Distorted probabilities and m-separable fuzzy measures

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Abstract— *Distorted probabilities are an important family of fuzzy measures. In a recent paper we introduced m-dimensional distorted probabilities, which generalize the former and permit us to have a smooth transition from distorted probabilities to unconstrained ones.*

In this paper we introduce the union condition and the strict union condition, and we show that when these conditions hold for a fuzzy measure, the fuzzy measure is a distorted probability. In addition, we present a few results that establish some relationships between other fuzzy measures.

Keywords: Fuzzy measures, distorted probabilities, m-dimensional distorted probabilities, m-symmetric fuzzy measures, union condition, strict union condition.

1 Introduction

Aggregation operators [28] are used in a large number of applications to combine information from different sources. Although the weighted mean is probably the most well known aggregation operator, other operators exist. E.g., the OWA, the WOWA and some fuzzy integrals. Choquet and Sugeno integrals are examples of such fuzzy integrals.

Fuzzy integrals permits the user to combine information when the sources supplying the information are not independent. To express this a priori knowledge about the sources, fuzzy integrals combine the input data with the information about the sources. Formally, the integrals integrate a function, which represents the data being aggregated, with respect to a fuzzy measure, which represents the a priori information about the sources.

A major difficulty for applying fuzzy integrals in real applications is that they are set functions, and thus, for any aggregation problem with n inputs, 2^n values should be defined. In fact, properly speaking, boundary conditions on the measure reduce this number to $2^n - 2$.

Real applications try to reduce the number of $2^n - 2$ required values using constrained measures. That is, measures that require less than $2^n - 2$ parameters. Sugeno λ-measures are probably the most used ones [22, 23, 20, 21]. Such measures solely require n values as well as an additional parameter λ, which can be deduced from the n values as [10] shows. k-order additive fuzzy measures are another family of measures with reduced complexity. This family, that has been extensively studied, is of special interest because the parameter k permits us to find a trade-off between expressiveness and complexity. In short, when $k = 1$ the measure has the lowest complexity (only $n - 1$ values are required) but it corresponds to a probability distribution. Instead, when $k = n$, any unconstrained fuzzy measure can be represented but at the cost of the highest complexity ($2^n - 2$ values should be defined).

In this paper we study m-dimensional distorted probabilities, another family of measures, introduced in [16], that generalizes distorted probabilities. Informally, distorted probabilities are measures that can be expressed in terms of a probability distribution and a function that distorts this distribution. Such measures, which were originated in psychology [19, 4, 5], have been extensively used. See, for example, the book by Aumann and Shapley (1974) [1] and their recent use in game theory. Recent research on such measures is on their determination either from examples (as in [27]) or from interviews [9]. Nevertheless, the modeling capabilities of distorted probabilities are limited. In [16] it was shown that the number of such measures with respect to the number of unconstrained ones is very small. Moreover, the larger is n, the smaller is the proportion of distorted probabilities. So, in most cases, fuzzy measures cannot be represented using distorted probabilities. To overcome this problem, we introduced in [16] m-dimensional distorted probabilities.

In this work we present some new results with respect to this family of measures. We show some conditions that, when fulfilled, imply distorted probabilities.

Other families of fuzzy measures have been studied in the literature. Two of them, that are relevant for the present paper, are the m-symmetric fuzzy measures [11, 12] and the hierarchically decomposable ones [26]. Some results establishing some relationships between these measures and distorted probabilities will be given.

The structure of the paper is as follows. In Sections 2, we review some concepts that are needed later on. In Section 3, we present the results establishing the connections among different kinds of fuzzy measures. The paper finishes with some conclusions.

2 Preliminaries

This section reviews some previous results in the literature that are needed in the rest of the paper. We start by defining fuzzy measures, and some of their families. Among them, we review m-dimensional distorted probabilities and a few results concerning these measures. The section finishes with a review of a few aggregation operators that are relevant for the purpose of this paper.

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2.1 Fuzzy measures

In this paper we will consider fuzzy measures on a finite universal set $X = \{x_1, \ldots, x_n\}$. For the sake of simplicity, when possible, we will consider $X := \{1, \ldots, n\}$. Now, we review the definition of fuzzy measure.

Definition 1 *A set function $\mu : 2^X \to [0,1]$ is a fuzzy measure if it satisfies the following axioms:*

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ *(boundary conditions)*

(ii) $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$ *(monotonicity)*

In order to distinguish measures satisfying (i) and (ii) with others that also satisfy some additional constraints (*e.g.* additivity $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$), we use the terms *unconstrained* fuzzy measures for the former ones and *constrained* fuzzy measures for the others.

2.2 m-symmetric fuzzy measures

The definition of these measures is based on the concept of *set of indifference*. Roughly speaking, a set of indifference is defined by elements that do not affect the value of the measure. That is, the elements of a set are indistinguishable with respect to the fuzzy measure.

Definition 2 *[11, 12] Given a subset A of X, we say that A is a set of indifference if and only if:*

$$\forall B_1, B_2 \subseteq A, |B_1| = |B_2|,$$

$$\forall C \subseteq X \setminus A \quad \mu(B_1 \cup C) = \mu(B_2 \cup C)$$

In this definition $|\cdot|$ corresponds to the cardinality of a set. We now consider m-symmetric fuzzy measures for the particular case of $m = 2$ and, then, we give the general definition.

Definition 3 *[11, 12] Given a fuzzy measure μ, we say that μ is an at most 2-symmetric fuzzy measure if and only if there exists a partition of the universal set $\{X_1, X_2\}$, with $X_1, X_2 \neq \emptyset$ such that both X_1 and X_2 are sets of indifference. An at most 2-symmetric fuzzy measure is 2-symmetric if X is not a set of indifference.*

Definition 4 *[11, 12] Given a fuzzy measure μ, we say that μ is an at most m-symmetric fuzzy measure if and only if there exists a partition of the universal set $\{X_1, \ldots, X_m\}$, with $X_1, \ldots, X_m \neq \emptyset$ such that $X_1, \ldots X_m$ are sets of indifference.*

The next proposition follows from this definition.

Proposition 1 *Every fuzzy measure μ is an at most n-symmetric fuzzy measure for $n = |X|$.*

So, all fuzzy measures can be considered as m-symmetric for a value of m large enough.

Definition 5 *[11, 12] Given two partitions $\{X_1, \ldots, X_p\}$ and $\{Y_1, \ldots, Y_r\}$ on the finite universal set X, we say that $\{X_1, \ldots, X_p\}$ is coarser than $\{Y_1, \ldots, Y_r\}$ if the following holds:*

$$\forall X_i \exists Y_j \text{ such that } Y_j \subseteq X_i$$

Definition 6 *Given a fuzzy measure μ, we say that μ is m-symmetric if and only if the coarsest partition of the universal set in sets of indifference contains m non empty sets. That is, the coarsest partition is of the form: $\{X_1, \ldots, X_m\}$, with $X_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$.*

Proposition 2 *[11, 12] Let μ be an m-symmetric measure with respect to the partition $\{X_1, \ldots, X_m\}$. Then, the number of values that are needed in order to determine μ is:*

$$\big[(|X_1| + 1) \cdots (|X_m| + 1)\big] - 2$$

An m-symmetric fuzzy measure can be represented in a $(|X_1| + 1) \cdots (|X_m| + 1)$ matrix M.

2.3 Hierarchically S-Decomposable Fuzzy Measures

[26] introduced Hierarchically S-Decomposable Fuzzy Measures. These measures (HDFM for short) can be seen as a generalization of S-decomposable measures. An important characteristic of S-decomposable fuzzy measures, is that the measure for any subset of X can be built from the measures on the singletons and a t-conorm S. When interactions among information sources are considered, such construction means that the interactions among pairs (or subsets) of sources can be expressed in a single and unique way. In particular, all interactions are modeled using the t-conorm S.

The so-called hierarchically S-decomposable fuzzy measures define a more general family of fuzzy measures as they permit us to express different kind of interactions between different subsets. This is achieved permitting us the use of different t-conorms for combining the measures of different singletons (and of different subsets).

This is obtained as follows: (i) the elements in X are structured in a hierarchy that gathers together elements that are *similar* (from the interactions point of view); (ii) each node of the hierarchy has associated a t-conorm to be used to *combine* the interactions. In this way, a richer variety of interactions can be expressed.

For example, if we have a fuzzy measure $\mu : 2^X \to [0,1]$ with $X = \{x_1, x_2, x_3, x_4, \ldots, x_n\}$ such that $\mu(\{x_1\}) = 0.2$, $\mu(\{x_2\}) = 0.4$, $\mu(\{x_3\}) = 0.3$ and $\mu(\{x_4\}) = 0.3$. Then, we have a negative interaction between x_1 and x_2 defining $\mu(\{x_1, x_2\}) = \max(\mu(\{x_1\}), \mu(\{x_2\}))$. Instead, for a positive interaction between x_3 and x_4 we define $\mu(\{x_3, x_4\}) = \min(1, \mu(\{x_3\}) + \mu(\{x_4\}))$. Both situations can be modeled with the t-conorms $S_1(x, y) = \max(x, y)$ and $S_2(x, y) = \min(1, x + y)$.

We give below the definition for the particular case of 2-level HDFM. That is, a measure where the hierarchy has only two levels.

Definition 7 *[26] Given a fuzzy measure μ, we say that μ is a 2-level Hierarchically Decomposable Fuzzy Measure (2-level HDFM) if there is a partition $\{X_1, \ldots, X_m\}$ on X (we denote the elements in X_i by $X_i = \{x_{i,1}, \ldots, x_{i,m_i}\}$) and t-conorms S, S_1, ..., S_m such that:*

$$\mu(A) = S(r_1(A), \ldots, r_m(A))$$

where

$$r_i(A) = S_i(\mu(\{x_{i,1}\} \cap A), \ldots, \mu(\{x_{1,m_i}\} \cap A))$$

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In the general case of HDFM, not presented here, a complete hierarchy is permitted and, then, the measure is defined recursively for each node using the t-conorm attached to the node, and the partition associated to the node.

2.4 Distorted probabilities

As briefly described in the introduction, distorted probabilities correspond to fuzzy measures that can be represented by a probability distribution and a distortion function. We formalize these measures as well as the required concepts below:

Definition 8 *Let $P : 2^X \to [0,1]$ be a probability measure. Then, we say that a function f is strictly increasing with respect to P if and only if*

$$P(A) > P(B) \textit{ implies } f(P(A)) > f(P(B))$$

Remark: Since we suppose that X is a finite set, when there is no restriction on the function f, a strictly increasing function f with respect to P can be regarded as a strictly increasing function on $[0,1]$. Note that with respect to increasingness only the points in $\{P(A)|A \in 2^X\}$ are essential, the others are not considered by $f(P(A))$.

Definition 9 *[1, 2] Let μ be a fuzzy measure. We say that μ is a distorted probability if there exists a probability distribution P and a strictly increasing function f with respect to P such that $\mu = f \circ P$.*

The next theorem gives the necessary and sufficient condition for a fuzzy measure μ to be a distorted probability. The theorem is based on *Scott's condition*:

Definition 10 *[16] Let μ be a fuzzy measure, μ satisfies* Scott's condition *when for all $A_i, B_i \in 2^X$ such that $\sum_{i=1}^n 1_{A_i} = \sum_{i=1}^n 1_{B_i}$ the condition below holds: $\mu(A_i) \leq \mu(B_i)$ for $i = 2, 3, \ldots, n$ implies $\mu(A_1) \geq \mu(B_1)$.*

Here 1_A represents the characteristic function of the set A. That is $1_A(x) = 1$ if and only if $x \in A$.

Using this condition, we can characterize distorted probabilities as follows:

Theorem 1 *[16] Let μ be a fuzzy measure; then, μ is a distorted probability if and only if Scott's condition holds.*

2.5 m-dimensional distorted probabilities

m-dimensional distorted probabilities were presented in [16] to overcome the limited expressiveness of distorted probabilities. They are defined as follows:

Definition 11 *[16] Let $\{X_1, X_2, \cdots, X_m\}$ be a partition of X; then, we say that μ is an at most m dimensional distorted probability if there exists a function f on $\mathbb{R}^m$ and probabilities P_i on $(X_i, 2^{X_i})$ such that:*

$$\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \cdots, P_m(A \cap X_m)) \quad (1)$$

where f on $\mathbb{R}^m$ is strictly increasing with respect to each variable.

We say that an at most m dimensional distorted probability μ is an m dimensional distorted probability if μ is not an at most $m-1$ dimensional.

The next proposition follows from the definition above.

Proposition 3 *Every fuzzy measure is an at most n-dimensional distorted probability with $n = |X|$.*

Note that for $n = |X|$, we are considering the following partition of X: $\{X_1 = \{x_1\}, \ldots, X_n = \{x_n\}\}$. So, $f(a_1, \ldots, a_n) = \mu(A)$ when $a_i = 1$ if and only if $x_i \in A$.

Also, we can prove that m-dimensional distorted probabilities define a family of measures with increasing complexity with respect to m. This means that increasing the value of m, the number of measures representable increases. The following proposition establishes this property.

Proposition 4 *Let $\mathcal{M}_k$ be the set of all fuzzy measures that are k-dimensional distorted probabilities and let $\mathcal{M}_0$ be the empty set. Then $\mathcal{M}_{k-1} \subset \mathcal{M}_k$ for all $k = 1, 2, \ldots, |X|$.*

Corollary 1 *Given a fuzzy measure μ, there exists a $k \in \{1, 2, \ldots, |X|\}$ such that $\mu \in \mathcal{M}_k$ and $\mu \notin \mathcal{M}_{k-1}$.*

Therefore, the proposed family of fuzzy measures permits us to cover the whole set of fuzzy measures.

2.6 Aggregation operators

Now we define the OWA and the WOWA operators. They will be of relevance in this work. As explained in detail in [25], the OWA operator permits to give importance to the data (with respect to their position) while the WOWA permits to give importance to the data (as the OWA operator) and also to the information sources (as the weighted mean does).

Definition 12 *[29, 30] Let $\mathbf{w}$ be a weighting vector of dimension n* (i.e., $w_i \geq 0$ *and* $\sum_{i=1}^n w_i = 1$)*, then a mapping OWA: $\mathbb{R}^n \to \mathbb{R}$ is an* Ordered Weighted Averaging (OWA) operator *of dimension n if*

$$OWA_{\mathbf{w}}(a_1, ..., a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}$$

where $\{\sigma(1), ..., \sigma(n)\}$ is a permutation of $\{1, ..., n\}$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = \{2, ..., n\}$ (i.e. $a_{\sigma(i)}$ is the i-th largest element in the collection $a_1, ..., a_n$).

Definition 13 *[25] Let $\mathbf{p}$ and $\mathbf{w}$ be two weighting vectors of dimension n, then a mapping WOWA: $\mathbb{R}^n \to \mathbb{R}$ is a* Weighted Ordered Weighted Averaging (WOWA) operator *of dimension n if*

$$WOWA_{\mathbf{p},\mathbf{w}}(a_1, ..., a_n) = \sum_{i=1}^{n} \omega_i a_{\sigma(i)}$$

where σ is defined as in the case of the OWA, and the weight ω_i is defined as:

$$\omega_i = w^*(\sum_{j \leq i} p_{\sigma(j)}) - w^*(\sum_{j < i} p_{\sigma(j)})$$

with w^* being a non-decreasing function that interpolates the points $\{(i/n, \sum_{j \leq i} w_j)\}_{i=1,...,n}$ together with the point $(0,0)$. The function w^* is required to be a straight line when the points can be interpolated in this way.

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Alternatively, it is possible to define the WOWA operator directly using the function w^*. This will be denoted by $WOWA_{\mathbf{p},w^*}(a_1,\ldots,a_n)$ or $WOWA_{\mathbf{p},w^*}(f)$ when f is a function $f : X \to \mathbb{R}$ such that $f(x_i) = a_i$. In this latter case, we will read $WOWA_{\mathbf{p},w^*}(f)$ as the $WOWA$ of f with respect to $\mathbf{p}$ and w^*.

We finish the section with the definition of the Choquet integral.

Definition 14 *[3] Let μ be a fuzzy measure, then the* Choquet integral *of a function $f : X \to \mathbb{R}^+$ with respect to the fuzzy measure μ is defined by:*

$$(C)\int f d\mu (= C_\mu(f)) = \sum_{i=1}^{n}[f(x_{s(i)}) - f(x_{s(i-1)})]\mu(A_{s(i)})$$

where $x_i \in X$ and where $f(x_{s(i)})$ indicates that the indices have been permuted so that
$0 \le f(x_{s(1)}) \le \cdots \le f(x_{s(n)}) \le 1$,
$A_{s(i)} = \{x_{s(i)},\ldots,x_{s(n)}\}$ *and* $f(x_{s(0)}) = 0$.

3 Distorted probabilities and symmetric fuzzy measures

3.1 A sufficient condition for distorted probablity

We have discussed above Scott's condition: Definition 10 for distorted probability. Nevertheless, this condition is not easy to check. We present below another condition under which a fuzzy measure is a distorted probability. The new condition is easier to check.

Definition 15 *We say that a fuzzy measure μ satisfies a union condition (for short UC), if $A \cap C = \emptyset, B \cap D = \emptyset$,*

$$\mu(A) \ge \mu(B), \mu(C) \ge \mu(D) \Rightarrow \mu(A \cup C) \ge \mu(B \cup D).$$

The next proposition is obvious from the definition.

Proposition 5 *Suppose that a fuzzy measure μ satisfies UC. We have $A \cap C = \emptyset, B \cap D = \emptyset$,*

$$\mu(A) = \mu(B), \mu(C) = \mu(D) \Rightarrow \mu(A \cup C) = \mu(B \cup D).$$

Definition 16 *We say that a fuzzy measure μ satisfies a strict union condition (for short SUC), if $A \cap C = \emptyset, B \cap D = \emptyset$,*

$$\mu(A) > \mu(B), \mu(C) \ge \mu(D) \Rightarrow \mu(A \cup C) > \mu(B \cup D).$$

Proposition 6 *Suppose that a fuzzy measure μ satisfies SUC. We have $A \cap C = \emptyset, B \cap C = \emptyset$,*

$$\mu(A) > \mu(B) \Rightarrow \mu(A \cup C) > \mu(B \cup C).$$

Applying Proposition 5,6,we have the next proposition.

Proposition 7 *Suppose that a fuzzy measure μ satisfies UC. There exists a function F on*

$$\{(x,y)|x = \mu(A), y = \mu(B), A \cap B = \emptyset, A, B \in 2^X\}$$

such that

$$\mu(A \cup B) = F(\mu(A), \mu(B))$$

for $A, B \in 2^X, A \cap B = \emptyset$, and

$$F(x,0) = x, F(F(x,y),z) = F(x,F(y,z)).$$

Moreover if μ satisfies both UC and SUC, F is strictly monotone with respect to each variable.

Suppose that a fuzzy measure μ satisfies both UC and SUC. Since F is strictly monotone on

$$\{(x,y)|x = \mu(A), y = \mu(B), A \cap B = \emptyset, A, B \in 2^X\},$$

the domain of F can be extended to $[0,1] \times [0,1]$ and F is monotone with respect to each variable and F is continuous. Then F can be represented by strictly monotone function φ on [0,1] as

$$F(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)).$$

That is,

$$F(\mu(A), \mu(B)) = \varphi^{-1}(\varphi(\mu(A)) + \varphi(\mu(B)))$$

for $A \cap B = \emptyset$. Therefore we have

$$\varphi(\mu(A \cup B)) = \varphi(\mu(A)) + \varphi(\mu(B)).$$

Let $P(A) := \varphi(\mu(A))$. Then, P is a probability and we have $\mu(A) = \varphi^{-1}(P(A))$. Therefore we have the next theorem.

Theorem 2 *A fuzzy measure μ is distorted probability if μ satisfies both UC and SUC.*

3.2 Symmetric fuzzy measure

We start showing that a 1-Symmetric fuzzy measure is a special case of distorted probabilities.

Proposition 8 *Let $\mu = f \circ P$ be a distorted probability. Then, μ is a 1-symmetric fuzzy measure if and only if $P(A) = |A|/|X|$.*

Now we show that all m-symmetric fuzzy measures are m-dimensional distorted probabilities. This implies that 1-symmetric fuzzy measures are distorted probabilities.

Proposition 9 *Let μ be an m-symmetric fuzzy measure with respect to the partition $\{X_1,\ldots,X_m\}$. Then, μ is an m-dimensional distorted probability.*

Although the reversal of this proposition is not true, the next proposition characterizes one case in which m-dimensional distorted probabilities are m-symmetric fuzzy measures.

Proposition 10 *Let μ be an m-dimensional distorted probability. If, $p_i(x_j) = p_i(x_k)$ for all $x_j, x_k \in X_i$ and for all $i = 1,\ldots,m$, then μ is an m-symmetric fuzzy measure.*

It is known that OWA operators are equivalent to Choquet integrals with respect to symmetric fuzzy measures. Therefore, m-symmetric fuzzy measures permit us to define a generalization of OWA operators. The m-dimensional OWA is defined below:

Definition 17 *The m-dimensional OWA is defined as the Choquet integral with respect to an m-symmetric fuzzy measure.*

As proven in [24], a Weighted OWA (WOWA) operator is equivalent to a Choquet integral with respect to a distorted probability. Therefore, a Choquet integral with an m-dimensional probability can be seen as a generalization of the WOWA operator. We define an m-dimensional WOWA as follows:

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Definition 18 *The m-dimensional WOWA is defined as the Choquet integral with respect to an m-dimensional distorted probability.*

Then, considering Definitions 17 and 18 above, we have the following corollary from Proposition 9:

Corollary 2 *An m-dimensional OWA is a particular case of an m-dimensional WOWA. In other words, a Choquet integral with respect to an m-symmetric fuzzy measure is a particular case of a Choquet integral with respect to an m-dimensional distorted probability.*

3.3 m-separable fuzzy measure

As shown in Proposition 9, m-symmetric fuzzy measures are one special class of the m-dimensional distorted probabilities. We present another special class of fuzzy measure

Definition 19 *Let μ be a fuzzy measure. Then, we say that μ is a m-separable fuzzy measure if there exists a function g and a partition $\{X_1, \ldots, X_m\}$ of X such that*

$$\mu(A) = g(\mu(A \cap X_1), \ldots, \mu(A \cap X_m)) \qquad (2)$$

where g is a m-dimensional function on $\mathbb{R}^m$. We say that g is a generating function for μ. We say that a generating function g is induced by h on $[0,1] \times [0,1]$ if $g(x_1, \ldots, x_m) = h(h(\ldots h(x_1, x_2), \ldots, x_{m-1}), x_m)$, $g(x_1, x_2, 0, \ldots, 0) = h(x_1, x_2)$.

Example 1 *Let $\{X_1, \ldots, X_m\}$ be a partition of X.*

1. *Suppose $g(x_1, \ldots, x_m) = x_1 + \cdots + x_m$, so g is induced by $h(x, y) = x + y$. Then we have*

   $$\mu(A) = \mu(A \cap X_1) + \cdots + \mu(A \cap X_m).$$

   This is an interadditivity defined in [14].

2. *Suppose $g(x_1, \ldots, x_m) = x_1 \vee \cdots \vee x_m$, so g is induced by $h(x, y) = x \vee y$. Then we have*

   $$\mu(A) = \mu(A \cap X_1) \vee \cdots \vee \mu(A \cap X_m).$$

3. *Suppose $g(x_1, \ldots, x_m) = (x_1^2 + \cdots + x_m^2)^{1/2}$, so g is induced by $h(x, y) = (x^2 + y^2)^{1/2}$. Then we have*

   $$\mu(A) = (\mu(A \cap X_1)^2 + \cdots + \mu(A \cap X_m)^2)^{1/2}.$$

Suppose that μ is a m-separable fuzzy measure generated by g and that g is induced by h. We say that h is associative if $h(h(x, y), z) = h(x, h(y, z))$. Suppose that h is strictly monotone and associative. Since g is symmetric, then h is symmetric, that is $h(x, y) = h(y, x)$. Then there exists a strictly monotone function φ such that $h(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$. Define φ- Möbius inverse m^φ by

$$m^\varphi(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \varphi(\mu(B)).$$

Let $\mathcal{P}$ be a partition of X for a m-separable fuzzy measure. Applying theorem in [14], $m^\varphi(A) = 0 \Rightarrow A \not\subset C, C \in \mathcal{P}$. Let $M := \{A | m^\varphi(A) \neq 0\}$ We have $\mathcal{P} \subset M$. Define $A_l := \{A | A \in M, x_l \in A\}$ for $X = \{x_1, \ldots, x_l, \ldots, x_n\}$ and $M_l := \max\{|A| | A \in A_l\}$. Since $A \subset C, C \in \mathcal{P}$ for $A \in A_l$, we have the next proposition.

Proposition 11 *Let μ be a m - separable fuzzy measure generated by g, and g be induced by a strict monotone and associative h on $[0,1]$. Then we have*

$$m \times \min_{l \in |1, \ldots, n|} M_l \leq n.$$

We say that a fuzzy measure μ is a φ $k-$order additive if $max\{|A| | A \in M\} = k$. If $\varphi(x) = x$, a φ $k-$order additive fuzzy measure is a k-order additive fuzzy measure [6, 7].

Proposition 12 *Let μ be a m-separable fuzzy measure generated by g, and g be induced by strict monotone and associative h on $[0,1]$. If μ is k- additive, then we have $m \times k \geq n$.*

Now, we consider the relationship between the m-separable fuzzy measures and some other families of measures.

Proposition 13 *2-level HDFMs with S_i Archimedean t-conorms are a m-separable fuzzy measures.*

Theorem 3 *Let $\{X_1, \ldots, X_m\}$ be a partition of X and μ_i $i = 1, \ldots m$ be distorted probabilities represented by f_i and P_i (i.e., $\mu_i = f_i \circ P_i$). Then, there exists a m-separable fuzzy measure μ such that*

$$\sum_{i=1}^{m} ((C) \int f df_i \circ P_i) = (C) \int f d\mu \qquad (3)$$

for all measurable function f.

As a corollary of this theorem, we have that the Choquet integral with respect to a m-separable fuzzy measure μ with $g(x_1, \ldots, x_m) = x_1 + \cdots + x_m$ can be represented as a two step Choquet integral.

Corollary 3 *Let μ be a m-separable fuzzy measure μ with $g(x_1, \ldots, x_m) = x_1 + \cdots + x_m$. Then the Choquet integral with respect to μ is represented as a two step Choquet integral of a 1st step integral with respect to a probability on $(1, \ldots, m)$. That is,*

$$(C) \int f d\mu = \int ((C) \int f df_i \circ P_i) dP(i). \qquad (4)$$

4 Conclusions

In this paper we have introduced two new conditions, the Union Condition (UC) and the Strict Union Condition (SUC), and we have studied distorted probability under these conditions. We have shown that a fuzzy measure is a distorted probability when both UC and SUC conditions are satisfied.

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References

[1] Aumann, R. J., Shapley, L. S., (1974), Values of Non-atomic Games, Princeton Univ. Press.

[2] Chateauneuf, A., Decomposable Measures, (1996), Distorted Probabilities and Concave Capacities, Mathematical Social Sciences 31, 19-37.

[3] Choquet, G., (1953/54), Theory of capacities, Ann. Inst. Fourier, 5 131-295.

[4] Edwards, W., (1953), Probability-Preferences in Gambling, American Journal of Psychology 66 349-364.

[5] Edwards, W., (1962), Subjective probabilities inferred from decisions, Psychological review 69 109-135.

[6] Grabisch, M., (1996), k-order additive fuzzy measures, Proc. 6th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU), 1345-1350, Granada, Spain.

[7] Grabisch,M. k-order additive discrete fuzzy measures and their representation. Fuzzy Sets and Systems 92 (1997), 167-189.

[8] Grabisch, M., Murofushi, T., Sugeno, M., (2000), Fuzzy Measures and Integrals: Theory and Applications, Physica-Verlag.

[9] Honda, A., Nakano, T., Okazaki, Y., (2002), Subjective evaluation based on distorted probability, Proc. of the SCIS/ISIS conference.

[10] Leszczyski, K., Penczek, P., Grochulski, W., (1985), Sugeno's fuzzy measure and fuzzy clustering, Fuzzy Sets and Systems, 15 147-158.

[11] Miranda, P., Grabisch, M., (2002), p-symmetric fuzzy measures, Proc. 8th IPMU, 545-552, Annecy, France.

[12] Miranda, P., Grabisch, M., P. Gil, (2002),p-symmetric fuzzy measures, Int. J. of Unc., Fuzz. and Knowledge-Based Systems, 10 (Supplement), 105-123.

[13] Murofushi, T., Sugeno, M., Machida, M., (1994), Non-monotonic fuzzy measures and the Choquet integral, Fuzzy Sets and Systems, 64:1 73-86.

[14] Murofushi, T., Sugeno, M., Fujimoto, K., (1997), Separated hierarchical decomposition of the Choquet integral, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 5:5 563-585.

[15] Narukawa, Y., Torra, V., (2004), On m-dimensional distorted probabilities and p-symmetric fuzzy measures, IPMU 2004 (ISBN 88-87242-54-2), vol. 2, 1279-1284, Perugia, Italy, July 4-9.

[16] Narukawa, Y., Torra, V., (2005), Fuzzy measure and probability distributions: distorted probabilities, IEEE Trans. on Fuzzy Systems, 13:5 617- 629.

[17] Narukawa, Y., Torra, V., (2005), On multidimensional distorted probability, IFSA 2005, Beijing, China.

[18] Ovchinnikov, S., (1998), An Analytic Characterization of Some Aggregation Operators, Int. J. of Intel. Systems, 13 59-68.

[19] Preston, M. G., Baratta, P., (1948), An Experimental Study of the Auction-Value of an Uncertain Outcome, American Journal of Psychology, 61 183-93.

[20] Soria-Frisch, A., (2003), Hybrid SOM and fuzzy integral frameworks for fuzzy classification, in Proc. IEEE Int. Conf. on Fuzzy Systems 840-845.

[21] Soria-Frisch, A., (2006), Unsupervised Construction of Fuzzy Measures through Self-Organizing Feature Maps and its Application in Color Image Segmentation, Int. J. of Approx. Reasoning, in press.

[22] Sugeno, M., (1974), Theory of fuzzy integrals and its applications, Ph. D. Dissertation, Tokyo Institute of Technology, Tokyo, Japan.

[23] Tahani, H., Keller, J. M., (1990), Information fusion in computer vision using the fuzzy integral, IEEE Trans. on Systems, Man and Cybernetics, 20:3 733-741.

[24] Torra, V., (1998), On some relationships between the WOWA operator and the Choquet integral, Proc. 7th IPMU 818-824, Paris, France.

[25] Torra, V., (1997), The Weighted OWA operator, Int. J. of Intel. Systems, 12, 153-166.

[26] Torra, V., (1999), On hierarchically S-decomposable fuzzy measures, Int. J. of Intel. Systems, 14:9, 923-934.

[27] Torra, V., (2000), Learning Weights for Weighted OWA Operators, Proc. IEEE Int. Conf. Industrial Elect., Control and Instrumentation (IECON 2000), Nagoya, Japan. (CD-ROM).

[28] Torra, V., Narukawa, Y., (2007), Modeling decisions: aggregation operators and information fusion, Springer, in press. ISBN: 978-3-540-68789-4.

[29] Yager, R. R., (1988), On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. Syst., Man, Cybern., 18 183-190.

[30] Yager, R. R., (1996), Quantifier Guided Aggregation Using OWA operators, Int. J. of Intel. Systems, 11, 49-73.

[31] Yager, R. R., (1999), Nonmonotonic OWA operators, Soft Computing 3 187-196.

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