On Some Classes of Aggregation Functions that are Migrative

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Abstract— *In this paper we introduce and describe two classes of aggregation functions with members that are migrative. First continuous triangular norms are studied that own the migrative property with respect to another fixed t-norm, in particular to the three prototypes minimum, product, and the Łukasiewicz t-norm. For classes of nilpotent and strict migrative t-norms the characterization and construction is carried out by solving functional equations for the generators. For the third case when the fixed t-norm is the minimum, an ordinal-sum-like construction is resulted. Then we formulate and study the migrative property for quasi-arithmetic means. The results are less permissive than for t-norms: we can hardly escape from the classical arithmetic mean.*

Keywords— Triangular norms, quasi-arithmetic means, functional equations, migrative property.

1 Introduction

In this paper we study the migrative property for some classes of binary *aggregation functions*. These are increasing functions $F\colon [a,b]\times[a,b]\to[a,b]$ such that $F(a,a)=a$ and $F(b,b)=b$.

It is well-known that some operations in fuzzy logic (like t-norms, t-conorms, uninorms, nullnorms) can be considered aggregation functions on the closed unit interval in the above sense. In addition, these operations own a characteristic property of associativity that can be formulated as a well-known functional equation. A function $F\colon [a,b]\times[a,b]\to[a,b]$ is called *associative* if

$$F(x,F(y,z))=F(F(x,y),z),\quad x,y,z\in[a,b]. \quad (1)$$

Under some additional conditions, the above-mentioned logical operations are the general solutions of the associativity equation (1).

Another important class of aggregation functions consists of *means*. One of the first authors who introduced the term "mean" in a mathematical sense was Cauchy [1]. In that spirit, a function $F\colon [a,b]\times[a,b]\to[a,b]$ is called a *mean* in the interval $[a,b]\subset\mathbb{R}$ if

$$\min(x,y)\le F(x,y)\le\max(x,y)$$

for all $x,y\in[a,b]$. Well-known examples are the median, the arithmetic mean, the geometric mean, the root-power mean, the harmonic mean and the like.

Although these means and t-norms, t-conorms, uninorms, nullnorms have rather different properties, there is one particular common property they all share. This is called *bisymmetry*, introduced by Aczél [2], and is defined for a function $M\colon [a,b]\times[a,b]\to[a,b]$ by

$$F(F(x,y),F(u,v))=F(F(x,u),F(y,v)), \quad (2)$$

for all $x,y,u,v\in[a,b]$.

It is easy to see that if a function $F\colon [a,b]\times[a,b]\to[a,b]$ is symmetric (i.e., $F(x,y)=F(y,x)$ for all $x,y\in[a,b]$) and associative then F is bisymmetric. The converse statement is not true in general. If, however, F is symmetric and has a neutral element, then bisymmetry of F implies that F is associative [3].

We give the definition of α-migrative two-variable functions in Definition 1 just as it was proposed originally in [5]. Since that notion has something to do, in essence, with triangular norms, first we attempt to catch hold of the very meaning of it for t-norms. This is expressed by equation (6), and two more equivalent forms in Theorem 2. That is, for t-norms we can choose any of the three forms of (8), (9) or (10). Equation (8) is obtained by considering the classical associativity functional equation, then fixing the t-norm (T_0) inside the equation, and finally fixing one of the variables' value at α. Equation (9) can be obtained similarly: the only difference is that we fixed the t-norm outside the equation. Equation (10) is just equivalent to (8) and (9) for t-norms.

Therefore, when we realized that Definition 1 can never hold for t-conorms, and it is very restrictive for quasi-arithmetic means, we tried to develop new definitions that are context dependent. The idea is to use the characteristic composite functional equation of the function class under study (e.g., the bisymmetry equation for means), then fix the function inside (or outside), and finally fix one of the variables at some α. Or alternatively, consider the equation that corresponds to (10). While for t-norms these three approaches are equivalent, for quasi-arithmetic means this is not the case at all.

According to our sketched ideas and investigations, in this paper we introduce and describe two classes of aggregation functions (triangular norms and quasi-arithmetic means) with members that are migrative. The underlying compact interval is $[0,1]$ in the study. In our formulation the migrative property is context-dependent. First we show the original property and its extensions for triangular norms. Then migrativity is reformulated and studied for quasi-arithmetic means. In the conclusions we highlight the essential differences between the two classes.

2 The migrative property for t-norms

In this section we highlight some essential aspects and consequences of the original α-migrative property and its possible extensions.

A *triangular norm* (t-norm for short) $T:[0,1]^2\to[0,1]$ is an associative, commutative, non-decreasing function such that $T(1,x)=x$ for all $x\in[0,1]$. Prototypes of t-norms are

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the minimum $T_{\mathbf{M}}(x,y) = \min(x,y)$, the product $T_{\mathbf{P}}(x,y) = xy$, and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. Obviously, the product t-norm $T_{\mathbf{P}}$ is α-migrative for any $\alpha \in]0,1[$.

As it is well-known, each continuous Archimedean t-norm T can be represented by means of a continuous additive generator (see e.g. [4]), i.e., a strictly decreasing continuous function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)), \tag{3}$$

where $t^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of t, and is given by

$$t^{(-1)}(u) = t^{-1}(\min(u, t(0))).$$

2.1 The Class of α-Migrative Functions

In [5] the authors introduced a new term – α-migrative – for a class of two-variable functions (binary operations) as follows.

Definition 1. Let α be in $]0,1[$. A binary operation $T\colon [0,1]^2 \to [0,1]$ is said to be *α-migrative* if we have

$$T(\alpha x, y) = T(x, \alpha y) \quad \text{for all } x,y \in [0,1]. \tag{4}$$

Although this definition seems to be rather general, the paper [5] deals with triangular norms and subnorms. Further progress can be seen in [6], where the authors studied a slightly modified definition of migrativity. They discussed also the associativity and bisymmetry of migrative aggregation functions.

Notice also that Definition 1 does not provide a meaningful notion for triangular conorms. Indeed, if S is a t-conorm then it is α-migrative if and only if $S(\alpha x, y) = S(x, \alpha y)$ holds for all $x, y \in [0,1]$. If we choose $y = 0$ then we must have $\alpha x = x$ for all $x \in [0,1]$, because S is α-migrative. This is impossible when $\alpha \neq 1$. Similarly, if $y = 1$ then we must have $S(x,\alpha) = 1$ for all $x \in [0,1]$, which is again impossible unless $\alpha = 1$. Therefore, even the correct definition of α-migrative t-conorms needs special care.

Obviously, the product t-norm $T_{\mathbf{P}}$ is α-migrative for any $\alpha \in]0,1[$. The following non-continuous t-norm T_β is also α-migrative:

$$T_\beta(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \beta xy & \text{otherwise.} \end{cases} \tag{5}$$

It was provided in [8] as a counterexample to the question: Is a strictly monotone t-norm always continuous?

We provide all *continuous* solutions of the equation (4) in Theorem 1. More details and proofs can be found in our paper [7].

Theorem 1. *Let α be in $]0,1[$. Suppose T is a continuous t-norm. Then the following statements hold.*

(a) T is α-migrative if and only if we have

$$T(\alpha, y) = \alpha y \quad \textit{for all } y \in [0,1]. \tag{6}$$

(b) If T is α-migrative then T is strict.

(c) Suppose t is an additive generator of T. Then T is α-migrative if and only if there exists a continuous, strictly decreasing function t_0 from $[\alpha,1]$ to the non-negative reals with $t_0(0) < +\infty$ and $t_0(1) = 0$ such that

$$t(x) = k \cdot t_0(\alpha) + t_0\left(\frac{x}{\alpha^k}\right) \quad \textit{if } x \in]\alpha^{k+1}, \alpha^k], \tag{7}$$

where k is any non-negative integer.

In the next subsection we extend the above definition by allowing any fixed t-norm instead of the product in the defining equation.

2.2 The Class of (α, T_0)-Migrative Functions

Following our plan, the next definition extends the migrative property as follows. Some more details can be found in [9] and in [10] with proofs.

Definition 2. Let α be in $]0,1[$ and T_0 a fixed triangular norm. A binary operation $T\colon [0,1]^2 \to [0,1]$ is said to be *α-migrative with respect to T_0* (shortly: (α, T_0)-migrative) if we have

$$T(T_0(\alpha,x), y) = T(x, T_0(\alpha, y)) \tag{8}$$

for all $x, y \in [0,1]$.

Obviously, Definition 1 is a particular case of Definition 2 with $T_0 = T_{\mathbf{P}}$.

In the next theorem we give a simple but very useful characterization of (α, T_0)-migrative t-norms.

Theorem 2. *Let α be in $]0,1[$ and T_0 a fixed triangular norm. Then the following statements are equivalent for a t-norm $T\colon [0,1]^2 \to [0,1]$.*

(i) T is α-migrative with respect to T_0;

(ii) T satisfies the following equation for all $x, y \in [0,1]$:

$$T_0(T(\alpha,x), y) = T_0(x, T(\alpha, y)); \tag{9}$$

(iii) T satisfies the following equation for all $x \in [0,1]$:

$$T(\alpha, x) = T_0(\alpha, x). \tag{10}$$

We do not recall the case when $T_0 = T_{\mathbf{L}}$. The interested reader can find the related results in [9, 10].

For illustrating the non-representable case $T_0 = T_{\mathbf{M}}$, we consider $(\alpha, T_{\mathbf{M}})$-migrative continuous triangular norms. In the present case the $(\alpha, T_{\mathbf{M}})$-migrative property is read as follows:

$$T(\min(\alpha,x), y) = T(x, \min(\alpha,y)),\ x,y \in [0,1]. \tag{11}$$

By Theorem 2 we know that the $(\alpha, T_{\mathbf{M}})$-migrative property is equivalent to the following functional equation:

$$T(\alpha, y) = \min(\alpha, y), \quad y \in [0,1]. \tag{12}$$

The description of all $(\alpha, T_{\mathbf{M}})$-migrative continuous triangular norms, as solutions to the equation (12), is given in the following theorem. For the proof see [10].

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Theorem 3. *A continuous t-norm T is $(\alpha, T_{\mathbf{M}})$-migrative if and only if there exist two continuous t-norms T_1 and T_2 such that T can be written in the following form:*

$$T(x,y) = \begin{cases} \alpha T_1\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) & x, y \in [0,\alpha], \\ \alpha + (1-\alpha) T_2\left(\frac{x-\alpha}{1-\alpha}, \frac{y-\alpha}{1-\alpha}\right) & x, y \in [\alpha, 1], \\ \min(x,y) & \textit{otherwise}. \end{cases}$$

Notice that equations (8) and (10) has something to do with the *generalized associativity equation* that has also been studied and solved under some additional conditions, see [11, 12]. It can be written as follows:

$$F(G(x,y),z) = H(x, K(y,z)). \tag{13}$$

In this general framework the particular form of $H = F$ and $K = G$ in (13) correspond to (8).

3 The class of α-migrative quasi-arithmetic means

Let $[a,b]$ be a compact real interval. Aczél [2] proved that a function $M : [a,b]^2 \to [a,b]$ is continuous, symmetric, strictly increasing in each argument, idempotent and fulfils the bisymmetry equation (2) if and only if

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \quad x, y \in [a,b] \tag{14}$$

with some continuous strictly monotonic function f. We also know that this result still holds for intervals of the form $]a,b]$, $[a,b[$, $]a,b]$ or even for any unbounded interval of the real line (see [13], pp 250–251, 280). A function M having a representation (14) is called a *quasi-arithmetic mean*.

When one tries to extend migrativity for quasi-arithmetic means, there are several options. One possibility is to choose a particular quasi-arithmetic mean M_0 (for instance, the arithmetic mean $M_{\mathbf{A}}(x,y) = \frac{x+y}{2}$), and consider the bisymmetry equation (2) extended in the following forms, with a fixed argument (say $y = \alpha$):

$$M(M_0(x,\alpha), M_0(u,v)) = M(M_0(x,u), M_0(\alpha,v)), \tag{15}$$
$$M_0(M(x,\alpha), M(u,v)) = M_0(M(x,u), M(\alpha,v)). \tag{16}$$

For motivations of these equations we refer to Theorem 2 and formulas (8)–(10) in case of t-norms.

We indeed chose the arithmetic mean, because of its key role in the representation of quasi-arithmetic means (14). In this case equations (15) and (16) are of the following respective forms:

$$M\left(\frac{x+\alpha}{2}, \frac{u+v}{2}\right) = M\left(\frac{x+u}{2}, \frac{\alpha+v}{2}\right), \tag{17}$$

$$\frac{M(x,\alpha)+M(u,v)}{2} = \frac{M(x,u)+M(\alpha,v)}{2}, \tag{18}$$

for a fixed $\alpha \in]0,1[$ and for all $x, u, v \in [0,1]$.

First we state that equation (18) is extremely restrictive: under the condition of idempotency, M must be the arithmetic mean.

Theorem 4. *Assume that an idempotent function $M\colon [0,1]^2 \to [0,1]$ satisfies equation (18). Then M is equal to the arithmetic mean.*

Therefore, if we want to define α-migrativity in a way that allows not only the arithmetic mean owning this property, we should try the other equation (17).

Lemma 1. *Let α be in $]0,1[$. Then*

$$M(\alpha, x) = \frac{\alpha+x}{2} \textit{ for all } x \in [0,1] \tag{19}$$

is a necessary condition for having (17) satisfied by an idempotent M.

This simple observation suggests that an appropriate definition of α-migrative quasi-arithmetic means might be the following one.

Definition 3. Let α be in $]0,1[$. A quasi-arithmetic mean $M\colon [0,1]^2 \to [0,1]$ is said to be *α-migrative* if (19) holds for all $x \in [0,1]$.

We solve equation (19) first, than we relate the solution to (17).

3.1 Quasi-Arithmetic Means that Satisfy (19)

Consider a quasi-arithmetic mean M in the form of (14) with a strictly increasing, continuous generator function f defined on $[0,1]$. Suppose M satisfies (19). That is, we have

$$f^{-1}\left(\frac{f(\alpha)+f(x)}{2}\right) = \frac{\alpha+x}{2} \quad \text{for all } x \in [0,1]. \tag{20}$$

Equation (20) implies that $f(0) = \lim_{x\downarrow 0} f(x)$ and $f(1) = \lim_{x\uparrow 1} f(x)$ are finite. Indeed, for instance, if $f(0) = -\infty$ then the left-hand side of (20) equals 0, while the right-hand side is $\frac{\alpha}{2}$, a contradiction. The other cases lead to a contradiction in a similar manner.

Therefore, (20) can be written in an equivalent form as

$$\frac{f(\alpha)+f(x)}{2} = f\left(\frac{\alpha+x}{2}\right) \quad \text{for all } x \in [0,1]. \tag{21}$$

The general solution of (21) is given in the next theorem.

Theorem 5. *Assume that $f : [0,1] \to \mathbb{R}$ is a strictly increasing, continuous function. Then f satisfies (21) if and only if there exist real numbers f_0, f_α, f_1 such that f can be written in the following form:*

$$f(x) = \begin{cases} \frac{f_\alpha - f_0}{\alpha} \cdot x + f_0 & \textit{if } x \in [0,\alpha] \\ \frac{f_1 - f_\alpha}{1-\alpha} \cdot x + \frac{f_\alpha - \alpha f_1}{1-\alpha} & \textit{if } x \in]\alpha, 1] \end{cases}. \tag{22}$$

That is, the most general quasi-arithmetic means that satisfy (19) are generated by a piece-wise linear generator function. It is linear on $[0,\alpha]$ and on $[\alpha, 1]$.

Next we show that the piecewise linear solution of (21) becomes linear when we feed it back into (17).

Theorem 6. *Let M be a quasi-arithmetic mean that satisfies (17). Then $M(x,y) = \frac{x+y}{2}$, i.e., M equals the arithmetic mean.*

Notice that this result is somewhat disappointing: we returned back again to quasi-arithmetic means.

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4 Conclusions

It is worth comparing results presented in this paper for t-norms and for quasi-arithmetic means. For t-norms a general form of (α, T_0)-migrativity resulted in two other equivalent formulations, and also infinite families of t-norms owning that property could be obtained. For quasi-arithmetic means idempotency alone implied the arithmetic mean as the only solution in (18). When we looked for quasi-arithmetic mean solutions of (17), it was also the arithmetic mean that satisfied the equation. The only exceptions are quasi-arithmetic mean solutions to (19) with piece-wise linear generator functions.

Notice that the general case when the fixed quasi-arithmetic mean $M_0(x, y) = f_0^{-1}\left(\frac{f_0(x) + f_0(y)}{2}\right)$ differs from the arithmetic mean can be handled in a similar way.

It might be worth studying other types of composite functional equations in the spirit of the present paper.

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