Operations for Real-Valued Bags and Bag Relations

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Abstract— *A generalization of bags is proposed and fundamental operations are described. That is, a real-valued bag is proposed which is a straightforward generalization of the traditional bags, but is necessary for complementation, and s-norm and t-norm operations. We mainly describe how complementation and s-norms are related. As a result a duality theorem between s-norm and t-norm is obtained. R-bag relations with max-s norm composition and max-t norm composition are also proposed. Further generalization that includes fuzzy bags and real-valued bags having membership of a region in a plane is moreover discussed.*

Keywords— bag relation, max-s composition, real-valued bag, set-valued bag, t-norm and conorm .

1 Introduction

Bags which are also called *multisets* have long been studied by computer scientists [5, 6]. In soft computing, Yager [19] have proposed fuzzy bags, and its theory and applications have been studied by several researchers [3, 4, 15, 16, 17, 18, 20]. In particular, the author [9, 10, 12] has redefined basic operations of the union and intersection for fuzzy bags. The definition anew by the authors is necessary in order to keep consistency between operations for ordinary fuzzy sets and those for fuzzy bags [12]. However, there are many problems that are left and should be studied.

One class of such problems is fundamental operations of bags including complementation and bag s-norms (and s-norms). Another class is bag relations that should have composition operations. In this paper we study these two problem classes. While studying these problems, we should generalize the traditional concept of bags, as described in this paper. We therefore introduce a generalization called R-bag which appears straightforward, but this generalization is necessary to consider complementation and bag t-norms. As a result, a duality theorem is derived using the complementation. Moreover a Minkwoski type bag s-norms are derived using a generating function.

Bag relations and their compositions for R-bags are then studied by introducing a max-s algebra and a max-t algebra. Here s and t stands for s-norm and t-norm.

Another generalization that includes R-bag and fuzzy bag is moreover mentioned and how the above consideration is extended to this generalized bag is suggested.

There are a number of propositions herein, but to prove propositions given in this paper is not difficult. Hence short notes on proofs are given in the appendix.

Although the purpose of this paper is to show theoretical properties of generalized bags, bags have many application possibilities, which are briefly mentioned in the conclusion.

2 Bags and R-Bags

The two terms of bags and multisets are used interchangeably, and we use *bags* throughout this paper.

2.1 Overview of crisp bags

Assume that the universal set $X = \{x_1, \dots, x_n\}$ is finite for simplicity. A crisp bag M of $X = \{x_1, \dots, x_n\}$ is characterized by a function $C_M(\cdot)$ (called count of M) whereby a natural number including zero $\boldsymbol{N} = \{0, 1, 2, \dots\}$ corresponds to each $x \in X$ ($C_M : X \to \boldsymbol{N}$).

For a crisp bag, different expressions such as

$$M = \{k_1/x_1, \dots, k_n/x_n\}$$

and

$$M = \{\overbrace{x_1, \dots, x_1}^{k_1}, \dots, \overbrace{x_n, \dots, x_n}^{k_n}\}$$

are used. In such a way, an element of X may appear more than once in a bag. In the above example x_i appears k_i times in M ($i = 1, \dots, n$).

Consider an example where $X = \{a, b, c, d\}$ and $C_M(a) = 2$, $C_M(b) = 1$, $C_M(c) = 3$, $C_M(d) = 0$. In other words, $M = \{a, a, b, c, c, c\}$. This means that a, b, c, and d are included 2, 1, 3, and 0 times, respectively, in M. We can write $M = \{2/a, 1/b, 3/c\}$ by ignoring an element of zero occurrence.

Basic relations and operations for crisp bags are as follows.

1. (inclusion):
$M \subseteq N \Leftrightarrow C_M(x) \le C_N(x), \quad \forall x \in X.$
2. (equality):
$M = N \Leftrightarrow C_M(x) = C_N(x), \quad \forall x \in X.$
3. (union):
$C_{M \cup N}(x) = \max\{C_M(x), C_N(x)\}.$
4. (intersection):
$C_{M \cap N}(x) = \min\{C_M(x), C_N(x)\}.$
5. (addition or sum):
$C_{M \oplus N}(x) = C_M(x) + C_N(x).$
6. (subtraction):
$C_{M \ominus N}(x) = \max\{0, C_M(x) - C_N(x)\}.$
7. (scalar multiplication):
$C_{\alpha M} = \alpha C_M(x)$, where α is a positive integer.
8. (Cartesian product):
Let P is a bag of Y.
$C_{M \times P}(x, y) = C_M(x) C_P(y).$

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We sometimes use $\vee$ and $\wedge$ for $\max$ and $\min$, respectively. We also note that the relations and operations are similar to those for fuzzy sets. However, bags have the addition which fuzzy sets do not have, and the Cartesian product for bags is different from that for fuzzy sets. Moreover, a complementation is not defined in general.

2.2 *R-bags*

An immediate generalization of crisp bag is to assume that $C_M(\cdot)$ has a nonnegative real-value instead of a natural number. For example, we admit $C_M(a) = 3.14$ or $C_N(a) = \pi$. Moreover we admit the value of infinity $+\infty$ for a technical reason. We thus assume $C_M : X \to [0, +\infty]$.

The above relations and operations 1–8 are used without any change except that α in the scalar multiplication can now be a positive real constant. For simplicity, a real-valued bag is called an R-bag here and the collection of all R-bags of X is denoted by $\mathcal{RB}(X)$.

Complementation of R-bags

A function $\mathcal{N} : [0, +\infty] \to [0, +\infty]$ with the next properties is used to define a complementation operation:

(i) $\mathcal{N}(0) = +\infty, \quad \mathcal{N}(+\infty) = 0.$

(ii) $\mathcal{N}(x)$ is strictly monotonically decreasing on $(0, +\infty)$.

A typical example is $\mathcal{N}(x) = const/x$ with $const > 0$.

Using this function, a natural operation for the complement is:

9.(complement):
$C_{\bar{M}}(x) = \mathcal{N}(C_M(x))$.

This operation justifies the generalization into R-bags. That is, even when we start crisp bags, the result of complementation is generally real-valued.

We immediately have the next two propositions of which the proof is easy and omitted.

Proposition 1 *For arbitrary $M, N \in \mathcal{RB}(X)$, the following properties hold:*

$$\overline{(\bar{M})} = M \tag{1}$$

$$\overline{M \cup N} = \bar{M} \cap \bar{N}, \qquad \overline{M \cap N} = \bar{M} \cup \bar{N}. \tag{2}$$

Proposition 2 *If we introduce an empty bag $\emptyset$ and the maximum bag* **Infinity** *in $\mathcal{RB}$ by*

$$C_\emptyset(x) = 0, \quad \forall x \in X, \tag{3}$$

$$C_{\textbf{Infinity}}(x) = +\infty, \quad \forall x \in X. \tag{4}$$

Then we have

$$\bar{\emptyset} = \textbf{Infinity}, \quad \overline{\textbf{Infinity}} = \emptyset. \tag{5}$$

s-norms and t-norms of R-bags

There have been studies on t-norms for crisp bags [7, 1], but generalization into R-bags admits a broader class of s-norms and t-norms. For this purpose we introduce two functions $t(a, b)$ and $s(a, b)$ as those in fuzzy sets, but the definitions are different.

Definition 1 *Two functions $t : [0, +\infty] \times [0, +\infty] \to [0, +\infty]$ and $s : [0, +\infty] \times [0, +\infty] \to [0, +\infty]$ having the following properties (I)–(IV) are called a t-norm and an s-norm for bags, respectively. An s-norm is also called a t-conorm for bags.*

(I)[monotonicity] For $a \le c$, $b \le d$,

$$t(a, b) \le t(c, d),$$
$$s(a, b) \le s(c, d).$$

(II)[symmetry]

$$t(a, b) = t(c, d), \quad s(a, b) = s(b, a).$$

(III)[associativity]

$$t(t(a, b), c) = t(a, t(b, c)),$$
$$s(s(a, b), c) = s(a, s(b, c)).$$

(IV)[boundary condition]

$$t(0, 0) = 0, \quad t(a, +\infty) = t(+\infty, a) = a,$$
$$s(+\infty, +\infty) = +\infty, \quad s(a, 0) = s(0, a) = a.$$

A purpose to introduce such norms for bags is to generalize the intersection and union operations. First we note that $s(a, b) = a + b$ and $s(a, b) = \max\{a, b\}$ satisfy the above conditions. The norm $t(a, b) = \min\{a, b\}$ also satisfies (I)–(IV). Thus, s-norms and t-norms can represent the three operations of addition, union, and intersection.

We moreover introduce a generating function $g(x)$ for s-norm.

Definition 2 *A function $g : [0, +\infty] \to [0, +\infty]$ is called a generating function for s-norm if it satisfies the next (i)–(iii):*
(i) it is strictly monotonically increasing,
(ii) $g(0) = 0, \quad g(+\infty) = +\infty$,
(iii) $g(x + y) \ge g(x) + g(y), \quad \forall x, y \in [0, +\infty]$.
We have

Proposition 3 *Let*

$$s(a, b) = g^{-1}(g(a) + g(b)). \tag{6}$$

Then $s(a, b)$ is an s-norm.

An example of the generation function is

$$g(x) = x^p \quad (p \ge 1). \tag{7}$$

Moreover, note the following.

Proposition 4 *Let $s(a, b)$ is an s-norm and $\mathcal{N}$ is the complementation operator. Then*

$$t(a, b) = \mathcal{N}(s(\mathcal{N}(a), \mathcal{N}(b))) \tag{8}$$

is a t-norm. Suppose $t(a, b)$ is a t-norm, then

$$s(a, b) = \mathcal{N}(t(\mathcal{N}(a), \mathcal{N}(b))) \tag{9}$$

is an s-norm.

If a pair of t-norm and s-norm has the above property, we say (s, t) has the *duality* of norm and conorm. The duality has the next property.

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Proposition 5 *Suppose $s_0(a,b)$ is an s-norm and $t_0(a,b)$ is derived from $s_0(a,b)$ by the operation (8). Let*

$$s(a,b) = \mathcal{N}(t_0(\mathcal{N}(a),\mathcal{N}(b)))$$

Then $s(a,b) = s_0(a,b)$.

Suppose also that $t_0(a,b)$ is a t-norm and $s_0(a,b)$ is derived from $t_0(a,b)$ by the operation (8). Let

$$t(a,b) = \mathcal{N}(s_0(\mathcal{N}(a),\mathcal{N}(b)))$$

Then $t(a,b) = t_0(a,b)$. In summary, applying (8) and (9) (resp. (9) and (8)) successively, we have the original norm.

Note that s-norm and t-norm are applied to define bag operations $M\mathcal{S}N$ and $M\mathcal{T}N$, respectively, for $M, N \in \mathcal{RB}(X)$.

$$C_{M\mathcal{S}N}(x) = s(C_M(x), C_N(x)). \quad (10)$$

$$C_{M\mathcal{T}N}(x) = t(C_M(x), C_N(x)). \quad (11)$$

Let us consider examples of s-norms and t-norms.

Example 1 The standard operators

$$s(a,b) = \max\{a,b\} \quad (12)$$

$$t(a,b) = \min\{a,b\} \quad (13)$$

are an s-norm and a t-norm, respectively. This pair has the duality stated in Proposition 5. Note, however, that s-norm (12) does not have a generating function that satisfies (6).

Second example uses the generating function.

Example 2 Let $g(x)$ be given by (7). Then we have

$$s(a,b) = (a^p + b^p)^{\frac{1}{p}}, \quad (14)$$

$$t(a,b) = (a^{-p} + b^{-p})^{-\frac{1}{p}}. \quad (15)$$

are an s-norm and a t-norm, respectively. This pair has the duality stated in Proposition 5 when $\mathcal{N} = const/x$ is used.

The second example has interesting properties. First, $s(a,b) = a+b$ is a particular case of (14) for $p = 1$. Moreover $s(a,b) = \max\{a,b\}$ and $t(a,b) = \min\{a,b\}$ are obtained from (14) and (15) when $p \to +\infty$.

New findings are included in the above example. First, the dual t-norm derived from $s(a,b) = a+b$ is the harmonic mean:

$$t(a,b) = (a^{-1} + b^{-1})^{-1} = \frac{ab}{a+b}.$$

(Note that the arithmetic mean $\frac{a+b}{2}$ and the geometric mean $\sqrt{ab}$ are not t-norms). Second, $s(a,b) = (a^2+b^2)^{\frac{1}{2}}$ with $p = 2$ which is similar to the Euclidean norm is an s-norm, and the dual t-norm is

$$t(a,b) = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}.$$

2.3 R-bags and fuzzy sets

Let us study what relations exist between fuzzy sets and R-bags. For this purpose we introduce another notation: the collection of all fuzzy sets of X is denoted by $\mathcal{FS}(X)$.

We introduce another function $h\colon [0,+\infty] \to [0,1]$ that is strictly monotonically increasing in $[0,+\infty)$ and

$$h(0) = 0, \quad h(+\infty) = 1.$$

An example of such a function is

$$h_0(x) = 1 - \exp(-K_0 x), \quad x \in [0,+\infty)$$

with constant $K_0 > 0$ and with the additional definition $h_0(+\infty) = 1$. Another example is

$$h_1(x) = 1 - \frac{K_1}{K_1 + K_2 x}, \quad x \in [0,+\infty)$$

with positive constants K_1, K_2 and with $h_1(+\infty) = 1$.

Since such a function has an inverse function $h^{-1}(y)$, we can define one to one correspondence between a fuzzy set and an R-bag:

$$\mu_{h(M)}(x) = h(C_M(x)). \quad (16)$$

For $M \in \mathcal{RB}(X)$, $h(M) \in \mathcal{FS}(X)$, and $h^{-1}(h(M)) = M$. Note that $h^{-1}(y)$ is also a strictly monotonically increasing.

We also have

$$h(\emptyset) = \emptyset, \quad h(\mathbf{Infinity}) = X;$$

$$h^{-1}(\emptyset) = \emptyset, \quad h^{-1}(X) = \mathbf{Infinity}.$$

Using these functions, we can generate operations for R-bags from those for fuzzy sets.

Proposition 6 *Let*

$$\mathcal{N}(x) = h^{-1}(1 - h(x)). \quad (17)$$

Then $\mathcal{N}(x)$ satisfies the properties of complementation operation. Hence for arbitrary $M \in \mathcal{RB}(X)$, $\bar{M}$ derived by

$$C_{\bar{M}}(x) = \mathcal{N}(C_M(x))$$

using (17) is a complement of M.

Note that this operation is derived from the complementation of fuzzy sets.

For the s-norm and t-norm, we have similar results. Let $s_F(a,b)$ and $t_F(a,b)$ are respectively an s-norm and a t-norm for *fuzzy sets*. We have the following.

Proposition 7 *For $a, b \in [0,+\infty]$, let*

$$s(a,b) = h^{-1}(s_F(h(a),h(b))) \quad (18)$$

$$t(a,b) = h^{-1}(t_F(h(a),h(b))) \quad (19)$$

Then $s(a,b)$ and $t(a,b)$ are an s-norm and a t-norm for R-bags.

we also have

Proposition 8 *Suppose $s(a,b)$ and $t(a,b)$ are an s-norm and a t-norm for R-bags. For $a, b \in [0,1]$, let*

$$s_F(a,b) = h(s(h^{-1}(a), h^{-1}(b))) \quad (20)$$

$$t_F(a,b) = h(t(h^{-1}(a), h^{-1}(b))) \quad (21)$$

Then $s_F(a,b)$ and $t_F(a,b)$ are an s-norm and a t-norm for fuzzy sets.

3 Bag Relations

What we consider in this section is a bag-relations and their composition. As a result we have max-s and max-t algebras for R-bags. The discussion in this section generalizes the max-plus algebra [2].

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3.1 max-s and max-t algebra

Let us use a particular notation of $\boxplus$ and $\boxdot$ for

$$a \boxplus b = \max\{a, b\}, \quad a \boxdot b = s(a, b) \tag{22}$$

where $s(a, b)$ is an s-norm for R-bags. We call this algebra as *max-s algebra*.

It is easy to see that the following properties hold.

$$a \boxplus b = b \boxplus a, \tag{23}$$
$$a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c, \tag{24}$$
$$a \boxplus 0 = a, \tag{25}$$
$$a \boxdot b = b \boxdot a, \tag{26}$$
$$a \boxdot (b \boxdot c) = (a \boxdot b) \boxdot c, \tag{27}$$
$$a \boxdot 0 = a. \tag{28}$$

Alternatively, we can define $\boxplus$ and $\boxdot$ for

$$a \boxplus b = \max\{a, b\}, \quad a \boxdot b = t(a, b) \tag{29}$$

where $t(a, b)$ is a t-norm for R-bags. We call this algebra as *max-t algebra*. We see that (23)–(27) hold, while (28) should be replaced by

$$a \boxdot +\infty = a. \tag{30}$$

We moreover note the next lemma.

Lemma 1 *Let a, b, c be real numbers. Then*

$$a \boxdot (b \boxplus c) = (a \boxdot b) \boxplus (a \boxdot c). \tag{31}$$

Thus the commutative, associative, and distributive properties hold for the max-s algebra. Noting Example 2, we observe that the max-t algebra includes the max-min algebra and the max-s algebra includes the max-plus algebra as particular cases.

3.2 Bag relations

We now define R-bag relations. The definition is simple. An R-bag relation R on $X \times Y$ is an R-bag R of $X \times Y$. The count function is denoted by $R(x, y)$ instead of $C_R(x, y)$ for simplicity. Sometimes an R-bag relation is also called bag relation for simplicity.

The reason why we call such a bag $R(x, y)$ a bag relation is that we can define composition operation.

Let X, Y, Z be three universes. Assume R is a bag relation of $X \times Y$ and S is a bag relation of $Y \times Z$. Then the max-s composition $R \circ S$ is defined as follows.

$$(R \circ S)(x, z) = \boxplus_{y \in Y} \{R(x, y) \boxdot S(y, z)\} \tag{32}$$

Note that

$$\boxplus_{y \in \{a_1, \ldots, a_L\}} = a_1 \boxplus a_2 \boxplus \cdots \boxplus a_L.$$

The max-t composition is defined by the same equation (32) except that $\boxdot$ uses a t-norm.

Note also that the addition is straightforward

$$(R_1 \boxplus R_2)(x, y) = R_1(x, y) \boxplus R_2(x, y), \tag{33}$$

for bag relations on $X \times Y$.

We have the following proposition.

Proposition 9 *The composition satisfies the associative property*

$$(R \circ S) \circ T = R \circ (S \circ T). \tag{34}$$

and the distributive property

$$(R_1 \boxplus R_2) \circ S = (R_1 \circ S) \boxplus (R_2 \circ S), \tag{35}$$
$$R \circ (S_1 \boxplus S_2) = (R \circ S_1) \boxplus (R \circ S_2). \tag{36}$$

We introduce the unit relations for the max-s and max-t compositions. Define O_X and Ω_X on $X \times X$:

$$O_X(x, y) = 0, \quad \forall x, y \in X, \tag{37}$$
$$\Omega_X(x, y) = +\infty, \quad \forall x, y \in X. \tag{38}$$

We define O_{XY} and Ω_{XY} on $X \times Y$ in the same way:

$$O_{XY}(x, y) = 0, \quad \forall (x, y) \in X \times Y, \tag{39}$$
$$\Omega_{XY}(x, y) = +\infty, \quad \forall (x, y) \in X \times Y. \tag{40}$$

Frequently we omit the subscripts like O and Ω when we have no ambiguity.

We then have

Proposition 10 *Assume that the max-s algebra is used. For arbitrary bag relation R on $X \times Y$,*

$$R \boxplus O = O \boxplus R = R, \tag{41}$$
$$R \circ O = O \circ R = R. \tag{42}$$

Proposition 11 *Assume that the max-t algebra is used. For arbitrary bag relation R on $X \times Y$,*

$$R \boxplus O = O \boxplus R = R, \tag{43}$$
$$R \circ \Omega = \Omega \circ R = R. \tag{44}$$

4 A Note on Further Generalization

The above generalization to R-bags does not include fuzzy bags [19, 10]. It appears that R-bags and fuzzy bags are inconsistent. However, there is further generalization discussed in [12] which uses a closed region in $\boldsymbol{R}^2$ as the count function for generalized bags. In this section we briefly note how the above discussion is extended into generalized bags which encompasses R-bags and fuzzy bags.

A generalized bag A has a count function as a region

$$\mathbf{N}(x, A) = C_A(x), \quad \forall x \in X \tag{45}$$

where $\mathbf{N}(x, A)$ is a closed set of $[0, +\infty)^2$. Let $\nu(y, z; x, A)$ is the characteristic function of the region $\mathbf{N}(x, A)$:

$$\nu(y, z; x, A) = \begin{cases} 1, & (y, z) \in \mathbf{N}(x, A), \\ 0, & (y, z) \notin \mathbf{N}(x, A). \end{cases}$$

This region should have the following properties:

(N1) For any $y \in [0, +\infty)$,

$$\eta(y) = \{z \in [0, +\infty) : \nu(y, z; x, A) = 1\}$$

is either an empty set or a closed interval $[0, \mu(y)]$ where $\mu(y) = \max\{y' : y' \in \eta(y)\}$.

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(N2) For all $y \in [0, +\infty)$, $\mu(y)$ is a monotonically decreasing function and $0 \leq \mu(y) \leq 1$.

(N3) For any $z \in [0, +\infty)$,

$$\theta(z) = \{y \in [0, +\infty) : \nu(y, z; x, A) = 1\}$$

is either an empty set or a closed interval $[0, \zeta(z)]$ where $\zeta(z) = \max\{z' : z' \in \theta(z)\}$.

(N4) For all $z \in [0, +\infty)$, $\zeta(z)$ is a monotonically decreasing function and $0 \leq \zeta(z) < +\infty$.

Roughly, a generalized bag is characterized by the pair $(\mu(y), \zeta(y))$ for each $x \in X$.

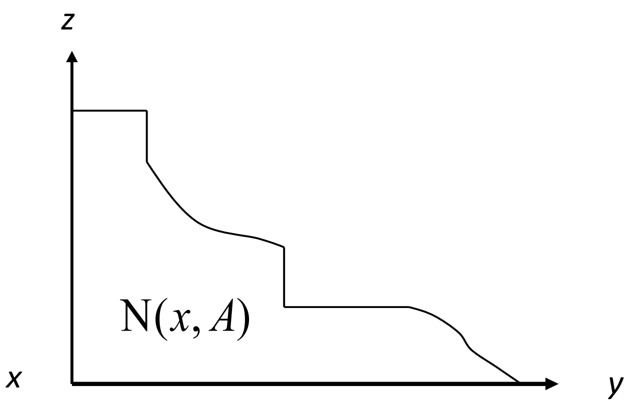

Figure 1: An illustration of $\mathbf{N}(x, A)$ as the count function for a generalized bag.

Two cuts which relate the generalized bag to R-bags and fuzzy sets are defined:

α-cut: Let $\alpha \in (0, 1]$. For the generalized bag A, $[A]_\alpha$ is an R-bag given by

$$C_{[A]_\alpha}(x) = \max\{z : \nu(\alpha, z; x, A) = 1\}.$$

ℓ-cut: Let $\ell \in (0, +\infty)$. For the generalized bag A, $\langle A \rangle_\ell$ is a fuzzy set given by

$$\mu_{\langle A \rangle_\ell}(x) = \max\{y : \nu(y, \ell; x, A) = 1\}.$$

It can be proved that the generalized bags are smallest generalization of fuzzy bags and R-bags [12]. Moreover, s-norms and t-norms can be used for the generalized bags using the α-cuts, while the complementation operation cannot be used for the generalized bags. Bag relations and their compositions can also be used for the generalized bags. The detailed consideration is too lengthy to give here and will be discussed elsewhere.

5 Conclusion

We have shown the operations of complementation and s-norm and t-norm for real-valued bags. Moreover R-bag relations and the max-s (or max-t) compositions are studied. Further generalization has moreover been proposed of which the details will be discussed in near future.

As seen herein, the theory of bags should be studied more deeply, since it has more ample mathematical structure than expected until now. More theoretical studies are thus needed. Relations between rough sets [14] and bags have been discussed [12], but there is much room for further studies. Application of bags such as those in information retrieval [8] and information mining [11, 13] will also be important.

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Appendix: Notes on Proofs

Since detailed proofs are too lengthy, we give short notes on how to prove the propositions given here. The proofs are mostly straightforward.

Notes on proofs of Propositions 1 and 2

These propositions are immediately proved by straightforward calculation using $\mathcal{N}$. The details are hence omitted.

Notes on proofs of Propositions 3 and 4

It is enough to check the conditions (I)–(IV) are satisfied.

Notes on proof of Proposition 5

$$\begin{aligned} s(a,b) &= \mathcal{N}(t_0(\mathcal{N}(a), \mathcal{N}(b))) \\ &= \mathcal{N}(\mathcal{N}(s_0(\mathcal{N}(\mathcal{N}(a)), \mathcal{N}(\mathcal{N}(b))))) \\ &= s_0(a,b). \end{aligned}$$

Notes on proof of Proposition 6

By a simple calculation, it is seen that $\mathcal{N}$ by (17) satisfies the conditions for the complementation.

Notes on proofs of Propositions 7 and 8

The conditions for s-norms and t-norms for fuzzy sets should be noted. Mostly it is easy to check that the conditions are satisfied, since the functions h and h^{-1} are monotone. The boundary conditions are satisfied from $h(0) = 0$ and $h(+\infty) = 1$.

Notes on proof of Lemma 1

Let $b \geq c$, then the left hand side is $s(a, b)$. The right hand side is

$$\max\{s(a, b), s(a, c)\} = s(a, b).$$

The case of $b < c$ is calculated likewise and we have the desired equality.

When we use a max-t algebra, we can prove the lemma in a similar way, since the monotone property is common to s-norm and t-norm. We omit the detail.

Notes on proof of Proposition 9

The detailed proof is lengthy, but the way to prove this proposition is just the same as the proof of the matrix properties for ordinary algebra, since those properties for ordinary algebra uses the commutative, associative, and distributive properties which are valid for the present algebra, since (23)–(28) (or (30)) and (31) are valid.

Notes on proofs of Propositions 10 and 11

Omitted as they are straightforward.

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