Lattice properties of discrete fuzzy numbers under extended min and max

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Abstract— *This paper proposes to study the lattice properties of two closed binary operations in the set of discrete fuzzy numbers. Using these operations to represent the meet and the join, we prove that the set of discrete fuzzy numbers whose support is a set of consecutive natural numbers is a distributive lattice. Finally, we demonstrate that the subsets of discrete fuzzy numbers, which have the same support, are distributive lattices too.*

Keywords— Fuzzy numbers, discrete fuzzy numbers, distributive lattice.

1 Introduction

It is possible to approach the theory of fuzzy numbers in different directions: theoretical[8, 9, 10, 13], geometric [1, 2], applications in engineering [13], social science [12], lattice theory[13, 20], etc.. Voxman [18] introduced the concept of a discrete fuzzy number as a fuzzy subset of real numbers with discrete support and analogous properties to a fuzzy number (convexity, normality). Also, like fuzzy numbers, it is possible to consider discrete fuzzy numbers from different points of view: theoretical [16, 18], applications in engineering [11, 19], social sciences [17], etc.

It is well known that, the arithmetic and lattice operations such as maximum and minimum on fuzzy numbers can be approached either by the direct use of the membership function (by the Zadeh's extension principle) or by the equivalent use of the *r-cuts* representation, for instance, [13, 14, 20].

Nevertheless, in the discrete case using the same methods, this process can yield a fuzzy subset that does not satisfy the conditions to be a discrete fuzzy number [3, 19]. In previous works [3, 4], we have presented a technique that allows us to obtain a closed addition on the set of discrete fuzzy numbers, DFN, and moreover, we focus on the addition of discrete fuzzy numbers whose support is an arithmetic sequence and even a subset of consecutive natural numbers. This type of numbers arise mainly when a fuzzy cardinality of a fuzzy set [5, 6] or a fuzzy multiset [15] is considered.

In [7] we define two closed binary operations in the set of discrete fuzzy numbers to obtain the maximum and the minimum of discrete fuzzy numbers. We prove as well that in the set $\mathcal{A}_1$, of discrete fuzzy numbers whose support is a set of consecutive natural numbers, these operations coincide with the function maximum and minimum obtained using the Zadeh's extension principle.

The aim of this paper is to continue studying the properties of these operations and if it is possible to obtain a structure of lattice using them. We will see that, in general, these binary operations only fulfill the associative, commutative and idempotent laws. We show as well that the triplets ($\mathcal{A}_1$,MAX_w,MIN_w) and (F_A,MIN_w,MAX_w) are distributive lattices, where F_A is the subset of discrete fuzzy numbers whose support is the support of A, with $A \in DFN$.

2 Preliminaries

In this section, we recall some definitions and the main results about discrete fuzzy numbers which will be used later.

Definition 2.1 *[13] A fuzzy subset u of $\mathbb{R}$ with membership mapping $u : \mathbb{R} \to [0, 1]$ is called* fuzzy number *if its* support *is an interval $[a, b]$ and there exist real numbers s, t with $a \leq s \leq t \leq b$ and such that:*

1. *u(x)=1 with $s \leq x \leq t$*
2. *$u(x) \leq u(y)$ with $a \leq x \leq y \leq s$*
3. *$u(x) \geq u(y)$ with $t \leq x \leq y \leq b$*
4. *$u(x)$ is upper semi-continuous.*

We will denote the set of fuzzy numbers by FN.

Definition 2.2 *[18] A fuzzy subset u of $\mathbb{R}$ with membership mapping $u : \mathbb{R} \to [0, 1]$ is called* discrete fuzzy number *if its support is finite, i.e., there are $x_1, ..., x_n \in \mathbb{R}$ with $x_1 < x_2 < ... < x_n$ such that supp(u) $= \{x_1, ..., x_n\}$, and there are natural numbers s, t with $1 \leq s \leq t \leq n$ such that:*

1. *$u(x_i)$=1 for any natural number and i with $s \leq i \leq t$ (*core*)*
2. *$u(x_i) \leq u(x_j)$ for each natural number i, j with $1 \leq i \leq j \leq s$*
3. *$u(x_i) \geq u(x_j)$ for each natural number i, j with $t \leq i \leq j \leq n$*

From now on, we will denote the set of discrete fuzzy numbers by DFN and $DFN(\mathbb{N})$ will stand for the set of discrete fuzzy numbers whose support is a subset of the set of Natural Numbers. Finally, a discrete fuzzy number will be denoted by *dfn*.

In general, the operations on fuzzy numbers f, g can be approached either by the direct use of their membership function, $\mu_f(x), \mu_g(x)$, as fuzzy subsets of $\mathbb{R}$ and the Zadeh's extension principle:

$$O(f, g)(z) = sup\{\mu_f(x) \wedge \mu_g(y) | O(x, y) = z\}$$

or by the equivalent use of the α-*cuts* representation[13]:

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$$O(f,g)^\alpha = O(f^\alpha, g^\alpha) = \{O(x,y)|x \in f^\alpha, y \in g^\alpha\}$$

and

$$O(f,g)(z) = sup\{\alpha \in [0,1]|z \in O(f,g)^\alpha\}$$

Nevertheless, in the discrete case, this process can yield a fuzzy subset that does not satisfy the conditions to be a discrete fuzzy number [3, 19]. For example, let $u = \{0.3/1, 1/3, 0.5/7\}$ and $v = \{0.4/2, 1/5, 1/6, 0.8/9\}$ be two discrete fuzzy numbers. If we use the Zadeh's extension principle to obtain their addition, it results the fuzzy subset $S = \{0.3/3, 0.4/5, 0.3/6, 0.3/7, 1/8, 1/9, 0.3/10, 0.8/12, 0.5/13, 0.5/16\}$ which doesn't fulfill the conditions to be a discrete fuzzy number, because the third property of the definition 2.2 fails. In a previous work [3, 4] we have presented an approach to a closed extended addition ($\oplus$) of discrete fuzzy numbers after associating suitable non-discrete fuzzy numbers, which can be used like a carrier to obtain the desired addition.
In a recent paper [4] we proved that a suitable carrier can be a discrete fuzzy number whose support is an arithmetic sequence and even a subset of consecutive natural numbers. Thus, we obtained the following results:

Proposition 2.3 *[4]*
Let $\mathcal{A}_r$ be the set $\{f \in DFN$ $(\mathbb{N})$, such that supp(f) is the set of terms of an arithmetic sequence with r as common difference$\}$. If $f, g \in \mathcal{A}_r$. The following facts:

1. $f \oplus g \in DFN(\mathbb{N})$
2. $f \oplus g \in \mathcal{A}_r$

hold.

Remark 2.4 *[4] Note that the set $\mathcal{A}_1$ is the set of discrete fuzzy numbers whose support is a set of consecutive natural numbers.*

Finally, we will use a kind of representation in the study of discrete fuzzy numbers:

Theorem 2.5 *[19] Let u be a dfn and let u^r be the r-cut $=\{x \in \mathbb{R}|u(x) \geq r\}$ for any $r \in (0,1]$. Let's u^0 denote the support of u. Then the following statements (1)-(4) hold:*

1. *u^r is a nonempty finite subset of $\mathbb{R}$, for any $r \in [0,1]$*
2. *$u^{r_2} \subset u^{r_1}$ for any $r_1, r_2 \in [0,1]$ with $0 \leq r_1 \leq r_2 \leq 1$*
3. *For any $r_1, r_2 \in [0,1]$ with $0 \leq r_1 \leq r_2 \leq 1$, if $x \in u^{r_1} - u^{r_2}$ we have $x < y$ for all $y \in u^{r_2}$, or x>y for all $y \in u^{r_2}$*
4. *For any $r_0 \in [0,1]$, there exist some real numbers r_0' with $0 < r_0' < r_0$ such that $u^{r_0'} = u^{r_0}$ (i.e. $u^r = u^{r_0}$ for any $r \in [r_0', r_0]$).*

And conversely, if for any $r \in [0,1]$, there exist $A^r \subset \mathbb{R}$ satisfying the following conditions (1)-(4):

1. *A^r is a nonempty finite for any $r \in [0,1]$*
2. *$A^{r_2} \subset A^{r_1}$, for any $r \in [0,1]$ with $0 \leq r_1 \leq r_2 \leq 1$*
3. *For any $r_1, r_2 \in [0,1]$ with $0 \leq r_1 \leq r_2 \leq 1$, if $x \in A^{r_1} - A^{r_2}$ we have $x < y$ for all $y \in A^{r_2}$, or x>y for all $y \in A^{r_2}$*
4. *For any $r_0 \in [0,1]$, there exists a real number r_0' with $0 < r_0' < r_0$ such that $A^{r_0'} = A^{r_0}$ (i.e. $A^r = A^{r_0}$, for any $r \in [r_0', r_0]$)*

then there exists a unique $u \in DFN$ such that $u^r = A^r$ for any $r \in [0,1]$.

3 Maximum and Minimum of discrete fuzzy numbers

It's well known, for example [13], that the set of fuzzy numbers is a distributive lattice using the following operations

$$MIN(u,v)(z) = \sup_{z=\min(x,y)} \min(u(x), v(y)), \forall z \in \mathbb{R} \quad (1)$$

$$MAX(u,v)(z) = \sup_{z=\max(x,y)} \min(u(x), v(y)), \forall z \in \mathbb{R} \quad (2)$$

for each couple $u, v \in FN$. If we use the same operations to obtain a similar structure in the set of discrete fuzzy numbers we see that this result is not possible. For instance, if we consider the discrete fuzzy numbers $u = \{0.3/1, 0.4/3, 1/4\}$ and $v = \{0.5/2, 1/5, 1/6\}$ and we use the previous definition to calculate the $MIN(u,v)$, we obtain the fuzzy subset $M = \{0.3/1, 0.5/2, 0.4/3, 1/4\}$ which doesn't satisfy the conditions of the definition 2.2. In [7], the authors study this drawback and we propose a new method to calculate them. Using this method, we will see later that, it is possible to provide different subsets of the set DFN with a structure of distributive lattice.

Definition 3.1 *[7] Let u, v be two dfn. For each $\alpha \in [0,1]$, let's consider the α-cut sets: $u^\alpha = \{x_1^\alpha, ..., x_p^\alpha\}$, $v^\alpha = \{y_1^\alpha, ..., y_k^\alpha\}$ for u and v respectively and the set $supp(u) \bigvee supp(v) = \{x \vee y | x \in supp(u), y \in supp(v)\}$. Let's define the set:*

$$A^\alpha = \{z \in supp(u) \bigvee supp(v) \text{ such that}$$
$$\min(u^\alpha \vee v^\alpha) \leq z \leq \max(u^\alpha \vee v^\alpha)\} =$$
$$= \{z \in supp(u) \bigvee supp(v) \text{ such that}$$
$$(\min u^\alpha \vee \min v^\alpha) \leq z \leq (\max u^\alpha \vee \max v^\alpha)\}$$

i.e.:

$$A^\alpha = \{z \in supp(u) \bigvee supp(v) | (x_1^\alpha \vee y_1^\alpha) \leq z \leq (x_p^\alpha \vee y_k^\alpha)\}$$

Proposition 3.2 *[7] The finite set A^α, as defined above, satisfies the properties 1,2,3 and 4 of theorem 2.5 and a discrete fuzzy number, $MAX_w(u,v)$, whose α-cuts are the finite set A^α exists and $MAX_w(u,v)(z) = \sup\{\alpha \in [0,1]$ such that $z \in A^\alpha\}$.*

Proposition 3.3 *[7] If $u, v \in \mathcal{A}_1$, then $MAX(u,v)$, defined through the extension principle, coincides with $MAX_w(u,v)$. So, if $u, v \in \mathcal{A}_1$, $MAX(u,v)$ is a discrete fuzzy number and $MAX(u,v) \in \mathcal{A}_1$.*

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Definition 3.4 *[7] Let u, v be two dfn. For each $\alpha \in [0,1]$, let's consider the α-cut sets: $u^\alpha = \{x_1^\alpha, ..., x_p^\alpha\}, v^\alpha = \{y_1^\alpha, ..., y_k^\alpha\}$ for u and v respectively and the set $supp(u) \bigwedge supp(v) = \{x \wedge y | x \in supp(u), y \in supp(v)\}$. Let's define the set:*

$$B^\alpha = \{z \in supp(u) \bigwedge supp(v) \text{ such that}$$

$$\min(u^\alpha \wedge v^\alpha) \leq z \leq \max(u^\alpha \wedge v^\alpha)\} =$$

$$= \{z \in supp(u) \bigwedge supp(v) \text{ such that}$$

$$(\min u^\alpha \wedge \min v^\alpha) \leq z \leq (\max u^\alpha \wedge \max v^\alpha)\}$$

i.e.:

$$B^\alpha = \{z \in supp(u) \bigwedge supp(v) | (x_1^\alpha \wedge y_1^\alpha) \leq z \leq (x_p^\alpha \wedge y_k^\alpha)\}$$

Remark 3.5 *In general, if "$*$" is a binary operation the equalities*

- $\min(u^\alpha * v^\alpha) = \min u^\alpha * \min v^\alpha$
- $\max(u^\alpha * v^\alpha) = \max u^\alpha * \max v^\alpha$

do not hold. For example, if we consider the sets $u^\alpha = \{-10, -9, -8, -7, -6, -1\}$, $v^\alpha = \{0, 1, 2, 3, 4\}$ and the usual product of real numbers as binary operation then $\min(u^\alpha \cdot v^\alpha) = -40$ and $\min(u^\alpha) \cdot \min(v^\alpha) = -10 \cdot 0 = 0$, and $\max(u^\alpha \cdot v^\alpha) = 0$ and $\max(u^\alpha) \cdot \max(v^\alpha) = -1 \cdot 4 = -4$

Proposition 3.6 *[7] The finite set B^α, as defined above, satisfies the properties 1,2,3 and 4 of Proposition 2.5 and a discrete fuzzy number, $MIN_w(u, v)$, whose α-cuts are the finite set B^α exists and $MIN_w(u, v)(z) = \sup\{\alpha \in [0,1]$ such that $z \in B^\alpha\}$.*

Proposition 3.7 *[7] If $u, v \in \mathcal{A}_1$, then $MIN(u, v)$, defined through the extension principle, coincides with $MIN_w(u, v)$. So, if $u, v \in \mathcal{A}_1$, $MIN(u, v)$ is a discrete fuzzy number and $MIN(u, v) \in \mathcal{A}_1$.*

We have seen, in the previous propositions, that the operations $MAX_w(u, v)$ and $MIN_w(u, v)$ are discrete fuzzy numbers.

Example 3.8 *If we use the method, as explained above, to calculate the minimum of $u = \{0.3/1, 0.4/3, 1/4\}$ and $v = \{0.5/2, 1/5, 1/6\}$, we obtain the following discrete fuzzy number $MIN_w(u, v) = \{0.3/1, 0.5/2, 0.5/3, 1/4\}$ where*

$$B^{0.3} = \{z \in \{1, 2, 3, 4\} \text{ such that } 1 \leq z \leq 4\} = \{1, 2, 3, 4\}$$

$$B^{0.4} = \{z \in \{1, 2, 3, 4\} \text{ such that } 2 \leq z \leq 4\} = \{2, 3, 4\}$$

$$B^{0.5} = \{z \in \{1, 2, 3, 4\} \text{ such that } 2 \leq z \leq 4\} = \{2, 3, 4\}$$

$$B^{1} = \{z \in \{1, 2, 3, 4\} \text{ such that } 4 \leq z \leq 4\} = \{4\}$$

Proposition 3.9 *Let A, B and C be three dfn such that their supports are $supp(A)$,$supp(B)$ and $supp(C)$ respectively. The following properties hold:*

1. *Commutativity*

$$supp(A) \bigwedge supp(B) = supp(B) \bigwedge supp(A)$$

and

$$supp(A) \bigvee supp(B) = supp(B) \bigvee supp(A)$$

2. *Associativity*

$$(supp(A) \bigwedge supp(B)) \bigwedge supp(C) =$$
$$= supp(A) \bigwedge (supp(B) \bigwedge supp(C))$$

and

$$(supp(A) \bigvee supp(B)) \bigvee supp(C) =$$
$$= supp(A) \bigvee (supp(B) \bigvee supp(C))$$

3. *Idempotence*

$$supp(A) \bigwedge supp(A) = supp(A)$$

and

$$supp(A) \bigvee supp(A) = supp(A)$$

4. *If the supports of A and B are the same or $supp(B) \subseteq supp(A)$ or $A, B \in \mathcal{A}_1$ then the next absorption laws*

$$supp(A) \bigwedge (supp(A) \bigvee supp(B)) = supp(A)$$

$$supp(A) \bigvee (supp(A) \bigwedge supp(B)) = supp(A)$$

hold.

5. *If the supports of A, B and C are the same or $A, B, C \in \mathcal{A}_1$ then the following distributive properties*

$$supp(A) \bigwedge (supp(B) \bigvee supp(C)) =$$

$$= (supp(A) \bigwedge supp(B)) \bigvee (supp(A) \bigwedge supp(C))$$

and

$$supp(A) \bigvee (supp(B) \bigwedge supp(C)) =$$

$$= (supp(A) \bigvee supp(B)) \bigwedge (supp(A) \bigvee supp(C))$$

hold.

6. *For any $A, B \in DFN$ then*

$$supp(A) \bigwedge (supp(B) \bigvee supp(C))) \subseteq$$

$$\subseteq (supp(A) \bigwedge supp(B)) \bigvee (supp(A) \bigwedge supp(C))$$

and

$$supp(A) \bigvee (supp(B) \bigwedge supp(C))) \subseteq$$

$$\subseteq (supp(A) \bigvee supp(B)) \bigwedge (supp(A) \bigvee supp(C))$$

7. *For any $A, B \in DFN$ then*

$$supp(A) \subseteq supp(A) \bigwedge (supp(A) \bigvee supp(B))$$

$$supp(A) \subseteq supp(A) \bigvee (supp(A) \bigwedge supp(B))$$

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Proof From the definition 2.2, the support of a discrete fuzzy number is a finite linearly ordered subset of real numbers. Moreover, it is well known that the set of real numbers is a distributive lattice with the usual operations maximum(max) and minimum(min). Let X, Y and Z denote the supports of $A, B, C \in DFN$ respectively.

1. Using the commutative property of the real functions maximum and minimum the proof is trivial.

2. Associativity

   If $z \in (X \bigwedge Y) \bigwedge Z$ then $z = \min(x, c)$ where $x = \min(a, b)$, $a \in X$, $b \in Y$ and $c \in Z$. So $z = \min(\min(a, b), c)$. Using the associativity of the function minimum, $z = \min(\min(a, b), c) = \min(a, \min(b, c))$. Then $z \in X \bigwedge (Y \bigwedge Z)$. Therefore

   $$(X \bigwedge Y) \bigwedge Z \subseteq X \bigwedge (Y \bigwedge Z)$$

   If $z \in X \bigwedge (Y \bigwedge Z)$ then $z = \min(a, x)$ where $x = \min(b, c)$, $a \in X$, $b \in Y$ and $c \in Z$. So $z = \min(a, \min(b, c))$. Using the associativity of the function minimum, $z = \min(a, \min(b, c)) = \min(\min(a, b), c)$. Then $z \in (X \bigwedge Y) \bigwedge Z$. Therefore

   $$X \bigwedge (Y \bigwedge Z) \subseteq (X \bigwedge Y) \bigwedge Z$$

   The proof of the other associative law is analogous.

3. Idempotence

   If $z \in X \bigwedge X$ then $z = \min(a, a')$ with $a, a' \in X$. Therefore $z = a \in X$ or $z = a' \in X$. This means that $z \in X$ and so $X \bigwedge X \subseteq X$. On the other hand, it is evident that $X \subseteq X \bigwedge X$ because the function minimum is idempotent and then for each $z \in X$, $z = \min(z, z)$.

   The proof of the other idempotence law is similar.

4. Absorption

   If $z \in X \bigwedge (X \bigvee Y)$ then $z = \min(a, \max(a', b))$ where $a, a' \in X$ and $b \in Y$. Then if $z = a$ or $z = a'$ obviously $z \in X$. But if $z = b$ we have that $a' \leq b \leq a$. Using the hypothesis of the proposition we obtain that $b \in X$. Therefore $X \bigwedge (X \bigvee Y) \subseteq X$.

   If $z \in X$ then $z = \min(z, \max(z, b))$ for all $b \in Y$. Because if $\max(z, b) = b$ then $\min(z, b) = z$. And, if $\max(z, b) = z$ then $\min(z, z) = z$. So, $X \subseteq X \bigwedge (X \bigvee Y)$

5. Distributivity

   Let's $A, B, C \in \mathcal{A}_1$ where $supp(A) = \{x_1, \cdots, x_n\}$,$supp(B) = \{y_1, \cdots, y_m\}$ and $supp(C) = \{z_1, \cdots, z_k\}$ respectively. Then

   $$supp(A) \bigwedge (supp(B) \bigvee supp(C))$$

   is the set of natural numbers z such that they belong to the set

   $$\{x_1, \cdots, x_n\} \bigwedge \{\max(y_1, z_1), \cdots, \max(y_m, z_k)\}$$

   , i.e., it's the set of natural numbers z such that $z \in \{\min(x_1, \max(y_1, z_1)), \cdots, \min(x_n, \max(y_m, z_k))\} =$ (using the distributive property of natural numbers with the usual order)

   $$= \{z \in \{\max(\min(x_1, y_1), \min(x_1, z_1)), \cdots$$
   $$\cdots, \max(\min(x_n, y_m), \min(x_n, z_k))\}\}$$

   Where the previous set is an interval of consecutive natural numbers. On the other hand,

   $$(supp(A) \bigwedge supp(B)) \bigvee (supp(A) \bigwedge supp(C)) =$$
   $$= \{z \in \{\{\min(x_1, y_1), \cdots, \min(x_n, y_m)\} \bigvee$$
   $$\{\min(x_1, z_1), \cdots, \min(x_n, z_k)\}\} =$$
   $$= \{z \in \{\max(\min(x_1, y_1), \min(x_1, z_1)), \cdots$$
   $$\cdots, \max(\min(x_n, y_m), \min(x_n, z_k))\}.$$

   The proof of the second distributive law is analogous.

6. If $z \in X \bigwedge (Y \bigvee Z))$ then $z = \min(a, \max(b, c))$ where $a \in X$, $b \in Y$ and $c \in Z$. Using the distributive property of the minimum and the maximum of real numbers, $z = \max(\min(a, b), \min(a, c))$. Then we obtain that $z \in (X \bigwedge Y) \bigvee (X \bigwedge Z)$ and so $X \bigwedge (Y \bigvee Z)) \subseteq (X \bigwedge Y) \bigvee (X \bigwedge Z)$.

   Analogously for the other inclusion.

7. If $z \in X$ then $z = \max(z, \min(z, b))$ for all $b \in Y$. Because if $\min(z, b) = b$ then $\max(z, b) = z$. And, if $\min(z, b) = z$ then $\max(z, z) = z$. So, $X \subseteq X \bigvee (X \bigwedge Y)$

Remark 3.10 *In general, the absorption and the distributive laws of the supports do not hold. For example, if* $X = \{4, 7, 9\}$,$Y = \{4, 6, 7\}$ *and* $Z = \{8, 9, 10\}$ *represent the supports of the discrete fuzzy numbers* A, B *and* C *respectively, then*

a) $X \bigwedge (X \bigvee Y) = \{4, 6, 7, 9\}$ *but* $X = \{4, 7, 9\}$.

b) $X \bigwedge (Y \bigvee Z) = \{4, 7, 8, 9\}$, *but* $(X \bigwedge Y) \bigvee (X \bigwedge Z) = \{4, 6, 7, 8, 9\}$.

The following theorem establishes some properties of the binary operations MAX_w and MIN_w using the same conditions and results that the previous proposition 3.9:

Theorem 3.11 *Let* MAX_w *and* MIN_w *be the binary operations on* DFN *defined by the propositions 3.2 and 3.6, respectively. Then, for any,* $A, B, C \in DFN$ *the following properties hold:*

1. Commutativity: $MIN_w(A, B) = MIN_w(B, A)$
$MAX_w(A, B) = MAX_w(B, A)$

2. Associativity:

$$MIN_w(MIN_w(A, B), C) =$$
$$= MIN_w(A, MIN_w(B, C))$$

and

$$MAX_w(MAX_w(A, B), C) =$$
$$= MAX_w(A, MAX_w(B, C))$$

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3. Idempotence: $MIN_w(A,A)=A$
   $MAX_w(A,A)=A$

If the supports of A and B are the same or $A,B,C\in\mathcal{A}_1$ *then*

4. Absorption: $MIN_w(A,MAX_w(A,B))=A$
   $MAX_w(A,MIN_w(A,B))=A$

5. Distributivity:

$$MIN_w(A,MAX_w(B,C))=$$
$$=MAX_w(MIN_w(A,B),MIN_w(A,C))$$

And

$$MAX_w(A,MIN_w(B,C))=$$
$$=MIN_w(MAX_w(A,B),MAX_w(A,C))$$

Proof Let A, B and C be three dfn. Let's consider the α-cut sets: $A^\alpha=\{x_1^\alpha,...,x_p^\alpha\}$, $B^\alpha=\{y_1^\alpha,...,y_k^\alpha\}$, $C^\alpha=\{w_1^\alpha,...,w_l^\alpha\}$ for A, B and C respectively.

1. We want to show that $MIN_w(A,B)=MIN_w(B,A)$

It is enough to prove that the discrete fuzzy numbers $MIN_w(A,B)$ and $MIN_w(B,A)$ are the same α-cut sets for each $\alpha\in[0,1]$.

By definition 3.4

$$MIN_w(A,B)^\alpha=$$
$$=\{z\in supp(A)\bigwedge supp(B)\text{ such that}$$
$$(\min A^\alpha\wedge\min B^\alpha)\leq z\leq(\max A^\alpha\wedge\max B^\alpha)\}=$$
$$=\{z\in supp(A)\bigwedge supp(B)|(x_1^\alpha\wedge y_1^\alpha)\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\}=$$
$$=\{z\in supp(A)\bigwedge supp(B)|(y_1^\alpha\wedge x_1^\alpha)\leq z\leq(y_p^\alpha\wedge x_k^\alpha)\}=$$
$$=MIN_w(B,A)^\alpha$$

Analogously for the other commutative law.

2. We want to see that

$$MIN_w(MIN_w(A,B),C)=MIN_w(A,MIN_w(B,C))$$

By definition 3.4

$$MIN_w(MIN_w(A,B),C)^\alpha=$$
$$=\{z\in supp(MIN_w(A,B))\bigwedge supp(C)\text{ such that}$$
$$\min MIN_w(A,B)^\alpha\wedge\min C^\alpha\leq z\leq\max MIN_w(A,B)^\alpha\wedge\max C^\alpha\}=$$
$$=\{z\in supp(MIN_w(A,B))\bigwedge supp(C)\text{ such that}$$
$$(x_1^\alpha\wedge y_1^\alpha)\wedge w_1^\alpha\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\wedge w_l^\alpha\}=$$
$$=\{z\in(supp(A)\bigwedge supp(B))\bigwedge supp(C)\text{ such that}$$
$$(x_1^\alpha\wedge y_1^\alpha)\wedge w_1^\alpha\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\wedge w_l^\alpha\}=$$
$$=\{z\in(supp(A)\bigwedge supp(B))\bigwedge supp(C)\text{ such that}$$
$$x_1^\alpha\wedge(y_1^\alpha\wedge w_1^\alpha)\leq z\leq x_p^\alpha\wedge(y_k^\alpha\wedge w_l^\alpha\})=$$
$$=\{z\in supp(A)\bigwedge(supp(B)\bigwedge supp(C))\text{ such that}$$
$$x_1^\alpha\wedge(y_1^\alpha\wedge w_1^\alpha)\leq z\leq x_p^\alpha\wedge(y_k^\alpha\wedge w_l^\alpha\})=$$
$$=MIN_w(A,MIN_w(B,C))^\alpha$$

The proof of the other associative law is similar.

3. We want to demonstrate that

$$MIN_w(A,A)=A$$

By definition 3.4

$$MIN_w(A,A)^\alpha=$$
$$=\{z\in supp(A)\bigwedge supp(A)\text{ such that}$$
$$(\min A^\alpha\wedge\min A^\alpha)\leq z\leq(\max A^\alpha\wedge\max A^\alpha)\}=$$
$$=\{z\in supp(A)|(x_1^\alpha\wedge x_1^\alpha)\leq z\leq(x_p^\alpha\wedge x_p^\alpha)\}=$$
$$=\{z\in supp(A)|x_1^\alpha\leq z\leq x_p^\alpha\}=$$
$$=A^\alpha$$

It is the same for the other idempotence law.

4. We want to demonstrate that

$$MIN_w(A,MAX_w(A,B))=A$$

By definition 3.4

$$MIN_w(A,MAX_w(A,B))^\alpha=$$
$$=\{z\in supp(A)\bigwedge supp(MAX_w(A,B))\text{ such that}$$
$$\min A^\alpha\wedge\min MAX_w(A,B)^\alpha\leq z\leq\max A^\alpha\wedge\max MAX_w(A,B)^\alpha\}=$$
$$=\{z\in supp(A)\bigwedge supp(MAX_w(A,B))\text{ such that}$$
$$x_1^\alpha\wedge(x_1^\alpha\vee y_1^\alpha)\leq z\leq x_p^\alpha\wedge(x_p^\alpha\vee y_k^\alpha\})=$$

(using the absorption law of real numbers)

$$=\{z\in supp(A)\bigwedge supp(MAX_w(A,B))\text{ such that}$$
$$x_1^\alpha\leq z\leq x_p^\alpha\}=\{z\in supp(A)\text{ such that }x_1^\alpha\leq z\leq x_p^\alpha\}=$$

$=A^\alpha$. (The previous equality is only possible if we use the same hypotheses and results of the proposition 3.9)

5. We want to show the first distributive law, ie,

$$MIN_w(A,MAX_w(B,C))=MAX_w(MIN_w(A,B),MIN_w(A,C))$$

By definition 3.4

$$MIN_w(A,MAX_w(B,C))^\alpha=$$
$$=\{z\in supp(A)\bigwedge supp(MAX_w(B,C))\text{ such that}$$
$$\min A^\alpha\wedge\min MAX_w(B,C)^\alpha\leq z\leq\max A^\alpha\wedge\max MAX_w(A,B)^\alpha\}=$$
$$=\{z\in supp(A)\bigwedge supp(MAX_w(B,C))\text{ such that}$$
$$x_1\wedge(y_1^\alpha\vee w_1^\alpha)\leq z\leq x_p^\alpha\wedge(y_k^\alpha\vee w_l^\alpha)\}=$$

(using the distributive law of real numbers)

$$=\{z\in supp(A)\bigwedge supp(MAX_w(B,C))\text{ such that}$$
$$(x_1^\alpha\wedge y_1^\alpha)\vee((x_1^\alpha\wedge w_1^\alpha)\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\vee(x_p^\alpha\wedge w_l^\alpha)\}=$$
$$=\{z\in supp(A)\bigwedge(supp(B)\bigvee supp(C))\text{ such that}$$
$$(x_1^\alpha\wedge y_1^\alpha)\vee((x_1^\alpha\wedge w_1^\alpha)\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\vee(x_p^\alpha\wedge w_l^\alpha)\}=$$
$$=\{z\in(supp(A)\bigwedge supp(B))\bigvee(supp(A)\bigwedge supp(C))$$
$$\text{such that }(x_1^\alpha\wedge y_1^\alpha)\vee((x_1^\alpha\wedge w_1^\alpha)\leq z\leq(x_p^\alpha\wedge y_k^\alpha)\vee(x_p^\alpha\wedge w_l^\alpha)\}=$$
$$=MAX_w(MIN_w(A,B),MIN_w(A,C))^\alpha$$

Analogously for the other distributive law. □

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Corollary 3.12 *The triplet ($\mathcal{A}_1$,MIN_w,MAX_w) is a distributive lattice, in which MIN_w and MAX_w represent the meet and join, respectively.*

Remark 3.13 *The lattice ($\mathcal{A}_1$,MIN_w,MAX_w) can also be expressed as the pair ($\mathcal{A}_1$,$\preceq$), where $\preceq$ is a partial ordering defined as:*
$A \preceq B$ if and only if $MIN_w(A, B) = A$
or, alternatively,
$A \preceq B$ if and only if $MAX_w(A, B) = B$
for any $A, B \in \mathcal{A}_1$.

Remark 3.14 *We can also define the partial ordering in terms of α-cuts:*
$A \preceq B$ if and only if $\min(A^\alpha, B^\alpha) = A^\alpha$
$A \preceq B$ if and only if $\max(A^\alpha, B^\alpha) = B^\alpha$
for any $A, B \in \mathcal{A}_1$ and $\alpha \in (0, 1]$, where A^α and B^α are finite sets (say , $A^\alpha = \{x_1^\alpha, \cdots, x_p^\alpha\}$,$B^\alpha = \{y_1^\alpha, \cdots, y_k^\alpha\}$). Then,

$$\min(A^\alpha, B^\alpha) = \{z \in supp(A) \bigwedge supp(B) \textit{ such that}$$

$$\min(x_1^\alpha, y_1^\alpha) \leq z \leq \min(x_p^\alpha, y_k^\alpha)\} = A^\alpha \textit{ and}$$

$$\max(A^\alpha, B^\alpha) = \{z \in supp(A) \bigvee supp(B) \textit{ such that}$$

$$\max(x_1^\alpha, y_1^\alpha) \leq z \leq \max(x_p^\alpha, y_k^\alpha)\} = B^\alpha$$

Example 3.15
Let $u = \{0.4/1, 1/2, 0.8/3, 0.6/4, 0.5/5, 0.4/6, 0.3/7\}$ and $v = \{0.3/4, 0.6/5, 0.7/6, 0.8/7, 1/8, 0.8/9\}$ be two discrete fuzzy numbers. It is easy to prove that $u \preceq v$ because $MAX_w(u, v) = v$ or equivalently $MIN_w(u, v) = u$.

Corollary 3.16 *For each finite subset X of real numbers, let's consider the subset F_X of DFN such that any element of F_X has as support X. Then, the triplet (F_X,MIN_w,MAX_w) is a distributive lattice, in which MIN_w and MAX_w represent the meet and join, respectively.*

Example 3.17 *Let $u = \{0.3/3, 0.4/5, 1/7, 0.7/8\}$ and $v = \{0.3/3, 1/5, 0.7/7, 0.6/8\}$ be two discrete fuzzy numbers. It is easy to prove that $v \preceq u$ because $MAX_w(u, v) = u$ or equivalently $MIN_w(u, v) = v$.*

Remark 3.18 *It is possible to find two discrete fuzzy numbers $A, B \in \mathcal{A}_1$ such that $MAX_w(A, B) \neq A$ and $MAX_w(A, B) \neq B$ or equivalenty $MIN_w(A, B) \neq A$ and $MIN_w(A, B) \neq B$. This means that the two discrete fuzzy numbers, A and B, are not comparable. For example, let's consider $A = \{0.3/4, 0.5/5, 0.8/6, 1/7, 0.9/8, 0.7/9\}$, $B = \{0.5/6, 1/7, 0.9/8, 0.6/9, 0.5/10\} \in \mathcal{A}_1$. Then, $MIN_w(A, B) = \{0.3/4, 0.5/5, 0.8/6, 1/7, 0.9/8, 0.6/9\} \neq A, B$.*

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