Type-2 Fuzzy Arithmetic using Alpha-planes

Hussam Hamrawi Simon Coupland

Centre for Computational Intelligence, Faculty of Technology, De Montfort University
Leicester, UK
Email: hussamw@dmu.ac.uk, simonc@dmu.ac.uk

Abstract— This paper examines type-2 fuzzy arithmetic using interval analysis. It relies heavily on alpha-cuts and alpha-planes. Furthermore, we discuss the use of quasi type-2 fuzzy sets proposed by Mendel and Liu and define quasi type-2 fuzzy numbers. Arithmetic operations of such numbers are defined and a worked example is presented.

Keywords— Fuzzy Arithmetic, Type-2 Fuzzy Arithmetic, Type-2 Fuzzy Numbers, Quasi Type-2 Fuzzy Numbers.

1 Introduction

Zadeh introduced the concept of type-2 fuzzy sets in the first of a trio of papers in 1975 [1]. In these papers he also defined linguistic variables, interval-valued fuzzy sets and their representations using the resolution identity (later known as α-cuts). Type-2 fuzzy logic has gained much attention recently due to its ability to handle uncertainty [2], and many advances appeared in both theory and applications [3, 4, 5]. Type-2 fuzzy numbers and the associated arithmetic operations have not received the same attention, only two main contributions appear in the literature that specificly target arithmetic operations using the representation theorem and the extension principle [6, 7]. Recent work on fuzzy systems has proposed an extension to α-cuts for type-2 fuzzy sets through the notion of α-planes [8]. Further work by Mendel and Liu [9] defined a quasi type-2 fuzzy logic system as a restricted special case of a type-2 fuzzy logic system represented by its α-planes.

In this paper we examine type-2 fuzzy arithmetic through α-planes. We use arithmetic operations on type-1 and interval type-2 fuzzy sets through their α-cut representation, moreover, we define quasi type-2 fuzzy numbers and derive their arithmetic operations.

This paper is organised as follows: Section 2 reviews some basic type-2 fuzzy set definitions used in this paper; Section 3 defines type-2 fuzzy numbers and quasi type-2 fuzzy numbers; Section 4 derives the arithmetic operations for these numbers; Section 5 provides a worked example; finally Section 6 provides a conclusion.

2 Type-2 Fuzzy Sets

We present a review of the basic terminology used in this paper. A type-2 fuzzy set (**T2FS**) [2, 10] is defined by equation (1)

$$\widetilde{A} = \int_{\forall x \in X} \int_{\forall u \in J_x \subseteq [0,1]} \mu_{\widetilde{A}}(x,u)/(x,u) \quad (1)$$

where $\int\int$ denotes the union over all admissible *domain values* x and *secondary domain values* u, and $\mu_{\widetilde{A}}(x,u)$ is a *type-2 membership function*, a T2FS is three dimensional (3D). The *Vertical Slice* (**VS**) is the two dimensional (2D) plane in the u and $\mu_{\widetilde{A}}(x,u)$ axes for a single value of $x = x'$, then the VS is defined by equation (2).

$$VS(x') = \mu_{\widetilde{A}}(x',u) \equiv \mu_{\widetilde{A}}(x') = \int_{u \in J_{x'}} f_{x'}(u)/u \quad (2)$$

where $f_{x'}(u) \in [0,1]$ is called the *secondary grade* and J_x represents the domain of the *secondary membership function* called the *secondary domain*, of course the VS is a *type-1 fuzzy set* (**T1FS**) in $[0,1]$. The *Vertical Slice Representation* (**VSR**) of T2FS is represented by the union of all the vertical slices

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | \forall x \in X\} \quad (3)$$

The *Footprint Of Uncertainty* (**FOU**) is derived from the union of all primary memberships

$$FOU(\widetilde{A}) = \int_{x \in X} J_x \quad (4)$$

Another important notion is the *principal membership function* (**PrMF**) defined as the union of all the primary memberships having secondary grades equal to 1

$$Pr(\widetilde{A}) = \int_{x \in X} u/x | f_x(u) = 1 \quad (5)$$

This notion allows us to view a T1FS as special case of T2FS in which its PrMF is obtained for only one primary membership having a secondary grade at unity [11, 12]. Based on [2] the FOU is described to be bounded by two membership functions a lower $\underline{\mu_{\widetilde{A}}(x)}$ and an upper $\overline{\mu_{\widetilde{A}}(x)}$. The FOU can be described in terms of its upper and lower membership functions (**MFs**)

$$FOU(\widetilde{A}) = \int_{x \in X} \left[\underline{\mu_{\widetilde{A}}(x)}, \overline{\mu_{\widetilde{A}}(x)}\right] \quad (6)$$

Interval type-2 fuzzy set (**IT2FS**) is defined to be a T2FS where all its secondary grades are of unity $\forall f_x(u) = 1$. A IT2FS can be completely determined using its FOU given in equation (6). Recalling from Zadeh [1] an *α-level set*, A_α, that comprises of elements $x \in X$ of a type-1 fuzzy subset A of universe X

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\} \quad (7)$$

where $\alpha \in [0,1]$, the fuzzy set A can be represented (decomposed) as

$$A = \bigcup_{\alpha \in [0,1]} \alpha . A_\alpha \quad (8)$$

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where $\mu_A(x) = \sup_{x \in A_\alpha} \alpha$ and (sup) denotes the supremum, this decomposition theorem [13] is called the *α-cut Representation*. The same analogy is used by Tahayori *et al.* [14]:

$$\widetilde{A}_{\widetilde{\alpha}} = \{(x,u)|f_x(u) \geq \widetilde{\alpha}\} \quad (9)$$

and the T2FS $\widetilde{A}$ can be represented (decomposed) as

$$\widetilde{A} = \bigcup_{\widetilde{\alpha}\in[0,1]} \widetilde{\alpha}.\widetilde{A}_{\widetilde{\alpha}} \quad (10)$$

where[1] $f_x(u) = \sup_{(x,u)\in\widetilde{A}_{\widetilde{\alpha}}} \widetilde{\alpha}$. Liu [8] calls equation (9) an *α-plane* due to its 2D nature and consequently equation (10) is called the *α-plane Representation*. Liu [8] and Wagner and Hagras [15] noted that when $\widetilde{\alpha} = 0$, $\widetilde{A}_{\widetilde{\alpha}}$ is actually the FOU and Liu and Mendel [9] generalise it to a footprint of uncertainty at plane $\widetilde{\alpha}$ denoted by $FOU(\widetilde{A}_{\widetilde{\alpha}})$. We are interested in the use of this representation as these α-planes can compute some of T2FS operations using IT2FS operations by applying Zadeh's extension principle [1] as follows

$$f(\widetilde{A}) = f(\bigcup_{\widetilde{\alpha}\in[0,1]} \widetilde{\alpha}.\widetilde{A}_{\widetilde{\alpha}}) = \bigcup_{\widetilde{\alpha}\in[0,1]} \widetilde{\alpha}.f(\widetilde{A}_{\widetilde{\alpha}}) \quad (11)$$

As Zadeh himself noted that this principle can be used with functions between sets, relations, etc. and it can extend any function from crisp sets to its fuzzy and obviously type-2 fuzzy counterparts. In [9] based on observations about the shape of the centroid Liu and Mendel defined the notion of *quasi type-2 fuzzy logic system* (QT2FLS). They propose approximating the T2FS using two α-planes ($\widetilde{A}_{\widetilde{\alpha}=0}$ and $\widetilde{A}_{\widetilde{\alpha}=1}$). Great use of these methods is made later in this paper.

3 Type-2 Fuzzy Numbers

A type-1 fuzzy number (T1FN) is defined as a fuzzy set that is both normal and convex [16, 17]. Normality is required in order to capture the concept of a fuzzy number being a set of real numbers close to a specific crisp number [13], in other words when all the uncertainty about a number disappears it reduces to a crisp number. Convexity is required as it allows meaningful arithmetic operations to be performed on fuzzy sets using the well established methods from interval analysis since α-level sets are closed intervals [13]. Coupland *et al.* [6, 7] define a type-2 fuzzy number (T2FN) as a type-2 fuzzy set having a numerical domain. Although no assumption for normality has been defined for these T2FN the examples used assumed normality. From another perspective within the framework of interval-valued fuzzy sets (IVFS) [18], which are equal to IT2FS [5, 19], interval-valued fuzzy numbers (IVFN) are defined to be convex and normal [20, 21]. Here it is desired to define a host of cases that may be conceived to qualify as T2FN. First normality is defined, recalling its definition within the context of T1FN.

Definition 3.1 (Normal T1FS) *A T1FS, A, is said to be normal if its height $h(A)$ is equal to 1 i.e. $\sup \mu_A(x) = 1$.*

Whenever a T1FS is not normal it is called *subnormal*. In IT2FS we differentiate between two cases of normality, one when both the upper and lower MFs are normal, and the other when only the upper T1FS is normal.

Definition 3.2 (Normal IT2FS) *A IT2FS, $\widetilde{A}$, is said to be normal if its upper MF is normal i.e. $\sup \overline{\mu_A(x)} = 1$.*

Note that there is a point in which a crisp number can be reached which is only depending on the upper MF, this can be seen as a less restrictive condition and is widely used in applications (e.g. Computing with words [22]).

Definition 3.3 (Perfectly Normal IT2FS) *A IT2FS, $\widetilde{A}$, is said to be perfectly[2] normal if both its upper and lower MFs are normal i.e. $\sup \overline{\mu_A(x)} = \sup \underline{\mu_A(x)} = 1$.*

Here the crisp number is reached when uncertainties about both the upper and lower MFs disappear, it is more restrictive, but conceptually more appealing since it generalises the concept of normal T1FS with only a single peak point where it is completely crisp. It can be seen as a special case of a normal IT2FS, this is clear in figure (1).

Definition 3.4 (Partially Normal T2FS) *A T2FS, $\widetilde{A}$, is said to be partially normal if its FOU is **a normal IT2FS.***

This is the weakest case in which a T2FS can qualify to be a number. There is an argument, this can not qualify to be a fuzzy number at all as the secondary membership function is clearly not about 1 at any point (i.e $f_x(u) \neq 1, \forall f_x(u)$).

Definition 3.5 (Normal T2FS) *A T2FS, $\widetilde{A}$, is said to be normal if its FOU is **a normal IT2FS** and it has a PrMF.*

Definition 3.6 (Perfectly Normal T2FS) *A T2FS, $\widetilde{A}$, is said to be perfectly normal if its FOU is a perfectly normal IT2FS and it has a PrMF which is normal (i.e. either a normal T1FS or a normal IT2FS).*

Second, recalling the general definition of a T1FN using piecewise-defined functions.

Definition 3.7 (T1FN[13]) *Let A be a fuzzy subset on real numbers. Then, A is a fuzzy number if and only if there exists a closed interval $[m_1, m_2] \neq \phi$ such that*

$$\mu_A(x) = \begin{cases} 1 & x \in [m_1, m_2] \\ l(x) & x \in [s, m_1) \\ r(x) & x \in (m_2, e] \\ 0 & x \in (-\infty, s); x \in (e, \infty) \end{cases} \quad (12)$$

where $l(x) \in [0,1]$ is monotonically increasing and continuous from the right; $r(x) \in [0,1]$ is monotonically decreasing and continuous from the left; $\langle s, m_1, m_2, e\rangle$ are the parameters that define the T1FN.

Observe that if A is subnormal it is not considered a number, but it is useful to define a *type-1 fuzzy sub-number* (T1FsN) which satisfies all the properties of a T1FN except it is subnormal, this can be defined for a fuzzy set A with height $h(A) = \sup_{\forall x} \mu_A(x) = h_A$ as follows

$$\mu_A(x) = \begin{cases} h_A & x \in [m_1, m_2] \\ l(x) & x \in [s, m_1) \\ r(x) & x \in (m_2, e] \\ 0 & x \in (-\infty, s); x \in (e, \infty) \end{cases} \quad (13)$$

[1] Here $\widetilde{\alpha}$ is used instead of α to distinguish that these are α-cuts in the third dimension.

[2] this term has been used by Kaufmann and Gupta [17] describing a perfectly triangular T2FN.

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Using equations (7) and (8) a T1FN can be represented as intervals using their α-cuts $A_\alpha = [a_1^\alpha, a_2^\alpha]$ as follows

$$A = \bigcup_{\forall\alpha} \alpha.[a_1^\alpha, a_2^\alpha] = \bigcup_{\forall\alpha} \alpha.[l^{-1}(\alpha), r^{-1}(\alpha)] \quad (14)$$

IT2FN can be represented by its lower and upper membership functions which themselves are T1FS. Here we use the terms above to define IT2FN. Let $FOU(\widetilde{A}) = [\underline{FOU(\widetilde{A})}, \overline{FOU(\widetilde{A})}]$ [3] to represent a perfectly normal IT2FS defined by its lower and upper membership functions, respectively. Then, using equation (14) individually for each of the memberships, it follows that

$$\underline{FOU(\widetilde{A})} = \bigcup_{\forall\alpha} \alpha.[\underline{a_1^\alpha}, \underline{a_2^\alpha}] = \bigcup_{\forall\alpha} \alpha.[\underline{l^{-1}}(\alpha), \underline{r^{-1}}(\alpha)]$$

$$\overline{FOU(\widetilde{A})} = \bigcup_{\forall\alpha} \alpha.[\overline{a_1^\alpha}, \overline{a_2^\alpha}] = \bigcup_{\forall\alpha} \alpha.[\overline{l^{-1}}(\alpha), \overline{r^{-1}}(\alpha)]$$

then

$$FOU(\widetilde{A}) = \bigcup_{\forall\alpha} \alpha. \left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}]\right] \quad (15)$$

for a normal IT2FS with lower membership function height, $h(\underline{FOU(\widetilde{A})}) = h_l$, a IT2FN can be represented as

$$FOU(\widetilde{A}) = \begin{cases} \bigcup_{\forall\alpha} \alpha. \left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}]\right] & \alpha \le h_l \\ \bigcup_{\forall\alpha} \alpha.[\overline{a_1^\alpha}, \overline{a_2^\alpha}] & \alpha > h_l \end{cases} \quad (16)$$

Then a IT2FN can be defined by definition (3.8)

Definition 3.8 (IT2FN)
Let $FOU(\widetilde{A}) = [\underline{FOU(\widetilde{A})}, \overline{FOU(\widetilde{A})}]$ be an interval type-2 fuzzy subset on real numbers. Then, $FOU(\widetilde{A})$ is an interval type-2 fuzzy number if:

- *$\underline{FOU(\widetilde{A})}$ and $\overline{FOU(\widetilde{A})}$ are T1FNs, in this case it is called a perfectly normal IT2FN.*
- *$\underline{FOU(\widetilde{A})}$ is a T1FsN [4] and $\overline{FOU(\widetilde{A})}$ is a T1FN, in this case it is called a normal IT2FN.*

This definition can be seen in figure (1), observe that both IT2FSs are IT2FNs. We can also consider an interval type-2 fuzzy sub-number (IT2FsN) when bothe the upper and lower MFs are T1FsNs. Now, it is desired to define T2FN using the same terminology. From equation (10) a T2FS

$$\widetilde{A} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{A}_{\widetilde{\alpha}})$$

and using equation (15) a perfectly normal T2FS

$$\widetilde{A} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.(\bigcup_{\forall\alpha} \alpha. \left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}]\right]) \quad (17)$$

and using equation (16) a normal T2FS

$$\widetilde{A} = \begin{cases} \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.(\bigcup_{\forall\alpha} \alpha. \left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}]\right]) & \alpha \le h_l \\ \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.(\bigcup_{\forall\alpha} \alpha.[\overline{a_1^\alpha}, \overline{a_2^\alpha}]) & \alpha > h_l \end{cases} \quad (18)$$

A partially normal T2FS will only require $\forall\widetilde{\alpha}; \widetilde{\alpha} \in [0, \sup_{\forall(x,u)} f_x(u)]$.

[3] FOU is used here since the FOU fully describes the IT2FS.

[4] see equation (13).

Definition 3.9 (T2FN) *Let $\widetilde{A} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{A}_{\widetilde{\alpha}})$ be a type-2 fuzzy subset on real numbers. Then, $\widetilde{A}$ is a type-2 fuzzy number if:*

- *$FOU(\widetilde{A}_{\widetilde{\alpha}=0})$ is a perfectly normal IT2FN, and $FOU(\widetilde{A}_{\widetilde{\alpha}=1}) = Pr(\widetilde{A})$ is normal (i.e. either a T1FN or a normal IT2FN), in this case it is called a perfectly normal T2FN.*
- *$FOU(\widetilde{A}_{\widetilde{\alpha}=0})$ is a normal IT2FN, and $FOU(\widetilde{A}_{\widetilde{\alpha}=1})$ ($Pr(\widetilde{A})$) exist, in this case it is called a normal T2FN.*
- *$FOU(\widetilde{A}_{\widetilde{\alpha}=0})$ is a normal IT2FN, in this case it is called a partially normal T2FN.*

Note that a special case of $FOU(\widetilde{A}_{\widetilde{\alpha}=1}) = Pr(\widetilde{A})$ is when all the vertical slices constructing $Pr(\widetilde{A})$ are triangular T1FN then $Pr(\widetilde{A})$ is T1FS, this is depicted in figures (2) and (3). We can also define a type-2 fuzzy sub-number (T2FsN) when its FOU is an IT2FsN. In some applications one may need to restrict a fuzzy set to a specific form, e.g. in computing with words Klir *et al.* [23, 24] defined a procedure to convert any given convex fuzzy set to a fuzzy interval that is expressed in some standard form using some specific criteria. We now examine a special form of T2FN based on some ideas from [9]. Some observations about the shape of a centroid led to the proposition of a quasi type-2 fuzzy logic system, similirly, a definition of a quasi T2FN (QT2FN) can be proposed.

Assumption 3.10 (QT2FS) *Consider the following propositions about a T2FN*

A1 The T2FN is Normal (i.e FOU is normal IT2FN and PrMF is normal).

A2 The FOU upper and lower membership functions, the PrMF, and the vertical slices are characterised by piece-wise functions.

A3 All the vertical slices that construct PrMF are T1FN.

A4 As for the rest of the vertical slices, they are T1FsN in which their supremum lay on piece-wise functions between the sides of the upper membership function of the FOU and the PrMF denoted $\widetilde{l}(x)$ on the left side, and $\widetilde{r}(x)$ on the right side.

A5 All the mid points of the FOU upper and lower membership functions, and the PrMF are at the same domain value.

these assumptions allow a T2FN to be completely determined by its FOU and PrMF, just like a T1FS based on certain assumptions can be completely determined by its core and support.

Definition 3.11 (QT2FN) *Let $\widetilde{A} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{A}_{\widetilde{\alpha}})$ be a type-2 fuzzy subset on real numbers. Then, $\widetilde{A}$ is a quasi type-2 fuzzy number if it is completely determined by its FOU and PrMF.*

Figures (3) and (4) are both QT2FNs.

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4 Arithmetic using Alpha-planes

In Kaufmann and Gupta [17] a comprehensive discussion on fuzzy numbers and arithmetic operations which is formulated and mostly dependent upon the interval of confidence (α-level sets). Recalling operations on intervals, let $[a_1, a_2]$ and $[b_1, b_2]$ be two interval numbers, then

$$[a_1, a_2] \oplus [b_1, b_2] = [a_1 \oplus b_1, a_2 \oplus b_2]$$

$$[a_1, a_2] \otimes [b_1, b_2] = [\min(a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_1, a_2 \otimes b_2) \\ , \max(a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_1, a_2 \otimes b_2)] \quad (19)$$

where $\oplus \in \{+, -\}$, $\otimes \in \{\times, \div\}$ and $0 \notin B$ if $\otimes = \div$. Interval operations are extended to T1FN. Let $A = \bigcup_{\forall\alpha} \alpha.[a_1^\alpha, a_2^\alpha]$ and $B = \bigcup_{\forall\alpha} \alpha.[b_1^\alpha, b_2^\alpha]$ be two T1FN, then

$$A \circ B = \bigcup_{\forall\alpha} \alpha.([a_1^\alpha, a_2^\alpha] \circ [b_1^\alpha, b_2^\alpha]) \quad (20)$$

where $\circ = \{+, -, \times, \div\}$. Also these operations are extended to IT2FN. Let $FOU(\widetilde{A}) = \bigcup_{\forall\alpha} \alpha. \left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}]\right]$ and $FOU(\widetilde{B}) = \bigcup_{\forall\alpha} \alpha. \left[[\overline{b_1^\alpha}, \underline{b_1^\alpha}], [\underline{b_2^\alpha}, \overline{b_2^\alpha}]\right]$ be two perfectly normal IT2FN, then [17]

$$FOU(\widetilde{A}) \circ FOU(\widetilde{B}) = \\ \bigcup_{\forall\alpha} \alpha.(\left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}] \circ [\overline{b_1^\alpha}, \underline{b_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}] \circ [\underline{b_2^\alpha}, \overline{b_2^\alpha}]\right]) \quad (21)$$

If $FOU(\widetilde{A})$ and $FOU(\widetilde{B})$ are normal IT2FN with lower membership function heights $h_{\underline{\widetilde{A}}}$ and $h_{\underline{\widetilde{B}}}$ respectively, then the following changes are made to equation (21)

- if $0 \leq \alpha \leq \min(h_{\underline{\widetilde{A}}}, h_{\underline{\widetilde{B}}})$ no changes are made.
- if $h_{\underline{\widetilde{A}}} < \alpha \leq h_{\underline{\widetilde{B}}}$ then $\underline{a_1^\alpha} = \underline{a_1^{h_{\widetilde{A}}}}$; $\underline{a_2^\alpha} = \underline{a_2^{h_{\widetilde{A}}}}$, and if $h_{\underline{\widetilde{B}}} < \alpha \leq h_{\underline{\widetilde{A}}}$ then $\underline{b_1^\alpha} = \underline{b_1^{h_{\widetilde{B}}}}$; $\underline{b_2^\alpha} = \underline{b_2^{h_{\widetilde{B}}}}$.
- if $\max(h_{\underline{\widetilde{A}}}, h_{\underline{\widetilde{B}}}) < \alpha \leq 1$ then equation (21) becomes $FOU(\widetilde{A}) \circ FOU(\widetilde{B}) = \bigcup_{\forall\alpha} \alpha.([\overline{a_1^\alpha}, \overline{a_2^\alpha}] \circ [\overline{b_1^\alpha}, \overline{b_2^\alpha}])$.

Wu and Mendel [25] noticed that such methods that result in discontinuous or nonconvex sets are neither desirable nor technically correct. Similarly such results appear for normal IT2FN arithmetic operations. According to Wu and Mendel [25] the following changes are made to equation (21)

$$FOU(\widetilde{A}) \circ FOU(\widetilde{B}) = \\ \begin{cases} \bigcup_{\forall\alpha} \alpha.(\left[[\overline{a_1^\alpha}, \underline{a_1^\alpha}] \circ [\overline{b_1^\alpha}, \underline{b_1^\alpha}], [\underline{a_2^\alpha}, \overline{a_2^\alpha}] \circ [\underline{b_2^\alpha}, \overline{b_2^\alpha}]\right]) \\ \quad , \text{if } 0 \leq \alpha \leq \min(h_{\underline{\widetilde{A}}}, h_{\underline{\widetilde{B}}}) \\ \bigcup_{\forall\alpha} \alpha.([\overline{a_1^\alpha}, \overline{a_2^\alpha}] \circ [\overline{b_1^\alpha}, \overline{b_2^\alpha}]) \\ \quad , \text{if } \min(h_{\underline{\widetilde{A}}}, h_{\underline{\widetilde{B}}}) < \alpha \leq 1 \end{cases} \quad (22)$$

Clearly T2FN arithmetic can be extended, let $\widetilde{A} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{A}_{\widetilde{\alpha}})$ and $\widetilde{B} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{B}_{\widetilde{\alpha}})$ be two T2FNs, then

$$\widetilde{A} \circ \widetilde{B} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{A}_{\widetilde{\alpha}}) \circ \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.FOU(\widetilde{B}_{\widetilde{\alpha}})$$

using the extension principle in equation (11) it follows that

$$\widetilde{A} \circ \widetilde{B} = \bigcup_{\forall\widetilde{\alpha}} \widetilde{\alpha}.(FOU(\widetilde{A}_{\widetilde{\alpha}}) \circ FOU(\widetilde{B}_{\widetilde{\alpha}})) \quad (23)$$

then we can use equation (21) for perfectly normal T2FNs or equation (22) for normal T2FNs. Next we examine QT2FN, let $\widetilde{A}^Q$ and $\widetilde{B}^Q$ be QT2FN completely determined by the FOU and PrMF, $\langle FOU(\widetilde{A}^Q_{\widetilde{\alpha}=0}), Pr(\widetilde{A}^Q)\rangle$ and $\langle FOU(\widetilde{B}^Q_{\widetilde{\alpha}=0}), Pr(\widetilde{B}^Q)\rangle$, respectively. Then the result of basic arithmetic operations between them is a QT2FN

$$\widetilde{C}^Q = \widetilde{A}^Q \circ \widetilde{B}^Q \quad (24)$$

completely determined by $\langle FOU(\widetilde{C}^Q_{\widetilde{\alpha}=0}), Pr(\widetilde{C}^Q)\rangle$ where $FOU(\widetilde{C}^Q_{\widetilde{\alpha}=0}) = FOU(\widetilde{A}^Q_{\widetilde{\alpha}=0}) \circ FOU(\widetilde{B}^Q_{\widetilde{\alpha}=0})$ and $Pr(\widetilde{C}^Q) = Pr(\widetilde{A}^Q) \circ Pr(\widetilde{B}^Q)$. This operation can only be performed for addition and subtraction as the functions $\widetilde{l}(x)$ and $\widetilde{r}(x)$ are preserved. In the case of multiplication and devision an approximation for these functions should be used. It has to be mentioned that methods to approximate a T1FN to some standard form is used in the literature (see Grzegorzewski [26]). Providing approximation to QT2FN is still an open question.

5 Worked Example

In this example we only consider QT2FN as it gives sufficient insight on T2FN. Let us consider the following triangular QT2FN $\widetilde{3}^Q = \langle FOU(\widetilde{3}^Q_{\widetilde{\alpha}=0}), Pr(\widetilde{3}^Q)\rangle$ depicted in figure (5) with parameters[5] $FOU(\widetilde{3}^Q_{\widetilde{\alpha}=0}) = \langle 1.5, 2.25, 3, 3.45, 4.75\rangle$, $h_{\underline{\widetilde{3}}} = 0.6$, and $Pr(\widetilde{3}^Q) = \langle 1.75, 3, 4.25\rangle$. And the QT2FN $\widetilde{12}^Q = \langle FOU(\widetilde{12}^Q_{\widetilde{\alpha}=0}), Pr(\widetilde{12}^Q)\rangle$ depicted in figure (6) with parameters $FOU(\widetilde{12}^Q_{\widetilde{\alpha}=0}) = \langle 10.25, 11.5, 12, 12.5, 14\rangle$, $h_{\underline{\widetilde{12}}} = 0.7$, and $Pr(\widetilde{12}^Q) = \langle 10.75, 12, 13.5\rangle$. When computing the addition $\widetilde{3}^Q + \widetilde{12}^Q$, first, we determine a suitable number of α-cuts along u for both FOU and PrMF[6]. In our case and for the sake of clarity we discretised u into 25 α-cuts. Then, we apply equation (24) to the decomposed IT2FS. This gives the result depicted in figure (7) with parameters $FOU(\widetilde{15}^Q_{\widetilde{\alpha}=0}) = \langle 11.75, 13.75, 15, 15.95, 18.75\rangle$, $h_{\underline{\widetilde{15}}} = 0.6$, and $Pr(\widetilde{15}^Q) = \langle 12.5, 15, 17.75\rangle$. This result we would expect, $\widetilde{3}^Q + \widetilde{12}^Q = \widetilde{15}^Q$.

6 Conclusions

In this paper we presented methods to perform type-2 fuzzy arithmetic operations using well known arithmetic operations on intervals. In order to better analyse our methods we defined a set of terms that describe different normality formations of IT2FS and T2FS, some of which, have already been used in the literature without a clear definition. Our methods relied heavily on interval analysis through α-cuts and α-planes that allow T2FSs be represented as a collection of intervals. Furthermore, we examined the use of quasi type-2 fuzzy sets proposed by Mendel and Liu and defined quasi type-2 fuzzy numbers as a special case of type-2 fuzzy numbers. Finally, we derived arithmetic operations for quasi type-2 fuzzy numbers and provided an illustration by a worked example.

[5] The parameters used here are $FOU(\widetilde{A}^Q_{\widetilde{\alpha}=0}) = \langle s_1, s_2, m_1, e_2, e_1\rangle$ derived from equation (12) where $\overline{FOU(\widetilde{A}^Q_{\widetilde{\alpha}=0})} = \langle s_1, m_1, e_1\rangle$ and $\underline{FOU(\widetilde{A}^Q_{\widetilde{\alpha}=0})} = \langle s_2, m_1, e_2\rangle$.

[6] In the case of T2FN we first discretise along $f_x(u)$ in order to determine a suitable number of α-planes, then, we discretise along u for each of the α-planes.

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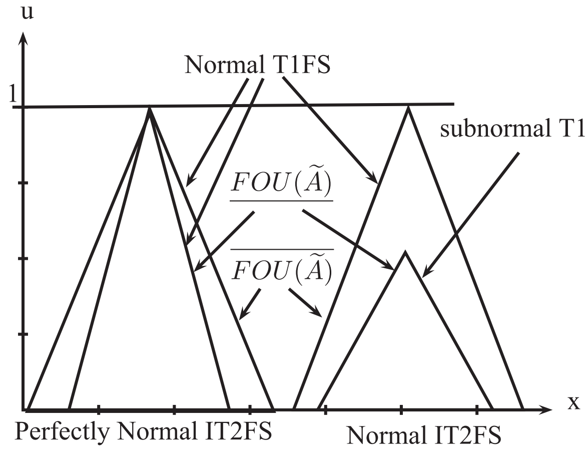

Figure 1: Different types of normal fuzzy sets

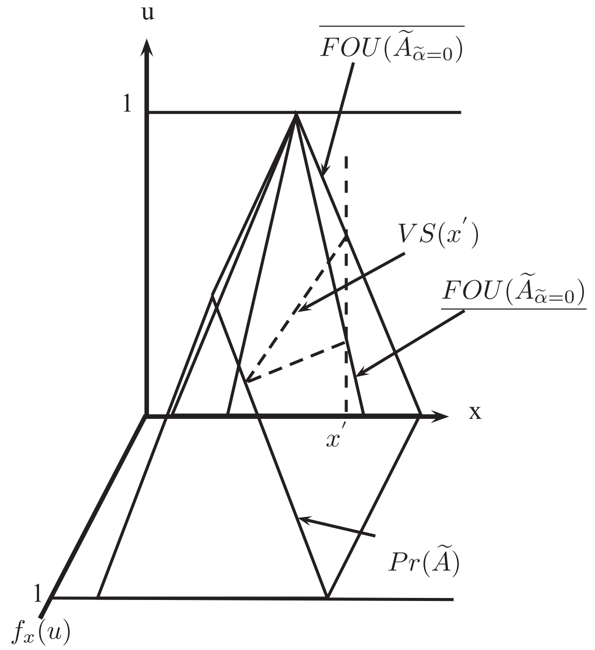

Figure 2: 3D representation of a perfectly normal T2FN, it is also a QT2FN.

We believe that this paper will form the basis for further work on type-2 fuzzy numbers and thier applications. Quasi type-2 fuzzy numbers may be viewed as a next step in the progress between IT2FN and T2FN. Arithmetic operations are already used in aggregation and averaging operations, measures that apply certain mathematical functions, and fusion functions between one system and another.

Further work with regards to the operations defined here will include approximation of standard forms of Quasi type-2 fuzzy numbers, and a thorough comparison between these methods and previous methods in terms of computational complexity, although, it is almost trivial that our methods are computationally sound as they are based on interval analysis.

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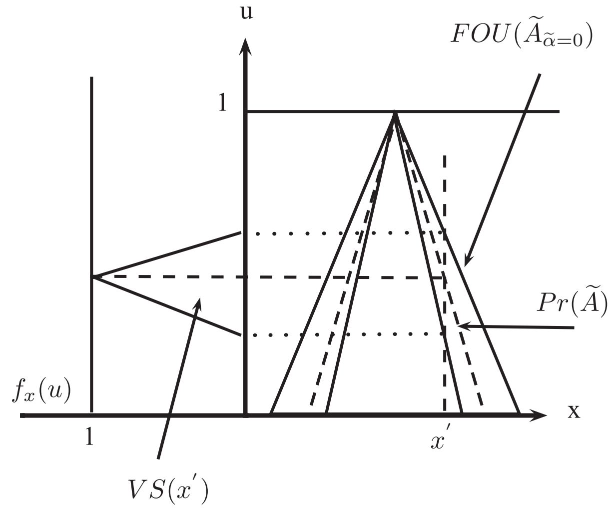

Figure 3: 2D representation of a perfectly normal T2FN of figure (2) with triangular vertical slices.

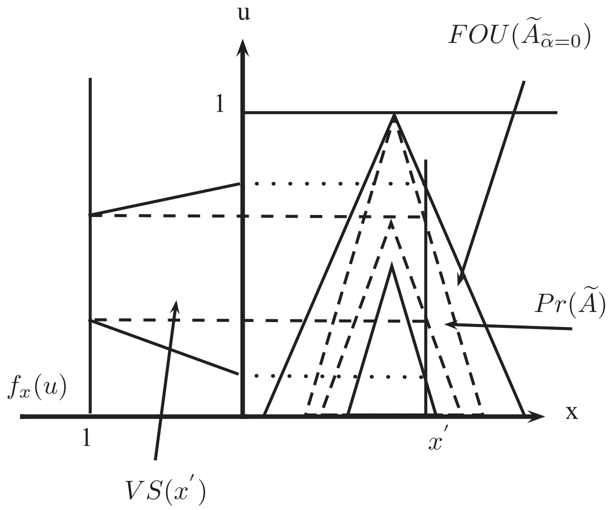

Figure 4: 2D representation of a normal T2FN with trapizoidal vertical slices, note that $Pr(\widetilde{A})$ is normal IT2FN.

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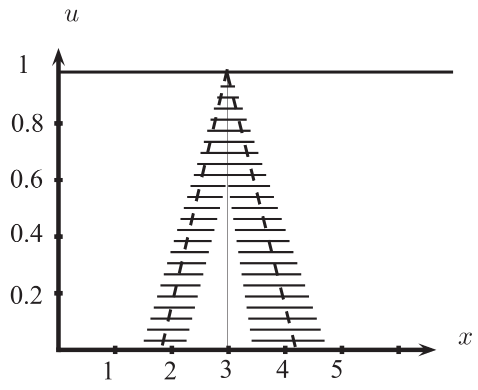

Figure 5: $\widetilde{3}^Q$ with discretised α-cuts, and the dashed line is $Pr(\widetilde{3}^Q)$

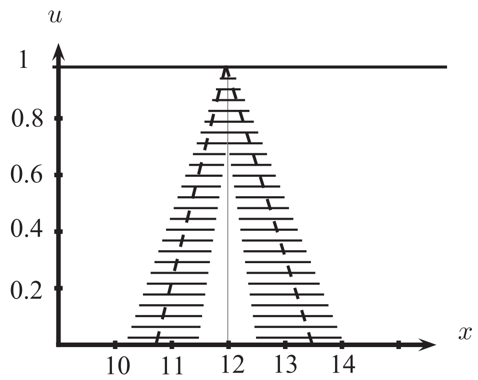

Figure 6: $\widetilde{12}^Q$ with discretised α-cuts, and the dashed line is $Pr(\widetilde{12}^Q)$

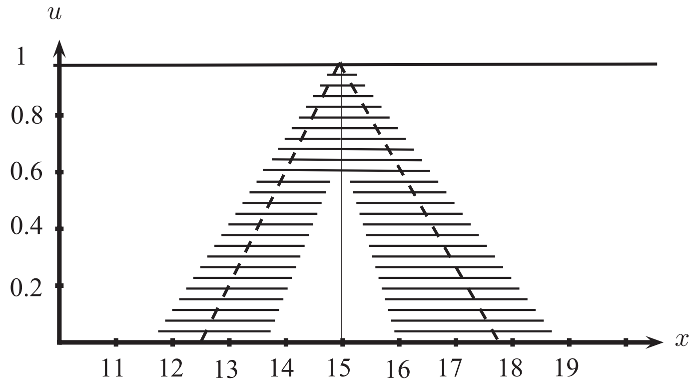

Figure 7: $\widetilde{15}^Q$ with discretised α-cuts, and the dashed line is $Pr(\widetilde{15}^Q)$

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