Ranking of fuzzy numbers, some recent and new formulas

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Abstract— *Ranking of fuzzy numbers plays a very important role in linguistic decision making and some other fuzzy application systems. Several strategies have been proposed for ranking of fuzzy numbers. Each of these techniques has been shown to produce non-intuitive results in certain cases. In this paper, some new approaches for ranking of trapezoidal fuzzy numbers are introduced.*

Keywords— Magnitude of fuzzy number, Parametric form of fuzzy number, Ranking of fuzzy numbers, Trapezoidal fuzzy number.

1 Introduction

Ranking of fuzzy numbers is an important component of the decision process in many applications. More than 30 fuzzy ranking indices have been proposed since 1976. In 1976 and 1977, Jain [1, 2] proposed a method using the concept of maximizing set to order the fuzzy numbers. Jain's method is that the decision maker considers only the right side membership function. A canonical way to extend the natural ordering of real numbers to fuzzy numbers was suggested by Bass and Kwakernaak [3] as early as 1977. Dubios and Prade 1978 [4], used maximizing sets to order fuzzy numbers. In 1979, Baldwin and Guild [5] indicated that these two methods have some disturbing disadvantages. Also, in 1980, Adamo [6] used the concept of α-level set in order to introduce α-preference rule. In 1981 Chang [7] introduced the concept of the preference function of an alternative. Yager in 1981 [8, 9] proposed four indices which may be employed for the purpose of ordering fuzzy quantities in $[0,1]$. Bortolan and Degani have been compared and reviewed some of these ranking methods [10]. Chen and Hwang [11] thoroughly reviewed the existing approaches, and pointed out some illogical conditions that arise among them. Chen [12], Choobineh [13], Cheng [14] have presented some methods , and also more recently numerous ranking techniques have been proposed and investigated by Chu, Tsao [15] and Ma, Kandel and Friedman [16]. Nowadays many researchers have developed methods to compare and to rank fuzzy numbers. Some of those methods are counter-intuitive and non discriminating [18, 19, 20, 21, 22] and recently some methods based on different distance functions have been introduced for ranking of fuzzy numbers [23, 24, 25, 26, 27].

2 Preliminaries

Though there are a number of ways of defining fuzzy numbers, for the purposes of this paper we adopt the following definition, we will identify the name of the number with that of its membership function for simplicity. Throughout this paper, $\mathbb{R}$ stands for the set of all real numbers, E stands the set of fuzzy numbers, $u(x)$ for the membership function of every $u \in E$ and $x \in \mathbb{R}$.

Definition 2.1 *[28] A fuzzy number is a fuzzy set like $u : \mathbb{R} \rightarrow I = [0,1]$ which satisfies:*

1. *u is upper semi-continuous,*
2. *$u(x) = 0$ outside some interval $[a,d]$,*
3. *There are real numbers a, b such that $a \le b \le c \le d$ and*
   a. *$u(x)$ is monotonic increasing on $[a,b]$,*
   b. *$u(x)$ is monotonic decreasing on $[c,d]$,*
   c. *$u(x) = 1,\ b \le x \le c$.*

The membership function u can be expressed as

$$u(x) = \begin{cases} u_L(x), & a \le x \le b, \\ 1, & b \le x \le c, \\ u_R(x), & c \le x \le d, \\ 0, & otherwise, \end{cases}$$

where $u_L : [a,b] \longrightarrow [0,1]$ and $u_R : [c,d] \longrightarrow [0,1]$ are left and right membership functions of fuzzy number u. An equivalent parametric form is also given in [29] as follows:

Definition 2.2 *[29] A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r),\ 0 \le r \le 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded monotonic increasing left continuous function,*
2. *$\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,*
3. *$\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.*

The trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with two defuzzifier x_0, y_0, and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is as

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma), & x_0 - \sigma \le x \le x_0, \\ 1 & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \le x \le y_0 + \beta, \\ 0, & otherwise, \end{cases}$$

and its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \qquad \overline{u}(r) = y_0 + \beta - \beta r.$$

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Let E_{TR} be the set of all trapezoidal fuzzy numbers on the real line. Provided that, $x_0 = y_0$ then u is a triangular fuzzy number, and we write $u = (x_0, \sigma, \beta)$. The support of fuzzy number u is defined as follows:

$$supp(u) = \overline{\{x \mid u(x) > 0\}},$$

where $\overline{\{x \mid u(x) > 0\}}$ is closure of set $\{x \mid u(x) > 0\}$.

The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented in [30, 31, 32] as follows. For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ we define addition $(u + v)$ and multiplication by scalar $k > 0$ as

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r), \quad (1)$$

$$(\underline{ku})(r) = k\underline{u}(r), \quad (\overline{ku})(r) = k\overline{u}(r). \quad (2)$$

To emphasis the collection of all fuzzy numbers with addition and multiplication as defined by (1) and (2) is denoted by E, which is a convex cone. The image (opposite) of $u = (x_0, y_0, \sigma, \beta)$, can be defined by $-u = (-y_0, -x_0, \beta, \sigma)$ (see [32, 33]).

Definition 2.3 *For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity*

$$D(u, v) = \left[\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 \, \mathrm{dr} + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 \, \mathrm{dr} \right]^{1/2},$$

is the distance between u and v, [16, 17, 34]. The function $D(u, v)$ is a metric in E and (E, D) is a complete metric space.

The ordering indices are organized into three categories by Wang and Kerre [35] as follows:

1- **Defuzzification method**: Each index is associated with a mapping from the set of fuzzy quantities to the real line. In this case fuzzy quantities are compared according to the corresponding real numbers.

2- **Reference set method**: In this case a fuzzy set as a reference set is set up and all the fuzzy quantities to be ranked are compared with the reference set.

3- **Fuzzy relation method**: In this case a fuzzy relation is constructed to make pairwise comparisons between the fuzzy quantities involved.

Let M be an ordering method on E. The statement "two elements u and v in E satisfy that u has a higher ranking than v when M is applied" will be written as "$u \succ v$ by M". "$u \sim v$" and "$u \succeq v$" are similarly interpreted. The following reasonable properties for the ordering approaches are introduced by Wang and Kerre [35].

Reasonable properties (axioms)

A_1 : For an arbitrary finite subset Γ of E and $u \in \Gamma$, $u \succeq u$.

A_2 : For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$, $u \succeq v$ and $v \succeq u$, we should have $u \sim v$.

A_3 : For an arbitrary finite subset Γ of E and $(u, v, w) \in \Gamma^3$, $u \succeq v$ and $v \succeq w$, we should have $u \succeq w$.

A_4 : For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$, $\inf supp(u) \geq \sup supp(v)$, we should have $u \succeq v$.

A'_4 : For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$, $\inf supp(u) > \sup supp(v)$, we should have $u \succ v$.

A_5 : Let Γ and Γ' be two arbitrary finite subsets of E also u and v are in $\Gamma \cap \Gamma'$. We obtain the ranking order $u \succ v$ by M on Γ' if and only if $u \succ v$ by M on Γ.

A_6 : Let $u,\ v,\ u + w$ and $v + w$ be elements of E. If $u \succeq v$, then $u + w \succeq v + w$.

A'_6 : Let $u,\ v,\ u + w$ and $v + w$ be elements of E. If $u \succ v$, then $u + w \succ v + w$, when $w \neq 0$.

A_7 : Let $u,\ v,\ uw$, and vw be elements of E and $w \geq 0$. If $u \succeq v$ then $uw \succeq vw$.

3 Some new and recent methods

3.1 Method of D-distance

Let all of fuzzy numbers are positive or negative. Without less of generality assume that all of them are positive. The membership function of $a \in R$ is $u_a(x) = 1$, if $x = a$; and $u_a(x) = 0$, if $x \neq a$. Hence if $a = 0$ we have the following

$$u_0(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

Since $u_0(x) \in E$, left fuzziness σ and right fuzziness β are 0, so for each $u \in E$

$$D(u, u_0) = \left[\int_0^1 (\underline{u}(r)^2 + \overline{u}(r)^2) \mathrm{dr} \right]^{1/2}.$$

Thus we have the following definition.

Definition 3.1 *For u and $v \in E$, define the ranking of u and v by saying*

$u > v \ iff \ d(u, u_0) > d(v, u_0)$,
$u < v \ iff \ d(u, u_0) > d(v, u_0)$,
$u \approx v \ iff \ d(u, u_0) = d(v, u_0)$.

Property 3.1. Suppose u and $v \in E$ are arbitrary there

(I) If $u = v$ then $u \approx v$.

(II) If $v \subseteq u$ and $\underline{u}(r)^2 + \overline{u}(r)^2 > \underline{v}(r)^2 + \overline{v}(r)^2$ for all $r \in [0, 1]$ then $v < u$.

Remark 3.1. (I) The distance triangular fuzzy number $u = (x_0, \sigma, \beta)$ of u_0 is defined as following

$$d(u, u_0) = \left[2x_0^2 + \sigma^2/3 + \beta^2/3 + x_0(\beta - \sigma)\right]^{1/2}.$$

(II) The distance trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ of u_0 is defined as following

$$d(u, u_0) = \left[2x_0^2 + \sigma^2/3 + \beta^2/3 - x_0\sigma + y_0\beta\right]^{1/2}.$$

(III) If $u \approx v$, it is not necessary that $u = v$. Since if $u \neq v$ and $(\underline{u}(r)^2 + \overline{u}(r)^2)^{1/2}$=$(\underline{v}(r)^2 + \overline{v}(r)^2)^{1/2}$ then $u \approx v$.

3.2 Method of min distance

In this subsection, we will propose the ranking of fuzzy numbers associated with the metric D in E.

Definition 3.2 *Let $\gamma = \{v_1, v_2, \ldots, v_n\} \subseteq E$ be is set of fuzzy numbers, we define a ranking on γ by min distance as*

$$v_i \succ v_j \ \text{ iff } \ D(v_i, \tilde{M}) > D(v_j, \tilde{M}),$$

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$$v_i \prec v_j \text{ iff } D(v_i, \tilde{M}) < D(v_j, \tilde{M}),$$

$$v_i \sim v_j \text{ iff } D(v_i, \tilde{M}) = D(v_j, \tilde{M}),$$

where $\tilde{M} = \widetilde{min}\{v_1, \ldots, v_n\}$.

Dubois and Prade [30] present rules for computing $\widetilde{min}$ and also comment on the properties of $\widetilde{min}$.

Remark 3.2. The min distance, has the properties $A_1, A_2, \ldots, A_4'$.

3.3 Method of sign distance

Definition 3.3 *For arbitrary fuzzy numbers* $u = (\underline{u}, \overline{u})$ *and* $v = (\underline{v}, \overline{v})$,*the function*

$$D_p(u, v) = \left[\int_0^1 |\underline{u}(r) - \underline{v}(r)|^p \mathrm{dr} + \int_0^1 |\overline{u}(r) - \overline{v}(r)|^p \mathrm{dr}\right]^{1/p}, \quad (p \geq 1)$$

is the distance between u *and* v.

Definition 3.4 *Let* $\gamma : E \longrightarrow \{-1, 1\}$ *be a function that is defined as follows:*

$$\forall\, u \in E : \gamma(u) = sign\left[\int_0^1 (\underline{u}(r) + \overline{u}(r))\mathrm{dr}\right],$$

where

$$\gamma(u) = \begin{cases} 1 & \text{if } sign\left(\int_0^1 (\underline{u} + \overline{u})(r)\mathrm{dr}\right) \geq 0, \\ -1 & \text{if } sign\left(\int_0^1 (\underline{u} + \overline{u})(r)\mathrm{dr}\right) < 0. \end{cases}$$

Remark 3.3. (I) If inf $supp(u) \geq 0$ or inf $\underline{u}(r) \geq 0$ then $\gamma(u) = 1$.

(II) If sup $supp(u) < 0$ or sup $\overline{u}(r) < 0$ then $\gamma(u) = -1$.

Definition 3.5 *For* $u \in E$,

$$d_p(u, u_0) = \gamma(u) D_p(u, u_0),$$

is called sign distance.

Definition 3.6 *For* u *and* $v \in E$, *define the ranking of* u *and* v *by* d_p *on* E, *i.e.*

$$u \succ v \text{ iff } d_p(u, u_0) > d_p(v, u_0),$$

$$u \prec v \text{ iff } d_p(u, u_0) < d_p(v, u_0),$$

$$u \sim v \text{ iff } d_p(u, u_0) = d_p(v, u_0).$$

Remark 3.4. (I) The function d_p, sign distance, has the properties $A_1, A_2, \ldots, A_5$.

(II) The function d_p, sign distance, for $p = 1$ has the properties A_6, A_6' if

$$\inf\{supp(u), supp(v), supp(u + w), supp(v + w)\} \geq 0$$

or

$$\sup\{supp(u), supp(v), supp(u + w), supp(v + w)\} \leq 0.$$

(III) Suppose u and $v \in E$ are arbitrary, then

(a) If $u = v$ then $u \sim v$,

(b) If $v \subseteq u$ and $\gamma(u)\left(|\underline{u}(r)|^p + |\overline{u}(r)|^p\right) > \gamma(v)\left(|\underline{v}(r)|^p + |\overline{v}(r)|^p\right)$ for all $r \in [0, 1]$ then $v \prec u$.

(IV) If $u \sim v$, it is not necessary that $u = v$. Since if $u \neq v$ and $\gamma(u)(|\underline{u}(r)|^p + |\overline{u}(r)|^p) = \gamma(v)(|\underline{v}(r)|^p + |\overline{v}(r)|^p)$ then $u \sim v$.

(V) If $u \preceq v$ then $-u \succeq -v$.

Therefore we can simply rank the fuzzy numbers by the defuzzification of $d_p(u, u_0)$. By Remark 3.4(V) we can logically infer ranking order of the images of the fuzzy numbers.

3.4 Method of H-distance

Definition 3.7 *A continuous function* $s : [0, 1] \longrightarrow [0, 1]$ *with the following properties is a* source *function*

1. $s(0) = 0$,
2. $s(1) = 1$,
3. $s(r)$ *is increasing.*
4. $\int_0^1 s(r)\mathrm{dr} = \frac{1}{2}$.

In fact, a reducing has the reflect of weighting the influence of the different r-cuts and diminishes the contribution of the lower r-levels. This is reasonable since these levels arises from values of membership function for which there is a considerable amount of uncertainty. For example, we can use $s(r) = r$.

Definition 3.8 *The* Value *and* Ambiguity *of a fuzzy number* $\widetilde{u}$ *are defined as follows, [36],*

$$Val_s(\widetilde{u}) = \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r)]\mathrm{dr},$$

$$Amb_s(\widetilde{u}) = \int_0^1 s(r)[\overline{u}(r) - \underline{u}(r)]\mathrm{dr}.$$

Definition 3.9 *For* $\widetilde{u}, \widetilde{v} \in E$, *we define* H***-distance*** *of* $\widetilde{u}$ *and* $\widetilde{v}$ *by*

$$D_H^s(\widetilde{u}, \widetilde{v}) = \frac{1}{2}\big\{\, |Val_s(\widetilde{u}) - Val_s(\widetilde{v})| + |Amb_s(\widetilde{u}) - Amb_s(\widetilde{v})| + d_H([\widetilde{u}]^1, [\widetilde{v}]^1)\big\},$$

where d_H *is the Hausdorff metric between intervals, and* $[.]^1$ *is the 1-cut representation of a fuzzy number.*

Property 3.2. The source distance, D_H^s, is a metric on E_{TR} and a pseudo-metric on E.

Remark 3.5. By the metric D_H^s and an arbitrary reference fuzzy set (like subsection 3.1 or 3.2), we can define a new ordering for fuzzy numbers.

3.5 Method of source distance

Definition 3.10 *For* $\widetilde{u}, \widetilde{v} \in E$, *we define* ***source distance*** *of* $\widetilde{u}$ *and* $\widetilde{v}$ *by*

$$D_s(\widetilde{u}, \widetilde{v}) = \frac{1}{2}\big\{\, |Val_s(\widetilde{u}) - Val_s(\widetilde{v})| + |Amb_s(\widetilde{u}) - Amb_s(\widetilde{v})| + \max\big\{\, |t_v - t_u|\,, |m_v - m_u|\big\}\big\},$$

where $[m_u, t_u]$ *and* $[m_v, t_v]$ *are the cores of fuzzy numbers* $\widetilde{u}$ *and* $\widetilde{v}$ *respectively.*

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Property 3.3. The source distance, D_s, is a metric on E_{TR} and a pseudo-metric on E.

Remark 3.6. By the metric D_s and an arbitrary reference fuzzy set (like subsection 3.1 or 3.2), we can define a new ordering for fuzzy numbers.

3.6 Method of magnitude

For an arbitrary trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with parametric form $u = (\underline{u}(r), \overline{u}(r))$, we define the *magnitude* of the trapezoidal fuzzy number u as

$$Mag(u) = \frac{1}{2}\Big(\int_0^1 (\underline{u}(r) + \overline{u}(r) + x_0 + y_0) f(r)\mathrm{dr}\Big),$$

where the function $f(r)$ is a non-negative and increasing function on $[0, 1]$ with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)\mathrm{dr} = \frac{1}{2}$. For example, we can use $f(r) = r$. The resulting scalar value is used to rank the fuzzy numbers. In the other words $Mag(u)$ is used to rank fuzzy numbers. The larger $Mag(u)$, the larger fuzzy number. Therefore for any two trapezoidal fuzzy numbers u and $v \in E$, we define the ranking of u and v by the $Mag(.)$ on E as follows:

1. $Mag(u) > Mag(v)$ if and only if $u \succ v$,
2. $Mag(u) < Mag(v)$ if and only if $u \prec v$,
3. $Mag(u) = Mag(v)$ if and only if $u \sim v$.

Then we formulate the order $\succeq$ and $\preceq$ as $u \succeq v$ if and only if $u \succ v$ or $u \sim v$, $u \preceq v$ if and only if $u \prec v$ or $u \sim v$. In the other words, this method is placed in the first class of Kerre's categories [35].

Remark 3.7. **(I)** If $\inf supp(u) \geq 0$ or $\inf \underline{u}(r) \geq 0$ then $Mag(u) \geq 0$.

(II) If $\sup supp(u) \leq 0$ or $\sup \overline{u}(r) \leq 0$ then $Mag(u) \leq 0$.

(III) For two arbitrary trapezoidal fuzzy numbers u and v, we have

$$Mag(u + v) = Mag(u) + Mag(v).$$

(IV) For all symmetric trapezoidal fuzzy numbers $u = (-x_0, x_0, \sigma, \sigma)$,

$$Mag(u) = 0.$$

(V) For any two symmetric trapezoidal fuzzy numbers $u = (x_0, y_0, \sigma, \sigma)$ and $v = (x_0, y_0, \beta, \beta)$,

$$Mag(u) = Mag(v).$$

Property 3.4. The function $Mag(.)$ has the properties $A_1, A_2, A_3, \ldots, A'_6$.

4 Conclusions

In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases. The proposed methods can effectively rank various fuzzy numbers and their images. These methods have some mathematical properties. Moreover some pseudo metric on the set of fuzzy numbers and metric on trapezoidal fuzzy numbers are introduced. We may conclude that these ordering methods are relatively reasonable for fuzzy numbers based on the introduced axioms.

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