# Fuzzy Arithmetic with Parametric LR Fuzzy Numbers

Luciano Stefanini, Laerte Sorini

DEMQ (Department of Economics and Quantitative Methods)
University of Urbino 'Carlo Bo', Italy.
Email: lucste@uniurb.it (L. Stefanini), laerte@uniurb.it (L. Sorini)

***Abstract***— *In this paper we suggest and describe a new family of parametric representations for LR fuzzy numbers and use them for fuzzy calculations and arithmetic.*

***Keywords***— LR Fuzzy Numbers, LU-Fuzzy Parametrization, Fuzzy Arithmetic.

## 1 Introduction

The arithmetical structure of fuzzy numbers have been developed in the 1980's and Dubois and Prade introduced the well known LR model and the corresponding formulas for the fuzzy operations (see the recent publication [1] and the references therein).

In general, the arithmetic operations on fuzzy numbers can be approached either by the direct use of the membership function (by the Zadeh extension principle) or by the equivalent use of the $\alpha - cuts$ representation. A solid result in fuzzy theory and practice is that calculations cannot be performed by using the same rules as in arithmetic with real numbers and in fact fuzzy calculus will not always satisfy the same properties (e.g. distributivity, invertibility and others).

A parametric representation of LR fuzzy numbers and the derivation of the corresponding arithmetic operator have been first introduced by Giachetti and Young (see [2], [3]). In this paper, we will see that it is possible to define a more flexible parametric representation of fuzzy numbers that allow a large variety of possible shapes and is very simple to implement (see [5] and [6] for a parametric $\alpha - cuts$ representation).

We suggest a parametrization for the LR fuzzy numbers, similar to the work done for the LU parametrization in [5] and [6]. In terms of the parameters representing the LR fuzzy numbers, it is possible to define the operators for the fuzzy arithmetic and we show that the errors of the approximations can be reduced to any small tolerance (by increasing the number of parameters); in fact, within the space of differentiable fuzzy numbers, the approximations form a dense subspace.

## 2 Fuzzy Numbers in LR Parametric Form

We will consider fuzzy quantities, i.e. fuzzy sets defined over the field $\mathbb{R}$ of real numbers and we will focus on fuzzy numbers, having a particular form of the membership function.

A general *fuzzy set* over a given set (or space) $\mathbb{X}$ of elements (the universe) is usually defined by its membership function $\mu : \mathbb{X} \longrightarrow \mathbb{T} \subseteq [0, 1]$ and a fuzzy (sub)set $u$ of $\mathbb{X}$ is uniquely characterized by the pairs $(x, \mu_u(x))$ for each $x \in \mathbb{X}$; the value $\mu_u(x) \in [0, 1]$ is the membership grade of $x$ to the fuzzy set $u$. So, a fuzzy set is given by

$$\begin{aligned} u &= \{(x, \mu_u(x))|x \in \mathbb{X}\} \\ \text{given } \mu_u &: \mathbb{X} \longrightarrow [0, 1] \end{aligned}$$

Fundamental concepts in fuzzy theory are the *support*, the *level-sets* (or *level-cuts*) and the *core* of a fuzzy set (or of its membership function): let $\mu_u$ be the membership function of a fuzzy set $u$ over $\mathbb{X}$. The support of $u$ is the (crisp) subset of points of $\mathbb{X}$ at which the membership grade $\mu_u(x)$ is positive:

$$supp(u) = \{x|x \in \mathbb{X}, \mu_u(x) > 0\}; \quad (1)$$

we always assume that $supp(u) \neq \emptyset$. For $\alpha \in ]0, 1]$, the $\alpha-$level cut of $u$ (or simply the $\alpha - cut$) is defined by

$$[u]_\alpha = \{x|x \in \mathbb{X},\ \mu_u(x) \geq \alpha\} \quad (2)$$

and for $\alpha = 0$ (or $\alpha \rightarrow +0$) by the closure of the support

$$[u]_0 = cl\{x|x \in \mathbb{X},\ \mu_u(x) > 0\}.$$

The core of $u$ is the set of elements of $\mathbb{X}$ having membership grade 1

$$core(u) = \{x|x \in \mathbb{X}, \mu_u(x) = 1\} \quad (3)$$

and we say that $u$ is normal if $core(u) \neq \emptyset$.

Of our interest are fuzzy sets when the space $\mathbb{X}$ is $\mathbb{R}$ (unidimensional real fuzzy sets).

Well-known properties of the $level - cuts$ are:

$$\begin{aligned} [u]_\alpha &\subseteq [u]_\beta \text{ for } \alpha > \beta, & (4) \\ [u]_\alpha &= \bigcap_{\beta<\alpha} [u]_\beta \text{ for } \alpha \in ]0, 1] & (5) \end{aligned}$$

and (if $x \in supp(u)$, otherwise $\mu_u(x) = 0$)

$$\mu_u(x) = \sup\{\alpha|\ \alpha \in ]0, 1] \text{ for which } x \in [u]_\alpha\}. \quad (6)$$

A particular class of fuzzy sets is when the support is a convex set ($C$ is said *convex* if $(1-t)x' + tx'' \in C$ for every $x', x'' \in C$ and all $t \in [0, 1]$) and the membership function is quasi-concave: we say that the membership function $\mu_u$ is quasi-concave if $\mu_u((1-t)x' + tx'') \geq \min\{\mu_u(x'), \mu_u(x'')\}$ for every $x', x'' \in supp(u)$ and $t \in [0, 1]$. Equivalently, $\mu_u$ is quasi-concave if the level-cuts $[u]_\alpha$ are convex sets for all $\alpha \in [0, 1]$.

A third property of the fuzzy quantities is related to the semi-continuity of the membership function and to the closedness of the level-cuts: $\mu_u$ is said to be upper semicontinuous if $\lim_{x \longrightarrow \widehat{x}} \sup \mu_u(x) = \mu_u(\widehat{x})$ at every $\widehat{x} \in supp(u)$ or, equivalently, if the level-cuts $[u]_\alpha$ are closed sets for all $\alpha \in [0, 1]$.

**Definition** (*fuzzy number*): $u$ is a fuzzy number if

(i.) $\mu_u$ has nonempty bounded support and is normal,
(ii.) $\mu_u$ is quasi-concave,
(iii.) $\mu_u$ is upper semicontinuous

ISBN: 978-989-95079-6-8

or, equivalently:

(a.) $[u]_\alpha$ are nonempty convex sets for all $\alpha \in [0,1]$

(b.) $[u]_\alpha$ are closed and bounded (compact) sets for all $\alpha \in [0,1]$

(c.) $[u]_\alpha$ satisfy conditions (4) and (5).

We will denote by $\mathcal{F}$ the set of fuzzy numbers. It can be structured by an addition and a scalar multiplication, defined either by the level sets or, equivalently, by the Zadeh extension principle. Let $u, v \in \mathcal{F}$ have membership functions $\mu_u$, $\mu_v$ and $\alpha - cuts$ $[u]_\alpha$, $[v]_\alpha$, $\alpha \in [0,1]$ respectively. The addition $u + v \in \mathcal{F}$ and the scalar multiplication $ku \in \mathcal{F}$ for $k \in \mathbb{R}\backslash\{0\}$ have membership functions (extension principle)

$$\mu_{u+v}(z) = \sup\{\min\{\mu_u(x), \mu_v(y)\} | z = x + y\} \quad (7)$$

$$\mu_{ku}(x) = \mu_u(\frac{x}{k}) \quad (8)$$

and level cuts

$$\begin{aligned} [u+v]_\alpha &= [u]_\alpha + [v]_\alpha \quad (9) \\ &= \{x + y | x \in [u]_\alpha,\ y \in [v]_\alpha\} \\ [ku]_\alpha &= k[u]_\alpha = \{kx | x \in [u]_\alpha\}. \quad (10) \end{aligned}$$

We denote by $[u^-_\alpha, u^+_\alpha]$, $\forall \alpha \in [0,1]$ the level-cuts of $u \in \mathcal{F}$ and by $[\widehat{u}^-, \widehat{u}^+]$ its core (level set at $\alpha = 1$). If $u^-_\alpha = \widehat{u}^-$ and $u^+_\alpha = \widehat{u}^+$, $\forall \alpha \in [0,1]$ we have a crisp interval or a crisp number. We say that $u$ is positive if $u^-_\alpha > 0$ , $\forall \alpha \in [0,1]$ and that $u$ is negative if $u^+_\alpha < 0, \forall \alpha \in [0,1]$.

**Definition** (*LR-fuzzy number*): An LR-fuzzy number (or interval) $u$ has membership function of the form

$$\mu_u(x) = \begin{cases} L(\frac{x-a}{b-a}) & \text{if } x \in [a,b] \\ 1 & \text{if } x \in [b,c] \\ R(\frac{d-x}{d-c}) & \text{if } x \in [c,d] \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

where $L, R : [0,1] \longrightarrow [0,1]$ are two non-decreasing *shape functions* such that $R(0) = L(0) = 0$ and $R(1) = L(1) = 1$. If $b = c$ we obtain a fuzzy number. The support is the compact interval $[a,d]$ and the core is $[b,c]$.

If $L$ and $R$ are invertible functions, then the $\alpha - cuts$ are obtained by

$$\begin{aligned} [u]_\alpha &= [u^-_\alpha, u^+_\alpha] \quad (12) \\ &\quad \text{with} \\ u^-_\alpha &= a + (b-a)L^{-1}(\alpha) \\ u^+_\alpha &= d - (d-c)R^{-1}(\alpha) \end{aligned}$$

The usual LR-fuzzy notation is $u = \langle a, b, c, d \rangle_{L,R}$ for an interval and $u = \langle a, b, c \rangle_{L,R}$ for a number. We refer to functions $L(.)$ and $R(.)$ as the left and right branches (shape functions) of $u$, respectively and to the functions $u^-_{(.)}$ and $u^+_{(.)}$ as the lower and upper branches on $u$, respectively. Note that, in the representation (12), we have $u^-_0 = a$, $u^-_1 = b$, $u^+_1 = c$ and $u^+_0 = d$.

### *2.1 Parametric LR-fuzzy numbers*

In this section we present the basic elements of a parametric representation of the shape functions based on monotonic Hermite-type interpolation. We first introduce some models for differentiable monotonic shape functions $p : [0,1] \longrightarrow [0,1]$ such that $p(0) = 0$ and $p(1) = 1$ with $p(t)$ differentiable and increasing on $[0,1]$; for any given nonnegative parameters $\beta_i \geq 0$, $i = 0, 1$ we consider functions $p : [0,1] \longrightarrow [0,1]$ satisfying the four Hermite interpolation conditions

$$\begin{aligned} p(0) &= 0,\ p(1) = 1 \\ p'(0) &= \beta_0,\ p'(1) = \beta_1. \end{aligned}$$

We obtain infinite many functions simply by fixing the two parameters $\beta_i$ that give the slopes (first derivatives) of the function at $t = 0$ and $t = 1$. To explicit the slope parameters we denote the interpolating function $p$ by

$$t \longrightarrow p(t; \beta_0, \beta_1) \text{ for } t \in [0,1].$$

We recall here two of the basic forms from [5]:

∘ (2,2)-rational spline:

$$p(t; \beta_0, \beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)}; \quad (13)$$

∘ mixed exponential spline:

$$\begin{aligned} p(t; \beta_0, \beta_1) &= \frac{1}{m}[t^2(3-2t) - \beta_0(1-t)^m \quad (14) \\ &\quad + \beta_0 + \beta_1 t^m] \\ \text{where } m &= 1 + \beta_0 + \beta_1. \end{aligned}$$

Both functions $p$ in (13) and (14) are strictly increasing on [0,1] and we obtain a linear $p(t) = t, \forall t \in [0,1]$ if $\beta_0 = \beta_1 = 1$ and a quadratic $p(t) = t^2 + \beta_0 t(1-t)$ if $\beta_0 + \beta_1 = 2$ with $\beta_0 \neq 1$.

The (parametric) monotonic functions like (13) and (14) can be used as models for the shape functions $L$ and $R$; in fact, if $a \leq b \leq c \leq d$ and $\beta_{0,L}, \beta_{1,L} \geq 0$, $\beta_{0,R}, \beta_{1,R} \geq 0$ are given, we obtain an LR-fuzzy number with the membership function

$$\mu_u(x) = \begin{cases} p(\frac{x-a}{b-a}; \beta_{0,L}, \beta_{1,L}) & \text{if } x \in [a,b] \\ 1 & \text{if } x \in [b,c] \\ p(\frac{x-d}{c-d}; \beta_{0,R}, \beta_{1,R}) & \text{if } x \in [c,d] \\ 0 & \text{otherwise} \end{cases}. \quad (15)$$

To make a uniform notation,we denote $a, b, c, d$ as $a = u_{0,L}$, $b = u_{1,L}$, $c = u_{1,R}$, $d = u_{0,R}$ and $\beta^*_{0,L} = \frac{\beta_{0,i}}{b-a}$, $\beta^*_{1,L} = \frac{\beta_{1,i}}{b-a}$, $\beta^*_{0,R} = \frac{\beta_{0,j}}{c-d}$, $\beta^*_{1,R} = \frac{\beta_{1,j}}{c-d}$; so, the eight parameters completely define $u$:

$$u = (u_{0,L}, \beta^*_{0,L}, u_{0,R}, \beta^*_{0,R}; u_{1,L}, \beta^*_{1,L}, u_{1,R}, \beta^*_{1,R}) \quad (16)$$

provided that $u_{0,L} \leq u_{1,L} \leq u_{1,R} \leq u_{0,R}$ and $\beta_{0,L}, \beta_{1,L} \geq 0$, $\beta_{0,R}, \beta_{1,R} \geq 0$.

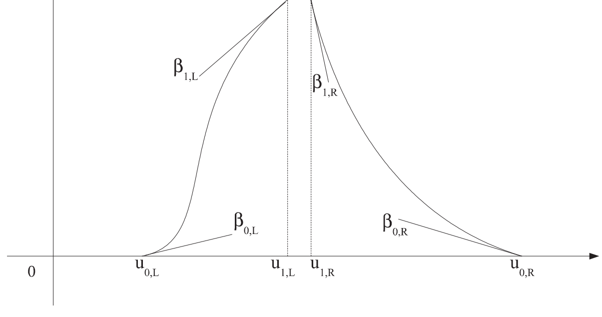


Figure 1: Typical LR parametric membership function

ISBN: 978-989-95079-6-8

**Remark**: As the nonnegative parameters $\beta_{i,L}$ and $\beta_{i,R}$ ($i = 1, 2$) are completely independent, each side of the membership function may have independent (monotonic) forms, e.g. concavity, convexity, inflextions etc. of left and right sides are independent, as illustrated in the figure. The shapes are simply "controlled" by the 4 parameters $\beta_{i,L}$ and $\beta_{i,R}$ ($i = 1, 2$).

Denote by $\mathbb{F}^{LR}$ the set of LR-fuzzy numbers defined by (16). The family of fuzzy numbers $\mathbb{F}^{LR}$, which include triangular and trapezoidal fuzzy numbers if all $\beta_{i,L} = \beta_{i,R} = 1$, are characterized by eight parameters and it appears that the inclusion of the slopes of left and right functions, even without generating piecewise monotonic approximations over subintervals is able to capture much more information than the linear approximation (see [5] and [6]).

The level sets included into the representation are the core and the support. Note that if we are interested to represent exactly the core, the support and the midpoint interval we require $N = 2$, $\alpha_0 = 0$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, the support $[u_{0,L}, u_{0,R}]$ (corresponding to $\alpha = \alpha_0$), the midpoint interval $[u_{1,L}, u_{1,R}]$ (corresponding to $\alpha = \alpha_1$), the core $[u_{2,L}, u_{2,R}]$ (corresponding to $\alpha = \alpha_0$) and the six parameters $\beta_{i,L}, \beta_{i,R} \geq 0$ to model the first derivatives of $\mu_u(x)$ at the extremal point of the three intervals above.

More generally, we can design differentiable parametric shape functions $L()$ and $R()$ by fixing $N + 1$ distinct level sets $0 = \alpha_0 < \alpha_1 < ... < \alpha_N = 1$ and assigning, for each $\alpha_i$ ($i = 0, 1, ..., N$), the four parameters needed for each level.

In this general case, we need the values $u_{i,L}$, $u_{i,R}$ but, instead of the parameters $\beta_{i,L}, \beta_{i,R}$, it is convenient to give directly the first derivatives of $\mu_u(x)$ at $u_{i,L}$, $u_{i,R}$ and calculate the $\beta_{i,L}, \beta_{i,R} \geq 0$ to use in the monotonic functions like (13) and (14) and model the membership function like (15) piecewise on each subinterval $[u_{i-1,L}, u_{i,L}]$ and $[u_{i,R}, u_{i-1,R}]$.

For $i = 0, 1, ..., N$, denote directly by $\delta u_{i,L} \geq 0$ and by $\delta u_{i,R} \leq 0$ the first derivatives of $\mu_u(x)$ at the points $x = u_{i,L}$ and $x = u_{i,R}$, respectively and suppose they are given. The membership function $\mu_u(x)$ is obtained by the following simple procedure:

**Assign** first $\mu_u(x) = 0$ for all $x$;
**for** all $x \in [u_{N,L}, u_{N,R}]$ set $\mu_u(x) = 1$;
**for** $i = 1, 2, ..., N$
  **for** $x \in [u_{i-1,L}, u_{i,L}]$ set
    $\beta_0 = \frac{u_{n,i} - u_{n-1,i}}{\alpha_n - \alpha_{n-1}} \delta u_{i-1,L}$
    $\beta_1 = \frac{u_{n,i} - u_{n-1,i}}{\alpha_n - \alpha_{n-1}} \delta u_{i,L}$
    $\mu_u(x) = \alpha_{i-1} + (\alpha_i - \alpha_{i-1}) p(\frac{x - u_{n-1,i}}{u_{n,i} - u_{n-1,i}}; \beta_0, \beta_1)$
  **end**
  **for** $x \in [u_{i,R}, u_{i-1,R}]$ set
    $\beta_0 = \frac{u_{n,j} - u_{n-1,j}}{\alpha_n - \alpha_{n-1}} \delta u_{i-1,R}$
    $\beta_1 = \frac{u_{n,j} - u_{n-1,j}}{\alpha_n - \alpha_{n-1}} \delta u_{i,R}$
    $\mu_u(x) = \alpha_{i-1} + (\alpha_i - \alpha_{i-1}) p(\frac{x - u_{n-1,j}}{u_{n,j} - u_{n-1,j}}; \beta_0, \beta_1)$
  **end**
**end**

Denote by $\mathbb{F}_N^{LR}$ the fuzzy numbers obtained in the form above; as each shape function is monotonic, the left and right branches are monotonic increasing on $[u_{0,L}, u_{N,L}]$ (the left of $\mu_u$) and decreasing on $[u_{N,R}, u_{0,R}]$ (the right of $\mu_u$). The number of parameters is $4N + 4$ and they are simply constrained to be $u_{0,L} \leq ... \leq u_{N,L} \leq u_{N,R} \leq ... \leq u_{0,R}$ and $\delta u_{i,L} \geq 0$ and $\delta u_{i,R} \leq 0$ to ensure monotonicity. In general, we will have $u_{i-1,L} < u_{i,L}$ and $u_{i-1,R} > u_{i,R}$ but it is easy to consider case of equality so that the graph of $\mu_u$ is a vertical line (discontinuity) at $u_{i,\cdot}$ if $u_{i-1,\cdot} = u_{i,\cdot}$.

In this general form, an LR fuzzy number is represented as follows:

$$u = (\alpha_i; u_{i,L}, \delta u_{i,L}, u_{i,R}, \delta u_{i,R})_{i=0,1,...,N}. \tag{17}$$

**Example**:
Consider a quasi-Gaussian membership function ($m \in \mathbb{R}$, $k, \sigma \in \mathbb{R}^+$; if $k \to +\infty$ the support is unbounded)

$$\mu_{qG}(x) = \begin{cases} \exp(-\frac{(x-m)^2}{2\sigma^2}) & \text{if } m - k\sigma \leq x \leq m + k\sigma \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Its LR parametrization for $m = 0$, $\sigma = 2$, $k = 4$, approximated with $N = 4$ (five points), is

Table 1. LR parametrization of fuzzy number (18)

| $\alpha_i$ | $u_{i,L}$ | $\delta u_{i,L}$ | $u_{i,R}$ | $\delta u_{i,R}$ |
|---|---|---|---|---|
| 0.0 | −8.0 | 0.00033 | 8.0 | −0.00033 |
| 0.25 | −3.33022 | 0.20814 | 3.33022 | −0.20814 |
| 0.5 | −2.35482 | 0.29435 | 2.35482 | −0.29435 |
| 0.75 | −1.51705 | 0.28445 | 1.51705 | −0.28445 |
| 1.0 | 0.0 | 0.0 | 0.0 | 0.0 |

It is interesting to note that, in our parametric representation (17) of LR-fuzzy numbers, the membership function has differentiable left and right branches; if we require that $\mu_u(x)$ be differentiable in all the (internal) points of the support of $u$, in particular also at $x = u_{N,L}$ and $x = u_{N,R}$, we simply need to set $\delta u_{N,L} = 0$ and $\delta u_{N,R} = 0$ in the parametric representation (17). This seems to be an important advantage in all the applications (e.g. fuzzy intelligent systems, neuro-fuzzy learning models) where overall differentiability of $\mu_u$ is required.

The generated LR fuzzy numbers form a subspace of the space of fuzzy numbers. In the case of differentiable membership functions, it is immediate to understand that the union of all parametric fuzzy numbers having the form (17) for all integer $N \geq 1$, i.e.

$$\mathcal{F}^{LR} = \bigcup_{N \geq 1} \mathbb{F}_N^{LR},$$

is dense into the space of (differentiable) fuzzy numbers. In fact, each fuzzy number (17) has the property of interpolate (exactly) the values $u_{i,L}, u_{i,R}$ ($i = 0, 1, ..., N$) and it is sufficient to refine the points $\alpha_i$ to obtain any desired precision. In our experience, approximations with values of $N$ from 5 to 20 have in general a very small error, of the order of $10^{-6} - 10^{-2}$.

Following [4], for $p \in [0, \infty]$, we can define a geometric distance $D_p(u, v)$ between fuzzy numbers $u, v \in \mathbb{F}_N^{LR}$, given by

$$D_p^{LR}(u, v) = \left( \sum_{i=0}^{N} |u_{i,L} - v_{i,L}|^p + |u_{i,R} - v_{i,R}|^p \right)^{1/p} + \\ + \left( \sum_{i=0}^{N} |\delta u_{i,L} - \delta v_{i,L}|^p + |\delta u_{i,R} - \delta v_{i,R}|^p \right)^{1/p}$$

and

ISBN: 978-989-95079-6-8

$$D_{\infty}^{LR}(u,v) = \max_{i=0,\ldots,N}\{|u_{i,L}-v_{i,L}|,|u_{i,R}-v_{i,R}|\}+ \\ + \max_{i=0,\ldots,N}\{|\delta u_{i,L}-\delta v_{i,L}|,|\delta u_{i,R}-\delta v_{i,R}|\}.$$

If we model the LR-fuzzy numbers by a (2,2)-rational spline $p(\alpha;\beta_0,\beta_1)$ like (13) the inverse $p^{-1}(t;\beta_0,\beta_1)$ can be computed analytically as we have to solve the quadratic equation (with respect to $\alpha$)

$$\alpha^2+\beta_0\alpha(1-\alpha) = t[1+(\beta_0+\beta_1-2)\alpha(1-\alpha)] \text{ i.e.}$$

$$\begin{aligned}(1+A(t))\alpha^2 - A(t)\alpha - t &= 0 \text{ where} \\ A(t) &= -\beta_0+\beta_0 t+\beta_1 t-2t.\end{aligned}$$

If $A(t)=-1$ then the equation is linear and the solution is $\alpha=t$. If $A(t)\neq -1$, then there exist two real solutions $\alpha_1 = \frac{2\sqrt{t}}{2+2A(t)}$, $\alpha_2 = \frac{2\sqrt{t}+2A(t)}{2+2A(t)}$ and we choose the one belonging to $[0,1]$.

## 3 Fuzzy arithmetic with LR parametrization

The fuzzy extension principle introduced by Zadeh is the basic tool for fuzzy calculus; it extends functions of real numbers to functions of fuzzy numbers and it allows the extension of arithmetic operations and calculus to fuzzy arguments. We have already defined the addition (7) and the scalar multiplication (8).

### *3.1 Basic arithmetic operators*

Let $\circ \in \{+,-,\times,/\}$ one of the four arithmetic operations and let $u,v\in\mathbb{F}_{\mathbb{I}}$ be given fuzzy intervals (or numbers), having $\mu_u(.)$ and $\mu_v(.)$ as membership functions and level-cuts representations $u=(u^-,u^+)$, $v=(v^-,v^+)$; the extension principle for the extension of $\circ$ defines the membership function of $w=u\circ v$ by

$$\mu_{u\circ v}(z) = \sup\{\min\{\mu_u(x),\mu_v(y)\}|z=x\circ y\}. \tag{19}$$

In terms of the $\alpha-cuts$, the four arithmetic operations and the scalar multiplication for $k\in\mathbb{R}$ are obtained by the well-known *interval arithmetic* (for all $\alpha\in[0,1]$)

Addition ($w=u+v$):
$[u+v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,

Scalar multiplication ($w=ku$):
$[ku]_\alpha = [\min\{ku_\alpha^-,ku_\alpha^+\}, \max\{ku_\alpha^-,ku_\alpha^+\}]$,

Subtraction ($w=u-v$):
$[u-v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$,

Multiplication ($w=uv$):

$$\begin{cases}(uv)_\alpha^- = \min\{u_\alpha^-v_\alpha^-, u_\alpha^-v_\alpha^+, u_\alpha^+v_\alpha^-, u_\alpha^+v_\alpha^+\} \\ (uv)_\alpha^+ = \max\{u_\alpha^-v_\alpha^-, u_\alpha^-v_\alpha^+, u_\alpha^+v_\alpha^-, u_\alpha^+v_\alpha^+\}\end{cases},$$

Division ($w=u/v$): if $0\notin[v_0^-,v_0^+]$

$$\begin{cases}(u/v)_\alpha^- = \min\left\{\frac{u_I^-}{v_I^-},\frac{u_I^-}{v_I^+},\frac{u_I^+}{v_I^-},\frac{u_I^+}{v_I^+}\right\} \\ (u/v)_\alpha^+ = \max\left\{\frac{u_I^-}{v_I^-},\frac{u_I^-}{v_I^+},\frac{u_I^+}{v_I^-},\frac{u_I^+}{v_I^+}\right\}\end{cases}.$$

In recent work (see [7]) the following generalized subtraction has been defined and analyzed:

**Definition:** Let $u,v$ be two fuzzy numbers; we define the generalized Hukuhara difference (gH-difference for short) of $u$ and $v$ as the fuzzy number $w$ such that

$$u\ominus_g v = w \iff \begin{cases} & (i) \quad u=v+w \\ \text{or} & (ii) \quad v=u-w\end{cases}. \tag{20}$$

The $\alpha$-cuts of both $u\ominus_g v$ can be expressed in terms of the $\alpha$-cuts of $u$ and $v$; but it is possible that $u\ominus_g v$ is not well defined; by the use of the parametrization (17) it is easy to test if the operation is well defined or not.

Consider two LR-fuzzy numbers $u$ and $v$ ($N=1$ for simplicity)

$$\begin{aligned} u &= (u_{0,L},\delta u_{0,L},u_{0,R},\delta u_{0,R}; \\ &\quad u_{1,L},\delta u_{1,L},u_{1,R},\delta u_{1,R}), \end{aligned} \tag{21}$$

$$\begin{aligned} v &= (v_{0,L},\delta v_{0,L},v_{0,R},\delta v_{0,R}; \\ &\quad v_{1,L},\delta v_{1,L},v_{1,R},\delta v_{1,R}). \end{aligned} \tag{22}$$

Note that in the formulas below $u$ and $v$ are not restricted to have the same $L(.)$ and $R(.)$ shape functions and changing the slopes will change the form of the membership functions (in all the cases below, eventually assume $\frac{0}{0}=0$).

The values and the slopes at the considered $\alpha$-cuts are computed exactly; in fact we use the exact standard formulae for the derivative of addition, multiplication, division and function composition. Corresponding to membership values $\alpha\neq\alpha_i$, $i=0,1,...,N$, the membership functions is approximated by using the rational (13) or the mixed (14) shape functions.

The approximate addition is the following

$$(u+v) = \begin{pmatrix} u_{0,L}+v_{0,L}, \frac{\delta u_{0,i}\,\delta v_{0,i}}{\delta u_{0,i}+\delta v_{0,i}}, \\ u_{0,R}+v_{0,R}, \frac{\delta u_{0,j}\,\delta v_{0,j}}{\delta u_{0,j}+\delta v_{0,j}}; \\ u_{1,L}+v_{1,L}, \frac{\delta u_{1,i}\,\delta v_{1,i}}{\delta u_{1,i}+\delta v_{1,i}}, \\ u_{1,R}+v_{1,R}, \frac{\delta u_{1,j}\,\delta v_{1,j}}{\delta u_{1,j}+\delta v_{1,j}} \end{pmatrix}.$$

Nothe that if the left and right shapes of $u$ and $v$ are the same (e.g. linear, quadratic) then the addition is exact.

The general algorithm for approximate LR addition is
**Algorithm** (*LR addition*) $w=u+v$
**for** $i=0,1,...,N$
$w_{i,L}=u_{i,L}+v_{i,L}$ , $w_{i,R}=u_{i,R}+v_{i,R}$
$\delta w_{i,L}=\frac{\delta u_{n,i}\,\delta v_{n,i}}{\delta u_{n,i}+\delta v_{n,i}}$ , $\delta w_{i,R}=\frac{\delta u_{n,j}\,\delta v_{n,j}}{\delta u_{n,j}+\delta v_{n,j}}$
**end**

The difference $w=u-v$ is similar
**Algorithm** (*LR subtraction*) $w=u-v$
**for** $i=0,1,...,N$
$w_{i,L}=u_{i,L}-v_{i,R}$ , $w_{i,R}=u_{i,R}-v_{i,L}$
$\delta w_{i,L}=\frac{\delta u_{n,i}\,\delta v_{n,j}}{\delta v_{n,j}-\delta u_{n,j}}$ , $\delta w_{i,R}=\frac{\delta u_{n,j}\,\delta v_{n,i}}{\delta v_{n,i}-\delta u_{n,j}}$
**end**

Also the approximate multiplication $w=uv$ can be obtained easily; in the case of two positive LR-fuzzy numbers $w$ is given by

$$w_{LR} = \begin{pmatrix} u_{0,L}v_{0,L}, \frac{\delta u_{0,i}\,\delta v_{0,i}}{v_{0,i}\,\delta v_{0,i}+u_{0,i}\,\delta u_{0,i}}, \\ u_{0,R}v_{0,R}, \frac{\delta u_{0,j}\,\delta v_{0,j}}{v_{0,j}\,\delta v_{0,j}+u_{0,j}\,\delta u_{0,j}}; \\ u_{1,L}v_{1,L}, \frac{\delta u_{1,i}\,\delta v_{1,i}}{v_{1,i}\,\delta v_{1,i}+u_{1,i}\,\delta u_{1,i}}, \\ u_{1,R}v_{1,R}, \frac{\delta u_{1,j}\,\delta v_{1,j}}{v_{1,j}\,\delta v_{1,j}+u_{1,j}\,\delta u_{1,j}} \end{pmatrix}.$$

ISBN: 978-989-95079-6-8

The general algorithm for approximate LR multiplication is

**Algorithm** (*LR multiplication*) $w = uv$
(eventually, use $\frac{0}{0} = 0$)
**for** $i = 0, 1, ..., N$
$m_i = \min\{u_{i,L}v_{i,L}, u_{i,L}v_{i,R}, u_{i,R}v_{i,L}, u_{i,R}v_{i,R}\}$
$M_i = \max\{u_{i,L}v_{i,L}, u_{i,L}v_{i,R}, u_{i,R}v_{i,L}, u_{i,R}v_{i,R}\}$
$w_{i,L} = m_i$ , $w_{i,R} = M_i$
**if** $u_{i,L}v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{n,i}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} + u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,L}v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{n,i}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} + u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,R}v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{n,j}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} + u_{n,j}\ \delta u_{n,j}}$
**elseif** $u_{i,R}v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{n,j}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} + u_{n,j}\ \delta u_{n,j}}$
**endif**
**if** $u_{i,L}v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{n,i}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} + u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,L}v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{n,i}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} + u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,R}v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{n,j}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} + u_{n,j}\ \delta u_{n,j}}$
**elseif** $u_{i,R}v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{n,j}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} + u_{n,j}\ \delta u_{n,j}}$
**endif**
**end**

The division is similar:

**Algorithm** (*LR division*) $w = u/v, 0 \notin [v]_0$
(eventually, use $\frac{0}{0} = 0$)
**for** $i = 0, 1, ..., N$
$m_i = \min\{u_{i,L}/v_{i,L}, u_{i,L}/v_{i,R}, u_{i,R}/v_{i,L}, u_{i,R}/v_{i,R}\}$
$M_i = \max\{u_{i,L}/v_{i,L}, u_{i,L}/v_{i,R}, u_{i,R}/v_{i,L}, u_{i,R}/v_{i,R}\}$
$w_{i,L} = m_i$ , $w_{i,R} = M_i$
**if** $u_{i,L}/v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{v_{n,i}^2\ \delta u_{n,i}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} - u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,L}/v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{v_{n,j}^2\ \delta u_{n,i}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} - u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,R}/v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{v_{n,i}^2\ \delta u_{n,j}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} - u_{n,j}\ \delta u_{n,j}}$
**elseif** $u_{i,R}/v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{v_{n,j}^2\ \delta u_{n,j}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} - u_{n,j}\ \delta u_{n,j}}$
**endif**
**if** $u_{i,L}/v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{v_{n,i}^2\ \delta u_{n,i}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} - u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,L}/v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{v_{n,j}^2\ \delta u_{n,i}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} - u_{n,i}\ \delta u_{n,i}}$
**elseif** $u_{i,R}/v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{v_{n,i}^2\ \delta u_{n,j}\ \delta v_{n,i}}{v_{n,i}\ \delta v_{n,i} - u_{n,j}\ \delta u_{n,j}}$
**elseif** $u_{i,R}/v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{v_{n,j}^2\ \delta u_{n,j}\ \delta v_{n,j}}{v_{n,j}\ \delta v_{n,j} - u_{n,j}\ \delta u_{n,j}}$
**endif**
**end**

The algorithm for the gH-difference $w = u \ominus_g v$ in LR form is also easy, if we recall that the $\alpha$-cuts of $w$ are the intervals (see [7]) $[\min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, \max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}]$:

**Algorithm** (*LR gH-subtraction*) $w = u \ominus_g v$
(eventually, use $\frac{0}{0} = 0$)
**for** $i = 0, 1, ..., N$
**define** $d_{i,L} = u_{i,L} - v_{i,L}$ and $d_{i,R} = u_{i,R} - v_{i,R}$
**if** $d_{i,L} < d_{i,R}$ **then**
$w_{i,L} = d_{i,L}, w_{i,R} = d_{i,R}$
$\delta w_{i,L} = \frac{\delta u_{n,i}\ \delta v_{n,i}}{\delta v_{n,i} - \delta u_{n,i}}, \delta w_{i,R} = \frac{\delta u_{n,j}\ \delta v_{n,j}}{\delta v_{n,j} - \delta u_{n,j}}$
**elseif** $d_{i,L} > d_{i,R}$ **then**
$w_{i,R} = d_{i,L}, w_{i,L} = d_{i,R}$
$\delta w_{i,R} = \frac{\delta u_{n,i}\ \delta v_{n,i}}{\delta v_{n,i} - \delta u_{n,i}}, \delta w_{i,L} = \frac{\delta u_{n,j}\ \delta v_{n,j}}{\delta v_{n,j} - \delta u_{n,j}}$
**else** (here $d_{i,L} = d_{i,R} = d$)
$w_{i,L} = d, w_{i,R} = d$
$\delta w_{i,L} = 0, \delta w_{i,R} = 0$
**endif**
**end**
**Test** if the calculated $w_{i,L}, w_{i,R}$ are ordered correctly
and $\delta w_{i,L} \geq 0, \delta w_{i,R} \leq 0$, then $w$ is $u \ominus_g v$.

The operations above are exact at the nodes $\alpha = \alpha_i$, $i = 0, 1, ..., N$, and have very small global errors for all $\alpha \in [0, 1]$ (if $N$ is sufficiently high, of the order of 5 to 20). Further, it is easy to control the error by using a sufficiently fine $\alpha-$decomposition and the results have shown that both the rational (13) and the mixed (14) models perform well.

Some parametric membership functions in the LR framework are present in many applications and the use of nonlinear shapes in increasing. Usually, one defines a given family, e.g. linear, quadratic, sigmoid, quasi gaussian, and the operations are performed within the same family.

Our proposed parametrization allows an extended set of flexible fuzzy numbers and is able to approximate all other forms with acceptable small errors and the additional advantage of producing good approximations to the results of the arithmetic operations even between LR fuzzy numbers having very different original shapes.

Furthermore, at the explicitly considered $\alpha-$cuts of the decomposition, all the values $w_{i,L},\ w_{i,R}$ and the slopes $\delta w_{i,L}$, $\delta w_{i,R}$ $(i = 0, 1, ..., N)$ of the membership function are exact.

### 3.2 *Computation of fuzzy-valued functions*

For a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, let $v = f(u_1, u_2, ..., u_n)$ denote its fuzzy extension, based on the application of the Zadeh's Extension Principle (EP for short). It is well known that the $\alpha - cuts$ $[v_\alpha^-, v_\alpha^+]$ of $v$ are obtained by solving the following box-constrained global optimization problems $(\alpha \in [0, 1])$

$$v_\alpha^- = \min\{f(x_1, ..., x_n) | x_k \in [u_k]_\alpha,\ k = 1, 2, ..., n\} \quad (23)$$

$$v_\alpha^+ = \max\{f(x_1, ..., x_n) | x_k \in [u_k]_\alpha,\ k = 1, 2, ..., n\} \quad (24)$$

where $[u_k]_\alpha = [u_{k,\alpha}^-, u_{k,\alpha}^+]$, $k = 1, 2, ..., n$, are the $\alpha - cuts$ of $u_k$.

The lower and upper values $v_\alpha^-$ and $v_\alpha^+$ of $v$ define equivalently (as $f$ is assumed to be continuous) the image of the cartesian product $\overset{n}{\underset{k=1}{\times}} [u_k]_\alpha$ via $f$, i.e. $[v_\alpha^-, v_\alpha^+] = f([u_1]_\alpha, ..., [u_n]_\alpha)$.

Except for simple elementary cases for which the optimization problems above can be solved analytically, the direct application of (23) and (24) is difficult and computationally expensive (see [6]).

At least if $f$ is differentiable, the advantages of the LR representation appear to be quite interesting, based on the fact that a small number of $\alpha$ points is in general sufficient to obtain good approximations (this is the essential gain in using the slopes to model fuzzy numbers), so reducing the number of constrained *min* (23) and *max* (24) problems to be solved directly. On the other hand, finding computationally efficient extension solvers is still an open research field in fuzzy calculations.

We now give the details of the fuzzy extension of general differentiable functions of only one variable, by the LR representation. The case of multidimensional differentiable functions can be approached in a similar way, by considering the

ISBN: 978-989-95079-6-8

partial derivatives of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the chain rules for the composition of multidimensional functions.

In all the computations we will adopt the EP method, but also if other approaches are adopted, the representation still remains valid.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$; its *(EP)*-extension $v = f(u)$ to a fuzzy argument $u = (u^-, u^+)$ has $\alpha - cuts$

$$[v]_\alpha = [\min\{f(x) \mid x \in [u]_\alpha\}, \max\{f(x) \mid x \in [u]_\alpha\}]. \quad (25)$$

If $f$ is monotonic increasing we obtain $[v]_\alpha = [f(u_\alpha^-), f(u_\alpha^+)]$ while, if $f$ is monotonic decreasing, $[v]_\alpha = [f(u_\alpha^+), f(u_\alpha^-)]$; for simplicity, in the monotonic case we assume that the derivative of $f$ is not null over the support of $u$, but it is possible to design the algorithm also in the case where, for some $\alpha$, $f'(u_\alpha^+) = 0$ or $f'(u_\alpha^-) = 0$.

The LR representation of $v = (\alpha_i; v_{i,L}, \delta v_{i,L}, v_{i,R}, \delta v_{i,R})_{i=0,1,...,N}$ is obtained by the following algorithm:

**Algorithm**: (1-dim monotonic extension)

Let $u = (\alpha_i; u_{i,L}, \delta u_{i,L}, u_{i,R}, \delta u_{i,R})_{i=0,1,...,N}$ be given and $f : [u_{0,L}, u_{0,R}] \rightarrow \mathbb{R}$ be differentiable and monotonic; calculate $v = f(u)$.

**for** $i = 0, 1, ..., N$
  **if** ($f$ is increasing) **then**
    $v_{i,L} = f(u_{i,L})$, $\delta v_{i,L} = \frac{\delta u_{n,i}}{f'(u_{n,i})}$
    $v_{i,R} = f(u_{i,R})$, $\delta v_{i,R} = \frac{\delta u_{n,j}}{f'(u_{n,j})}$
  **elseif** ($f$ is decreasing) **then**
    $v_{i,L} = f(u_{i,R})$, $\delta v_{i,L} = \frac{\delta u_{n,j}}{f'(u_{n,j})}$
    $v_{i,R} = f(u_{i,R})$, $\delta v_{i,R} = \frac{\delta u_{n,j}}{f'(u_{n,j})}$
  **endif**
**end**

As an example, the monotonic exponential function $f(x) = \exp(x)$ has LR-fuzzy extension

$$\exp(u) = \begin{array}{c}(\alpha_i; \exp(u_{i,L}), \exp(u_i^-)\,\delta u_{i,L}/\exp(u_{i,L}), \\ \exp(u_{i,R}), \delta u_{i,R}/\exp(u_{i,R}))_{i=0,1,...N}\end{array}$$

In the non monotonic (differentiable) case, we have to solve the optimization problems in (25) for each $\alpha = \alpha_i$, $i = 0, 1, ..., N$, i.e.

$$(EP_i): \begin{cases} v_{i,L} = \min\{f(x) | x \in [u_{i,L}, u_{i,R}]\} \\ v_{i,R} = \max\{f(x) | x \in [u_{i,L}, u_{i,R}]\}. \end{cases}$$

The $\min$ (or the $\max$) can occur either at a point which is coincident with one of the extremal values of $[u_{i,L}, u_{i,R}]$ or at a point which is internal; in the last case, the derivative of $f$ is null and $\delta v_{i,L} = +\infty$ (or $\delta v_{i,R} = -\infty$).

In the calculations, we "approximate" $\pm\infty$ by $\pm BIG$ where $BIG$ is a big positive number.

**Algorithm**: (1-dim non monotonic LR extension)

Let $u = (\alpha_i; u_{i,L}, \delta u_{i,L}, u_{i,R}, \delta u_{i,R})_{i=0,1,...,N}$ be given and $f : [u_{0,L}, u_{0,R}] \rightarrow \mathbb{R}$ be differentiable and monotonic; calculate $v = f(u)$.

**for** $i = 0, 1, ..., N$
  **solve** $\min\{f(x) | x \in [u_{i,L}, u_{i,R}]\}$
  **let** $\widehat{x}_i = \arg\min\{f(x) | x \in [u_{i,L}, u_{i,R}]\}$
  **if** $\widehat{x}_i = u_{i,L}$ **then** $v_{i,L} = f(u_{i,L})$, $\delta v_{i,L} = \frac{\delta u_{n,i}}{f'(u_{n,i})}$
  **elseif** $\widehat{x}_i = u_{i,R}$ **then** $v_{i,L} = f(u_{i,R})$, $\delta v_{i,L} = \frac{\delta u_{n,j}}{f'(u_{n,j})}$
  **else** $v_{i,L} = f(\widehat{x}_i)$, $\delta v_{i,L} = BIG$
  **endif**
  **solve** $\max\{f(x) | x \in [u_{i,L}, u_{i,R}]\}$
  **let** $\widehat{x}_i = \arg\max\{f(x) | x \in [u_{i,L}, u_{i,R}]\}$
  **if** $\widehat{x}_i = u_{i,L}$ **then** $v_{i.R} = f(u_{i,L})$, $\delta v_{i,R} = \frac{\delta u_{n,i}}{f'(u_{n,i})}$
  **elseif** $\widehat{x}_i = u_{i,R}$ **then** $v_{i,R} = f(u_{i,R})$, $\delta v_{i,R} = \frac{\delta u_{n,j}}{f'(u_{n,j})}$
  **else** $v_{i,R} = f(\widehat{x}_i)$, $\delta v_{i,R} = -BIG$
  **endif**
**end**

## 4 Conclusion and further work

We suggest a parametrization of LR fuzzy numbers $u$ by the use of parametric monotonic (simple) functions to model the left and the right branches of $u$. The obtained parametrizations form a subspace of the space of fuzzy numbers and can be possibly be refined to become a dense subspace. Within the parametrizations, we define arithmetic operators for the basic arithmetic and for the fuzzy extension of functions by the application of Zadeh's extension principle.

**References**


[1] D. Dubois, H. Prade (ed.), *Fundamentals of Fuzzy Sets*, Kluwer, Boston, The Handbooks of Fuzzy Sets Series, 2000.

[2] R. E. Giachetti and R.E. Young: A Parametric Representation of Fuzzy Numbers and their Arithmetic Operators, *Fuzzy Sets and Systems,* 91, (1997) 185-202.

[3] R. E. Giachetti and R.E. Young: Analysis of the Error in the Standard Approximation used for Multiplication of Triangular and Trapezoidal fuzzy Numbers and the Development of a New Approximation, *Fuzzy Sets and Systems,* 91, (1997) 1-13.

[4] S. Heilpern, Representation and application of fuzzy numbers, *Fuzzy Sets and Systems*, 91, 1997, 259-278.

[5] L. Stefanini, L. Sorini, M. L. Guerra, Parametric representation of fuzzy numbers and application to fuzzy calculus, *Fuzzy Sets and Systems*, 157, 2006, 2423 – 2455

[6] L. Stefanini, L. Sorini, M.L. Guerra, Fuzzy Numbers and Fuzzy Arithmetic, in W. Pedrycz et al. (Eds) "Handbook of Granular Computing" , Chapter 12, J. Wiley & Sons, 2008.

[7] L. Stefanini, A Generalization of Hukuhara Difference, in D. Dubois et al. (Eds.), Soft Methods for Handling Variability and Imprecision, ASC 48, Springer Verlag, 2008, 203-210.


ISBN: 978-989-95079-6-8