\operatorname{Man} T_1 \neq \operatorname{Man} T_2$. Then no non-trivial convex combination F of T_1 and T_2 is a t-norm.*

Corollary 5.2 *Combining the result of Theorem 5.1 with the result given by Jenei [7], a non-trivial convex combination of two distinct nilpotent t-norms is never a t-norm.*

Corollary 5.3 *A non-trivial convex combination of a strict and a nilpotent t-norm is never a t-norm. This is an alternative proof of the result given earlier by Ouyang, Fang, and Li [10].*

Acknowledgment

The first author was supported by the Grant Agency of the Czech Republic under Project 401/09/H007.

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