A value via posets induced by graph-restricted communication situations

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Abstract— *This paper provides a new value (solution concept or allocation rule) of cooperative games via posets induced by graphs. Several values in a graph-restricted communication situation have been proposed or introduced by Myerson, Borm, and Hamiache... However, these values have been subjected to some criticisms in certain types of games. The value proposed in this paper withstands these criticisms. Moreover, these existing values have been defined only in situations represented by undirected graphs, while the notion of the value proposed in this paper can be extended to situations represented by directed graphs.*

Keywords— graph-restricted situations, communication situations, values, posets, cooperative games.

1 Introduction and Preliminaries

Throughout the paper, N denotes the universal set of n elements. For convenience, we often number the elements such that the universal set is $N = \{1, 2, \ldots, n\}$. A real-valued function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game*. A monotone game (i.e., $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$) is called a *capacity* or *a fuzzy measure*. We often call the pair (N, v), rather than v, a game or a capacity. The set of all games on N is denoted by $\mathcal{G}^N$. A real vector-valued function $\Phi : \mathcal{G}^N \to \mathbb{R}^{|N|}$ is called a *value*. In cooperative game theory, N is considered to be the set of all players. For every subset S of N, often called a *coalition*, $v(S)$ represents the (transferable) utility/profits that players in S can obtain if they decide to cooperate. For every game (N, v), the value $\Phi(N, v)$ represents an allocation rule, which provides an assessment of the benefits for each player from participating in a game v. For the sake of simplicity, we mainly discuss games in terms of various set functions (e.g., games, capacities, fuzzy measures, and so forth.) on N.

To avoid cumbersome notations, we often omit braces for singletons, e.g., by writing $v(i)$, $U \setminus i$ instead of $v(\{i\})$, $U \setminus \{i\}$. Similarly, for pairs, we write ij instead of $\{i, j\}$. Furthermore, cardinalities of subsets $S, T, \ldots$, are often denoted by the corresponding lower case letters $s, t, \ldots$, otherwise by the standard notation $|S|, |T|$,...

1.1 Games and capacities with graph restricted situations

In ordinary cooperative game theory it is implicitly assumed that all coalitions of N can be formed; however, this is generally not the case. For players to coordinate their actions, they must be able to communicate. The bilateral communication channels between players in N are described by a *communication network*. Such a network can be represented by an *undirected graph* (N, L), which has the set of players as its *nodes* $S \subseteq N$ and in which the players are connected by the set of *links* $L \subseteq \{ij \mid i, j \in N, i \neq j\}$; i.e., players i and j can communicate (directly) with each other if $ij \in L$. This paper will deals with only situations induced by communication networks described by undirected graphs. Many other approaches to the situations can be seen via the literatures [1, 2].

Definition 1.1 (communication situation)
The triple (N, v, L), which reflects a situation consisting of a game v on N and a communication network (N, L), is called a *communication situation*. We denote the set consisting of all communication situations on N by CS^N. For a coalition $T \subseteq N$, the restriction of (N, L) to T is denoted by $(T, L(T))$ and defined by $L(T) := \{ij \in L \mid ij \subseteq T\}$.

Definition 1.2 (feasible coalition) We say that players j and k are *connected in* $S \subseteq N$ if $j = k$ or there exists a subset $\{i_1, \cdots, i_m\} \subseteq S$ such that $j = i_1$, $k = i_m$, and $\{i_t, i_{t+1}\} \in L$ for all $t \in \{1, \cdots, m-1\}$. Then we denote $j \sim_S k$. Clearly, this relation $\sim_S$ is an equivalence relation. Hence, the notion of connectedness in S induces a partition $S/L := S/\sim_S$ of S. A coalition $S \subseteq N$ is said to be *feasible* in the communication network (N, L) if any two players, $j \in S$ and $k \in S$, are connected in S (i.e., $S/L = \{S\}$).

Example 1.1
Consider the communication situation (N_1, v, L_1) with $N_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and $L_1 = \{12, 15, 26, 37, 47, 56\}$ (Fig.1). Then, all the players in $\{1, 2, 6\}$ can communicate with other;

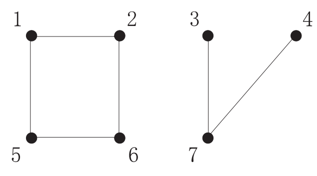

Figure 1: Communication network(N, L_1).

i.e., the coalition $\{1, 2, 6\}$ is feasible. Hence, they can fully coordinate their actions and obtain the value $v(\{1, 2, 6\})$. On the other hand, in the coalition $\{1, 2, 3, 4\}$, players 1 and 2 can communicate with each other, but players 3 and 4 cannot communicate with any other players in $\{1, 2, 3, 4\}$. Thus, feasible subcoalitions of $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{3\}$, and $\{4\}$ (i.e., forming the coalition $\{1, 2, 3, 4\}$ is unfeasible). Hence, the value attainable by the players in $\{1, 2, 3, 4\}$ should be $v(\{1, 2\}) + v(\{3\}) + v(\{4\})$. In general, the value attainable by the players in $S \in N$ under a communication situation (N, v, L) is represented by

$$\sum_{T \in S/L} v(T). \quad (1)$$

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Definition 1.3 (network-restricted game [3]) The *network-restricted game* (N, v^L) associated with (N, v, L) is defined as

$$v^L(S) := \sum_{T \in S/L} v(T) \quad \text{for each } S \subseteq N. \tag{2}$$

Note that if (N, L) is the complete graph (i.e., $L = \{ij \mid i, j \in N, i \neq j\}$), the network-restricted game v^L is equal to the original game v.

The network-restricted game evaluates the possible gains from cooperation in a communication situation from the viewpoint of the players. The next example focuses on the importance of communication channels and links in a communication situation.

Example 1.2 In the communication situation L_1 depicted in Fig.1, the value obtainable by the players in the grand coalition N is

$$v^{L_1}(N) = v(\{1, 2, 5, 6\}) + v(\{3, 4, 7\}), \tag{3}$$

since $N/L_1 = \{\{1, 2, 5, 6\}, \{3, 4, 7\}\}$. If for some reason the communication link between players 4 and 7 is lost, the communication network L_1 becomes the new communication network $L_2 = \{12, 15, 26, 37, 56\}$. Then, $N/L_2 = \{\{1, 2, 5, 6\}, \{4\}, \{3, 7\}\}$ and the value obtainable by the players in the grand coalition N becomes

$$v^{L_2}(N) = v(\{1, 2, 5, 6\}) + v(\{4\}) + v(\{3, 7\}). \tag{4}$$

Then,

$$v^{L_1}(N) - v^{L_2}(N) \tag{5}$$

can be interpreted as a type of marginal contribution of the link $\{4, 7\} \in L_1$ to the communication network L_1.

Definition 1.4 (link game [4]) The *link game* (L, γ^v) associated with (N, v, L) consisting of a zero-normalized game v is a game on L defined by

$$\gamma^v(M) := v^M(N) = \sum_{T \in N/M} v(T) \quad \text{for each } M \subseteq L. \tag{6}$$

Note that, for an ordinary game v, γ^v is not a game on L since $\gamma^v(\emptyset) = \sum_{T \in N/\emptyset} v(T) = \sum_{i \in N} v(\{i\}) \neq 0$.

The link game $\gamma^v(M)$ represents the worth of the communication network $M \subseteq L$ as the worth of the grand coalition in the communication situation (N, v, M) through the network-restricted game v^M.

Definition 1.5 (Möbius Transform [5]**)** The *Möbius transform* of a game $v : 2^N \to \mathbb{R}$ (resp. $\gamma : 2^L \to \mathbb{R}$) is a game on N (resp. L) denoted by $\Delta^v : 2^N \to \mathbb{R}$ (resp. $\Delta^\gamma : 2^L \to \mathbb{R}$) and is defined by

$$\Delta^v(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T) \quad \text{for each } S \in 2^N. \tag{7}$$

$$(resp.\ \Delta^\gamma(M) := \sum_{K \subseteq M} (-1)^{|M \setminus K|} \gamma(K) \quad \text{for each } M \in 2^L). \tag{8}$$

Equivalently, we have that

$$v(S) = \sum_{T \subseteq S} \Delta^v(T) \quad \forall S \in 2^N. \tag{9}$$

$$(resp.\ \gamma(M) = \sum_{K \subseteq M} \Delta^\gamma(K) \quad \forall M \in 2^L). \tag{10}$$

Thus, the worth $v(S)$ (resp. $\gamma(M)$) of a coalition S (resp. communication network M) is equal to the sum of the Möbius transform of all its subcoalitions (subnetworks). This gives a recursive definition of the Möbius transform. The Möbius transform of every singleton is equal to its worth, while recursively, the Möbius transform of every coalition (resp. communication network) of at least two players (resp. links) is equal to its worth minus the sum of the Möbius transform of all its proper subcoalitions (resp. subnetworks). In this sense, the Möbius transform of a coalition S (resp. communication network M) can be interpreted as the extra contribution of the cooperation/synergy among the players in S (resp. links in M) that they did not already achieve by smaller coalitions (resp. networks). In fact, in the context of *interaction indices* (e.g.,[6, 7]), the Möbius transform $\Delta^v(S)$ is called the *internal interaction index* of S, which represents the magnitude of a type of interaction among the elements in S. The Möbius transform is also occasionally called the *Harsanyi dividends*[8].

Definition 1.6 (unanimity game) The *unanimity game* for a non-empty coalition $T \subseteq N$ is denoted by u_T and defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

For any game $v : 2^N \to \mathbb{R}$, v can be represented as

$$v(S) = \sum_{T(\neq \emptyset) \in 2^N} \Delta^v(T) \cdot u_T(S) \quad \forall S(\neq \emptyset) \in 2^N. \tag{12}$$

2 Values for communication situations

In this section, we briefly introduce the Shapley value for ordinary cooperative games and three existing values for communication situations that appear in the literatures [3, 4, 9], *the Myerson value, the position value, and the Hamiache value*.

Definition 2.1 (the Shapley value [10]) The *Shapley value* $\Phi : \mathcal{G}^N \to \mathbb{R}^{|N|}$ for a game $(N, v) \in \mathcal{G}^N$ is defined by

$$\Phi_i(N, v) := \sum_{T \ni i} \frac{1}{|T|} \Delta^v(T) \quad \text{for each } i \in N. \tag{13}$$

Definition 2.2 (the Myerson value [3]) The *Myerson value* $\Psi : CS^N \to \mathbb{R}^{|N|}$ for a communication situation $(N, v, L) \in CS^N$ is defined by

$$\Psi(N, v, L) := \Phi(N, v^L). \tag{14}$$

The Myerson value is the allocation rule that assigns to every communication situation (N, v, L) the Shapley value of the network-restricted game (N, v^L). Note that $\Psi(N, v, L) = \Phi(N, v)$ if (N, L) is the complete graph.

Definition 2.3 (position value [4]) The *position value* $\pi : CS^N \to \mathbb{R}^{|N|}$ for a communication situation $(N, v, L) \in CS^N$ is defined by

$$\pi_i(N, v, L) := \frac{1}{2} \sum_{\substack{l \in L \\ l \ni i}} \Phi_l(L, \gamma^v) \quad \text{for each } i \in N. \tag{15}$$

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The Shapley value $\Phi_l(L, \gamma^v)$ of a link $l \in L$, which is induced via (13) for the link game (L, γ^v), can be interpreted as a type of *expected marginal contribution* of the link l to all communication networks containing l. Then, the value is divided equally between the two players at the ends of the considered link $l \in L$. The position value of a given player $i \in N$ is obtained as the sum of all these shares.

We focus to a third value for communication situations, introduced by Hamiache [9]. Given a communication situation (N, v, L) and $S \subseteq N$, we denote by S^* the set of all nodes of the communication network (N, L) that are adjacent to at least one of the nodes of S,

$$S^* := \{i \in N \mid \exists j \in S \text{ such that } ij \in L\}. \tag{16}$$

Definition 2.4 (associated game [9]) For a value ϕ on CS^N (i.e., $\phi : CS^N \to \mathbb{R}^{|N|}$), the *associated game* v^*_ϕ of v with respect to ϕ is defined for $S \subseteq N$, by

$$v^*_\phi(S) := \begin{cases} v(S) + \displaystyle\sum_{j \in S^* \setminus S} \Big(\phi_i(S^{+j}, v|_{S^{+j}}, L(S^{+j})) - v(j)\Big) & \text{if } |S/L| = 1, \\ \displaystyle\sum_{T \in S/L} v^*_\phi(T) & \text{otherwise,} \end{cases} \tag{17}$$

where $S^{+j} := S \cup \{j\}$ and $v|_{S^{+j}}$ is the restriction of v to S^{+j}.

Hamiache [9] claims that there is a unique value ϕ, the so-called *Hamiache value*, for communication situations satisfying the following five properties, *component-efficiency*, *linearity w.r.t. games*, *independence of irrelevant players*, *positivity*, and *associated consistency*:

Component-efficiency :
For any (N, v, L) and any $S \in N/L$,

$$\sum_{i \in S} \phi_i(N, v, L) = v(S). \tag{18}$$

Linearity w.r.t. games :
For any $\alpha, \beta \in \mathbb{R}$ and $(N, v, L), (N, w, L) \in CS^N$,

$$\phi(N, \alpha v + \beta w, L) = \alpha\phi(N, v, L) + \beta\phi(N, w, L). \tag{19}$$

Independence of irrelevant players :
For any (N, L) and for any two feasible coalitions $R \subseteq T$,

$$\phi_i(N, u_R, L) = \phi_i(T, u_R, L(T)) \ \ \forall i \in T. \tag{20}$$

Positivity :
For any feasible coalition $T \subseteq N$,

$$\phi_i(T, u_T, L(T)) \geq 0 \ \ \forall i \in T. \tag{21}$$

Associated consistency:
For any $(N, v, L) \in CS^N$,

$$\phi(N, v, L) = \phi(N, v^*_\phi, L). \tag{22}$$

Note that $\phi(N, v, L) = \Phi(N, v)$ if (N, L) is the complete graph.

3 Posets induced by communication networks

3.1 Communication networks and posets

In this subsection, we consider and introduce a subposet of $B(n) := (2^N, \subseteq)$ induced by a communication network (N, L).

For a communication network (N, L), the set of all feasible coalitions in (N, L) is denoted by $P(N, L)$. i.e.,

$$P(N, L) := \{S \subseteq N \mid |S/L| = 1\}. \tag{23}$$

The set $P(N, L)$, together with set inclusion $\subseteq$ as an order on $P(N, L)$, is called the *poset induced by the communication network* (N, L).

Example 3.1 Let $N = \{1, 2, 3\}$, $L_a = \{12, 13, 23\}$, $L_b = \{13, 23\}$, and $L_c = \{12\}$. Then the posets induced by communication networks (N, L_a), (N, L_b), and (N, L_c), as shown in (a) – (c) in Fig. 2, are represented as shown in (a) – (c) in Fig. 3, respectively.

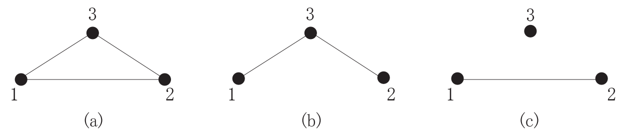

Figure 2: Communication networks on $N = \{1, 2, 3\}$.

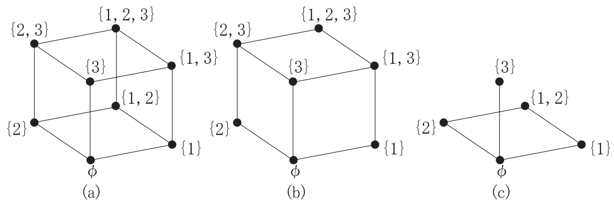

Figure 3: Posets corresponding to networks in Fig. 2.

Definition 3.1 (Möbius transform on posets)
Let $P := (N, \leq)$ be a poset. For a function $v : P \to \mathbb{R}$, the *Möbius transform* Δ^v of v is a function on P satisfying the following equation:

$$v(x) = \sum_{y \leq x} \Delta^v(y) \quad \forall x \in P. \tag{24}$$

Definition 3.2 (representation functions)
The *representation function of a communication situation* (N, v, L) is a function v^P on the poset $P(N, L)$ defined by

$$v^P(S) = v(S) \quad \text{for each } S \in P(N, L). \tag{25}$$

Then, the Möbius transform Δ^{v^P} of v^P is represented as

$$\Delta^{v^P}(S) := \sum_{\substack{T \in P(N,L) \\ T \subseteq S}} (-1)^{|S \setminus T|} v^P(T) \quad \forall S \in P(N, L). \tag{26}$$

Conversely,

$$v^P(S) := \sum_{\substack{T \in P(N,L) \\ T \subseteq S}} \Delta^{v^P}(T) \quad \forall S \in P(N, L). \tag{27}$$

Definition 3.3 (poset representation) The *poset representation of a communication situation* (N, v, L) is the pair $(P(N, L), \Delta^{v^P})$ of the poset induced by (N, v, L) and the Möbius transform Δ^{v^P} of representation function v^P of (N, v, L).

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4 A new value in communication situations

In this section, we introduce a new value for communication situations.

4.1 An interpretation of the Shapley value

Now, we consider the case $N = \{1,2\}$; the Shapley value $\Phi_1(N, v)$ of player 1 in a game v is obtained, from (13), as

$$\Phi_1(N, v) = \frac{1}{1}\Delta^v(\{1\}) + \frac{1}{2}\Delta^v(\{1,2\}). \tag{28}$$

This can be interpreted as an allocation rule of *Harsanyi dividends* (i.e., the Möbius transform) described as follows:

Allocation rule of Harsanyi dividends : We consider a process to form the coalition $\{1,2\}$. Then, there are two shortest paths from $\emptyset$ to $\{1,2\}$ in Fig. 2. One is the path $\emptyset \to \{1\} \to \{1,2\}$; another is the path $\emptyset \to \{2\} \to \{1,2\}$. The path $\emptyset \to \{1\} \to \{1,2\}$ can be interpreted as follows: *Player 1 makes an offer to player 2 for forming the coalition* $\{1,2\}$. *Player 2 accepts the offer and adds to the coalition* $\{1\}$ *to form the new coalition* $\{1,2\}$. Among these two paths, the only path that passes through $\{1\}$ is $\emptyset \to \{1\} \to \{1,2\}$. That is, the number of paths from $\emptyset$ to $\{1,2\}$ is 2, while of the number of paths via $\{1\}$ is 1. Then player 1 obtains $\frac{1\ path}{2\ paths}$ of the amount of the Harsanyi dividend $\Delta^v(\{1,2\})$ (i.e., $\frac{1}{2}\Delta^v(\{1,2\})$). In the same way, player 1 obtains $\frac{1}{1}\Delta^v(\{1\})$ and $\frac{0}{1}\Delta^v(\{2\})$. The Shapley value of player 1 is obtained as the sum of all these shares.

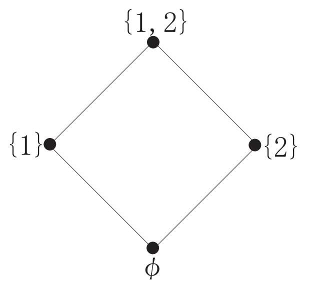

Figure 4: The Boolean lattice $B(2)$ on $N = \{1,2\}$.

This allocation rule can be extended to the case $N = \{1,2,3\}$ (Fig. 5).

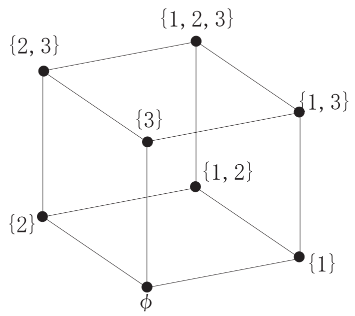

Figure 5: The Boolean lattice on $N = \{1,2,3\}$.

Indeed,

$$\begin{aligned}\Phi_1(N, v) &= \frac{1}{1}\Delta^v(\{1\}) + \frac{1}{2}\Delta^v(\{1,2\}) + \frac{1}{2}\Delta^v(\{1,3\}) \\ &+ \frac{0}{2}\Delta^v(\{2,3\}) + \frac{2}{6}\Delta^v(\{1,2,3\}). \end{aligned} \tag{29}$$

For instance, there are six shortest paths from $\emptyset$ to $\{1,2,3\}$. Among them, two paths pass through $\{1\}$, as shown in Fig. 6.

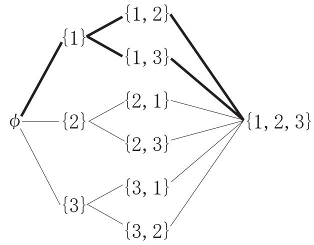

Figure 6: Shortest paths from $\emptyset$ to $\{1,2,3\}$.

4.2 An interpretation of the Myerson value

The Myerson value of (N, v, L) is the Shapley value of the network-restricted game (v^L, N). That is, the Myerson value is obtained by applying the above allocation rule to Harsanyi dividends $\{\Delta^{v^L}\}$ of v^L. Then Δ^{v^L} is given as follows:

Proposition 4.1 Let $(N, v, L) \in CS^N$ be a communication situation and $(B(n), \Delta^{v^L})$ the poset representation of the network-restricted game (N, v^L) associated with (N, v, L). Then,

$$\Delta^{v^L}(S) = \begin{cases} \Delta^v(S) & \text{if } S \in P(N, L), \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

4.3 Criticisms of existing values

Each of the existing values for communication situations, the Myerson value, the position value, and the Hamiache value, has been subject to criticisms, as follows.

The Myerson value :

$$\Psi_i(N, u_S, L) = \Psi_i(N, u_S, M) = \frac{1}{|S|} \quad \forall i \in N \tag{31}$$

whenever S is a feasible coalition in both (N, L) and (N, M). For example, consider the communication situation with $L = \{ij \subseteq N \mid j \in N \setminus i\}$ (i.e., L is a star with a central player i); then every player receives the same value (see Example $\Psi(N, v, L_e)$ in Example 5.3).

The position value :

Irrelevant null players often have positive values (see Example 5.2), where a null player $i \in N$ of the game (N, v) is a player satisfying $v(S \cup i) = v(S)$ for any $S \subseteq N$.

The Hamiache value :

It is very complex to compute the Hamiache value. Not only that, associated consistency is rather technical.

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4.4 A new value in communication situations

In this section, we propose a new value for communication situations that withstands all these criticisms.

Definition 4.1 (chain, saturated chain) A *chain* (or a *totally ordered set* or *linear ordered set*) is a poset in which any two elements are comparative. That is, a subset C of $P(N, L)$ is called a chain if $S \subseteq T$ or $T \subseteq S$ for any $S, T \in C$. The chain C of $P(N, L)$ is *saturated* (or *unrefinable*) if there does not exist $W \in P(N, L) \setminus C$ such that $S \subsetneq W \subsetneq T$ for some $S, T \in C$ and that $C \cup W$ is a chain.

Definition 4.2 (shortest path) For two feasible coalitions S, $T \in P(N, L)$, a saturated chain $\mathcal{P}$ of $P(N, L)$ is called a *shortest path* from S to T if $S, T \in \mathcal{P}$ and $S \subseteq W \subseteq T$ for any $W \in \mathcal{P}$. Then, we denote the set of all shortest paths from S to T by $\{S \to T\}$.

In the following, we propose a new value in communication situations, based on the interpretation of the Shapley value mentioned in Subsection 4.1.

Definition 4.3 We now propose a new value $\sigma(N, v, L)$ of a communication situation (N, v, L), as follows.

$$\sigma_i(N, v, L) := \sum_{S \in P(N,L)} \frac{|\{i \to S\}|}{|\{\emptyset \to S\}|} \Delta^{v^P}(S) \quad \text{for each } i \in N. \tag{32}$$

The number $|\{\emptyset \to S\}|$ of all shortest paths from $\emptyset$ to S indicates the number of all processes in which the feasible coalition S is formed. Also, $|\{i \to S\}|$ indicates the number of all processes in which the feasible coalition S is formed by the initiator $i \in N$. Then, the player $i \in N$ obtains $\frac{|\{i \to S\}|}{|\{\emptyset \to S\}|}$ of the amount of $\Delta^{v^P}(S)$ if $\Delta^{v^P}(S)$ is allocated in proportion to the frequency with which the player i initiates the formation of the feasible coalition S. The value proposed here of a given player $i \in N$ is obtained as the sum of all these shares.

Now we show an example that supports the naturalness of the definition of this value.

Example 4.1 We consider the communication situation (N, v, L) with $N = \{1, 2, 3\}$, $L = \{13, 23\}$, and $\Delta^{v^P}(S) \geq 0$ for any $S \in P(N, L)$. The value $\sigma_i(N, v, L)$ proposed here of player $i \in N$ is represented as the values of the ammeters A_i in the electric circuit with current sources $I_S = \Delta^{v^P}(S)$, as shown in Fig.7.

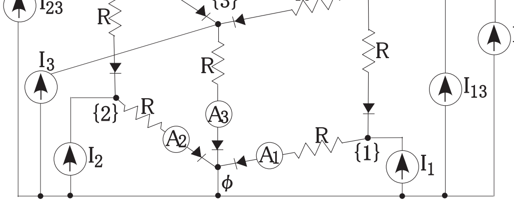

Figure 7: Electric circuit representing (N, v, L).

Property 1 The value σ proposed here satisfies *component-efficiency*, *linearity w.r.t. games*, *independence of irrelevant players*, and *positivity*.

Property 2 Let (N, u_N, L_c^*) be a communication situation with $L_c^* = \{cj \mid j \in N \setminus c\}$, $c \in N$. Then,

$$\sigma_i(N, u_N, L_c^*) = \begin{cases} \frac{1}{2} & \text{if } i = c, \\ \frac{1}{2(n-1)} & \text{otherwise.} \end{cases} \tag{33}$$

That is, if the communication network (N, L) is a star-graph with central player $c \in N$, in the unanimity game u_N, the central player obtains a half of the total amount of $u_N(N) = 1$ and the rest of the amount are shared out equally among the other players (see L_b, L_e in example 5.3).

However, we have not found any axiomatic characterization of the value proposed in this paper yet.

5 Comparison of existing values

In this section, we compare the existing four values (the Shapley, Myerson, position, and Hamiache values) and the value proposed in this paper. Examples 5.1 and 5.2 not only compare them but also illustrate the criticisms against the Shapley, Myerson, and position values, respectively.

Example 5.1 Consider the communication situation (N, v, L) with $N = \{1, 2, 3\}$, $L = \{13, 23\}$ ((b) in Fig. 2), and

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 30 & \text{if } |S| = 2 \\ 36 & \text{if } S = N. \end{cases} \tag{34}$$

Then,

$$\Phi(N, v) = (12, 12, 12), \quad \Psi(N, v, L) = (7, 7, 22),$$
$$\pi(N, v, L) = (9, 9, 18), \quad \phi(N, v, L) = (9, 9, 18),$$
$$\sigma(N, v, L) = (9, 9, 18).$$

Example 5.2 Consider the communication situation (N, v, L) with $N = \{1, 2, 3\}$, $L = \{12, 13, 23\}$ (L_d in Fig. 8), and

$$v(S) = \begin{cases} 12 & \text{if } S \supseteq \{1, 2\} \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Then,

$$\Phi(N, v) = (6, 6, 0), \quad \Psi(N, v, L) = (6, 6, 0),$$
$$\pi(N, v, L) = (5, 5, 2), \quad \phi(N, v, L) = (6, 6, 0),$$
$$\sigma(N, v, L) = (6, 6, 0).$$

Example 5.3 Consider communication situations (N, u_N, L) with $2 \leq |N| \leq 4$, $|N/L| = 1$ (i.e., (N, L) is connected). Fig.8 shows all connected graphs (up to isomorphism) with $2 \leq n \leq 4$ nodes. Then, for any such communication situations (N, u_N, L),

$$\Phi_i(N, u_N, L) = \Psi_i(N, u_N, L) = \frac{1}{|N|} \quad \forall i \in N. \tag{36}$$

Table 1 shows comparisons of the remaining values (i.e., the position value π, the Hamiache value ϕ, and the value σ proposed in this paper), and illustrates that the value σ does not always coincide with the Hamiache value ϕ.

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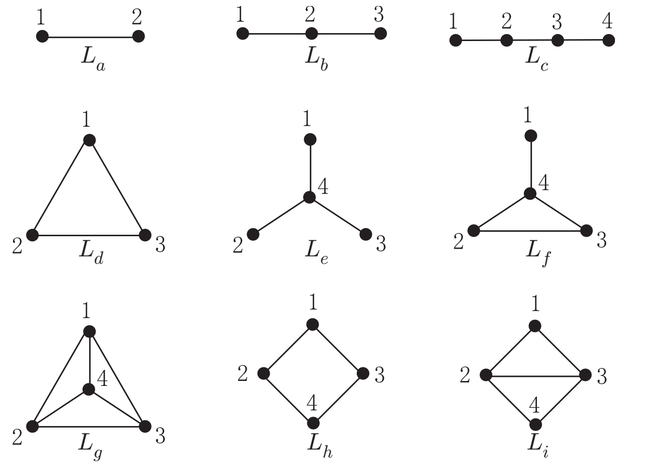

Figure 8: Graphs with at most four nodes.

Table 1: Comparison of existing values.

	π	ϕ	σ
L_a	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$
L_b	$(\frac{1}{4},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{2},\frac{1}{4})$
L_c	$(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6})$	$(\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8})$	$(\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8})$
L_d	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
L_e	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{3}{6})$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{3}{6})$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{3}{6})$
L_f	$(\frac{3}{12},\frac{2}{12},\frac{2}{12},\frac{5}{12})$	(0.172, 0.190, 0.190, 0.448)	$(\frac{2}{14},\frac{3}{14},\frac{3}{14},\frac{6}{14})$
L_g	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
L_h	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
L_i	$(\frac{13}{60},\frac{17}{60},\frac{17}{60},\frac{13}{60})$	$(\frac{3}{14},\frac{4}{14},\frac{4}{14},\frac{3}{14})$	$(\frac{2}{10},\frac{3}{10},\frac{3}{10},\frac{2}{10})$

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