Approximation properties of inverse fuzzy transforms over a residuated lattice

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Abstract— *This contribution focuses on inverse fuzzy transforms (shortly inverse F-transforms) over residuated lattices introduced by I. Perfilieva and their approximation properties. We will try to reduce some requirements used in the original work to prove Approximation theorem. Moreover, we show in which sense F-transforms are the best approximations.*

Keywords— Fuzzy transform, Approximation, Extensional relation.

1 Introduction

In [1], F-transforms were introduced basically in three forms: the first (ordinary) was constructed over the ordinary algebra of reals[1], while the other two were constructed over the residuated lattice. Note that up to now, most of the applications of F-transforms, e.g. ordinary/partial differential equations [2]/[3], image compression/fusion (see [4]/[5, 6]), data analysis [7] etc., employ only the ordinary one. The main reason follows from the character of the inverse F-transforms, where the ordinary F-transform average the function values. Whereas the other two approximate them from above and from below, respectively. And since these F-transforms (over the residuated lattice) can be viewed as formalizations of a collection of *graded fuzzy rules* [8], they are primarily suitable for applications, which use fuzzy "IF-THEN" rules.

In this contribution, we will focus on the inverse F-transforms over the residuated lattice and show their approximation properties. Moreover, we will provide a simpler way to get the approximation theorem than the one in [1] (Theorem 10 and 11). Thus, we would like to brush up this topic and make it more attractive for a broader community.

Note that we present mainly an excerpt of a larger manuscript [9] that has been submitted for publication. For background information and proof details, readers are referred to this upcoming article.

2 Basic definitions and overview of the known results.

2.1 Basic definitions

Definition 1 *A residuated lattice on L is an algebra*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1}\rangle \qquad (1)$$

with four binary operations and two constants such that

- $\mathcal{L} = \langle L, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ *is a lattice with the largest element* $\mathbf{1}$ *and the least element* $\mathbf{0}$ *w.r.t. the lattice ordering* $\leq$,
- $\mathcal{L} = \langle L, *, \mathbf{1}\rangle$ *is a commutative semigroup with the unit element* $\mathbf{1}$*, i.e.* $*$ *is commutative, associative, and* $\mathbf{1} * x = x$ *for all* $x \in L$,
- $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.*

$$z \leq (x \rightarrow y) \quad \textit{iff} \quad x * z \leq y \quad \textit{for all } x, y, z \in L.$$

In the sequel, let us assume $\mathcal{L}$ be a residuated lattice of the form (1). Moreover, we define the biresidual operation (biresiduum)

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

Throughout the whole text, we will deal with fuzzy relations whose membership functions take values from the support of $\mathcal{L}$ and we denote this fact by $\subsetsim$. Let $M_1, \dots, M_n$ be some nonempty sets of objects, $\mathcal{R}_I = \{R_i \subsetsim M_i^2\}_{i \in I}$ be a system of binary fuzzy relations ordered by $I = \{1, \dots, n\}$, $f \subsetsim M_1 \times \dots \times M_n$ be an n-ary fuzzy relation.

Convention 2 *For the sake of brevity, we will denote $M_1 \times \cdots \times M_n$ by $M^{(n)}$. Moreover, we will write $R(\bar{x}, \bar{y}) = R_1(x_1, y_1) * \cdots * R_n(x_n, y_n)$.*

Definition 3 *We say that f is extensional w.r.t. $\mathcal{R}_I$ if*

$$R(\bar{x}, \bar{y}) * f(\bar{x}) \leq f(\bar{y}),$$

for each $\bar{x}, \bar{y} \in M^{(n)}$.

Let us recall that a fuzzy $*$-equivalence (also known as similarity or indistinguishability) $R \subsetsim M^2$ (M be some nonempty set of objects) is reflexive, symmetric and transitive w.r.t. $*$. Observe that $\leftrightarrow$ is $*$-equivalence on L. In the following we will omit specification $*$ whenever it will be clear from the context.

Additionally, we recall notions used in the original approximation theorems in [1].

Definition 4 *A system of fuzzy sets $\{A_1, \dots, A_k\}$ on a nonempty set M establishes a semi-partition if*

$$\bigvee_{x \in M} A_i(x) * A_j(x) \leq \bigwedge_{x \in M} A_i(x) \leftrightarrow A_j(x),$$

for each $i, j \in \{1, \dots, k\}$.

[1]Due to a nature of the used operations, it can be reinterpreted over ŁΠ-algebra.

ISBN: 978-989-95079-6-8

2.2 $F^{\uparrow(\downarrow)}$-transforms

Let us recall inverse F-transforms from [1] generalized for the n-dimensional case[2]:

$$\begin{aligned}&\text{Assume } f \underset{\sim}{\subseteq} M^{(n)}, C \subseteq M^{(n)}, A_{j_i} \underset{\sim}{\subseteq} M_i,\\ &\quad \forall j \in J = \{1,\dots,k\}, \forall i \in I = \{1,\dots,n\},\\ &\quad \text{and } \mathbf{j} \text{ be abbreviation for } [j_1,\dots,j_n],\\ &f^{\downarrow}_{F,k}(\bar{x}) = \bigvee_{\mathbf{j}\in J^n} A_{\mathbf{j}}(\bar{x}) * [\bigwedge_{\bar{c}\in C} A_{\mathbf{j}}(\bar{c}) \to f(\bar{c})],\\ &f^{\uparrow}_{F,k}(\bar{x}) = \bigwedge_{\mathbf{j}\in J^n} A_{\mathbf{j}}(\bar{x}) \to [\bigvee_{\bar{c}\in C} A_{\mathbf{j}}(\bar{c}) * f(\bar{c})], \text{ where}\\ &A_{\mathbf{j}}(\bar{x}) = A_{j_1}(x_1) * \dots * A_{j_n}(x_n), \forall \mathbf{j} \in J^n.\end{aligned}$$

In [1], $f^{\downarrow}_{F,k}$ is called inverse F$^{\downarrow}$-transform and $f^{\uparrow}_{F,k}$ inverse F$^{\uparrow}$-transform. Moreover, the expression in the square brackets above is denoted by $F^{\uparrow(\downarrow)}_{\mathbf{j}}$ in the case of $f^{\uparrow(\downarrow)}_{F,k}$ and known as $\mathbf{j}$-th component of F$^{\uparrow(\downarrow)}$-transform.

Below, we overview known results from [1] that are in the center of our interest:

1. From the inequality $a * (a \to b) \le b \le a \to (a * b)$, it follows that for an arbitrary f:

$$f^{\downarrow}_{F,k}(\bar{x}) \le f(\bar{x}) \le f^{\uparrow}_{F,k}(\bar{x}),\ \forall \bar{x} \in C. \tag{2}$$

2. The $\mathbf{j}$-th components of F$^{\uparrow}$-transform and F$^{\downarrow}$-transform are the least elements of the following respective sets:

$$S_{\mathbf{j}} = \{a \in M^{(n)} |\ A_{\mathbf{j}}(\bar{x}) \le f(\bar{x}) \to a,\ \forall \bar{x} \in C\}, \tag{3}$$
$$T_{\mathbf{j}} = \{a \in M^{(n)} |\ A_{\mathbf{j}}(\bar{x}) \le a \to f(\bar{x}),\ \forall \bar{x} \in C\}. \tag{4}$$

3. A core of the approximation theorems inheres in the following inequality:

$$\bigvee_{\mathbf{j}\in J^n} A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{x}) \le f^{\uparrow(\downarrow)}_{F,k}(\bar{x}) \leftrightarrow f(\bar{x}),\ \forall \bar{x} \in C. \tag{5}$$

Remark that a closeness of elements is expressed using $\leftrightarrow$, which is the dual concept to pseudo-metric as shown in [10], and naturally, the estimation goes there from below.

Below, we summarize the requirements that were used in [1] to prove (5), i.e., the estimation of the equivalence of $f^{\downarrow(\uparrow)}_{F,k}$ to the original function f:

	Requirements in [1] (Theorems 10 and 11)
1.	$\mathcal{L}$ be a BL-algebra,
2.	each $A \in \{A_{j_i}\}_{j\in J, i\in I}$ be normal,
3.	$\{A_{1_i},\dots,A_{k_i}\}$ forms a semi-partition $\forall i \in I$,
4.	f be extensional w.r.t. $\mathcal{P}_I = \{P_i\}_{i\in I}$ defined by $P_i(x,y) = \bigwedge_{j\in J} A_{j_i}(x) \leftrightarrow A_{j_i}(y)$, for all $i \in I$,
5.	$E = \{\bar{e}_j \in M^{(n)} |\ P_i(x, e_{j_i}) = A_{j_i}(x), i \in I, j \in J\}$ be subset of C (see p. 1013 in [1]).

[2] Note that in the original work, f was a mapping from a finite subset of L into L (support of the fixed residuated lattice $\mathcal{L}$), where $L = [0, 1]$. Since we measure a precision of the approximation using values of L and we do not use specific properties of $[0, 1]$, this determination is not needed in our work.

3 Approximation properties

3.0.1 Formulation of the problem and basic properties

Let $\mathcal{L}$ and $\mathcal{R}_I = \{R_i \underset{\sim}{\subseteq} M_i^2\}_{i\in I}$ be as specified in Subsection 2.1, and $\varepsilon \in L$. Moreover, we assume to have only a partial information about $f \underset{\sim}{\subseteq} M^{(n)}$ in the form of a set of samples $\{f(\bar{c})|\ \bar{c} \in C \subseteq M^{(n)}\}$. The problem (ApproxF) that we are going to solve is formulated as follows:

$$\begin{gathered}\text{Find } D \subseteq M^{(n)} \text{ and } \tilde{f} \in \mathbb{M}^D_{\vee(\wedge)} \text{ such that}\\ \varepsilon \le f(\bar{x}) \leftrightarrow \tilde{f}(\bar{x}),\ \forall \bar{x} \in C,\end{gathered}$$

where

$$\mathbb{M}^D_{\vee} = \Big\{ \bigvee_{\bar{d}\in D} (R(\bar{d},\bar{x}) * g(\bar{d})) \mid\ g \underset{\sim}{\subseteq} M^{(n)} \Big\} \quad \text{and}$$
$$\mathbb{M}^D_{\wedge} = \Big\{ \bigwedge_{\bar{d}\in D} (R(\bar{x},\bar{d}) \to g(\bar{d})) \mid\ g \underset{\sim}{\subseteq} M^{(n)} \Big\}.$$

Due to terminology introduced in [11], we approach this problem from the view point of fuzzy relations of a special structure, the so-called *normal forms*:

$$\mathrm{DNF}^P_h(\bar{x}) = \bigvee_{\bar{p}\in P} (R(\bar{p},\bar{x}) * h(\bar{p})) \quad \text{and} \tag{6}$$
$$\mathrm{CNF}^P_h(\bar{x}) = \bigwedge_{\bar{p}\in P} (R(\bar{x},\bar{p}) \to h(\bar{p})), \tag{7}$$
$$\text{where } h \underset{\sim}{\subseteq} M^{(n)}, P \subseteq M^{(n)}, \bar{x} \in M^{(n)}, \tag{8}$$

are called the disjunctive and conjunctive normal forms for h w.r.t. P, respectively. Hence,

$$\mathbb{M}^D_{\vee} = \Big\{ \mathrm{DNF}^D_g \mid\ g \underset{\sim}{\subseteq} M^{(n)} \Big\} \quad \text{and}$$
$$\mathbb{M}^D_{\wedge} = \Big\{ \mathrm{CNF}^D_g \mid\ g \underset{\sim}{\subseteq} M^{(n)} \Big\}.$$

It is possible to rewrite inverse F-transforms using normal forms if we take $\{R_i\}_I$ and D:

- Assume $M^{(n)}$ contains sufficiently many elements to define set D.
- Take $D = \{\bar{d}_j \in M^{(n)} |\ d_{j_i} \ne d_{j'_i},\ \forall j, j' \in J, \forall i \in I\}$.
- Define $R(\bar{x}, \bar{d}_j) = R(\bar{d}_j, \bar{x}) = A_{\mathbf{j}}(\bar{x}), \forall j \in J, \mathbf{j} \in J^n$ and $R(\bar{x},\bar{y}) = \mathbf{0},\ \forall \bar{y} \in M^{(n)} \setminus D$.

In the case of lacking elements to construct set D, which can hypothetically arise because there is no restriction to the number of sets A_{j_i}, we can artificially extend $M^{(n)}$ to reach the required cardinality. Then we obtain that $\mathrm{DNF}^D_{\mathrm{CNF}^C_f}$ corresponds to an inverse $F^{\downarrow}$-transform $f^{\downarrow}_{F,k}$ and $\mathrm{CNF}^D_{\mathrm{DNF}^C_f}$ to an inverse $F^{\uparrow}$-transform $f^{\uparrow}_{F,k}$.

For an arbitrary $f \underset{\sim}{\subseteq} M^{(n)}$, we can certainly show[3]

$$\bigvee_{\bar{d}\in D} (R(\bar{d},\bar{x}) * f(\bar{x})) \le f(\bar{x}) \le \bigwedge_{\bar{d}\in D} (R(\bar{x},\bar{d}) \to f(\bar{x})), \forall \bar{x} \in C.$$

[3] Considering $\bigvee_{\bar{d}\in D} R(\bar{d},\bar{x}) = \bigvee_{\bar{d}\in D} R(\bar{x},\bar{d}) = \mathbf{1}$ gives the following equality $f(\bar{x}) = \bigvee_{\bar{d}\in D}(R(\bar{d},\bar{x}) * f(\bar{x})) = \bigwedge_{\bar{d}\in D}(R(\bar{x},\bar{d}) \to f(\bar{x}))$, for all $\bar{x} \in C$, because in an arbitrary residuated lattice $\mathbf{1} * a = \mathbf{1} \to a = a$, for all $a \in L$. Hence, it is reasonable to apply this requirement for the partitioning of the domain of f in the practise.

ISBN: 978-989-95079-6-8

The task stands in the replacement of $f(\bar{x})$ by suitable values so that the approximation is the best possible w.r.t. the available knowledge. In the following proposition, we are going to show that the combinations of $\{\mathrm{DNF}^D_f, \mathrm{CNF}^C_f\}$ and $\{\mathrm{CNF}^D_f, \mathrm{DNF}^C_f\}$ have this property.

Theorem 5 – DNF^C_f *is the least element of the following set:*

$$S = \{g \subsetsim M^{(n)} \mid f(\bar{x}) \leq \bigwedge_{d \in D} (R(\bar{x}, \bar{d}) \to g(\bar{d})),\ \forall \bar{x} \in C\}. \tag{9}$$

– CNF^C_f *is the greatest element of the following set:*

$$S' = \{g \subsetsim M^{(n)} \mid \bigvee_{d \in D} (R(\bar{d}, \bar{x}) * g(\bar{d})) \leq f(\bar{x}),\ \forall \bar{x} \in C\}. \tag{10}$$

As it has been shown above, there cannot be better lower approximation of f from the set $\mathbb{M}^D_\vee$ than $\mathrm{DNF}^D_{\mathrm{CNF}^C_f} \in \mathbb{M}^D_\vee$. And analogously, $\mathrm{CNF}^D_{\mathrm{DNF}^C_f} \in \mathbb{M}^D_\wedge$ is the best upper approximation of f. In other words, the best choice for coefficients in the inverse formula of $\mathrm{F}^{\uparrow(\downarrow)}$-transform is to take just the coefficients of the direct $\mathrm{F}^{\uparrow(\downarrow)}$-transform. Both results do not consider any boundary for an $\varepsilon-$precision of the approximations. It is dependent on the suitable choice of $\mathcal{R}_I$ and the set of nodes D arising from the following estimation in terms of a $*$-equivalence:

Theorem 6 *Let $f \subsetsim M^{(n)}$ be extensional w.r.t. $\mathcal{S}_I = \{S_i \subsetsim M_i^2\}_{i \in I}$ as follows:*

$$S_i(x, y) = \bigvee_{\bar{d} \in D} R_i(x, d_i) * R_i(d_i, y). \tag{11}$$

Then we can prove

$$\bigvee_{\bar{d} \in D} R(\bar{x}, \bar{d}) * R(\bar{d}, \bar{x}) \leq \begin{array}{l} \mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \\ \mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \end{array} \quad \forall \bar{x} \in C. \tag{12}$$

Corollary 7 *Let $\overline{\mathcal{R}}_I = \{\bar{R}_i\}_{i \in I}$ be defined by*

$$\bar{R}_i(x, y) = \bigvee_{j \in J} A_{j_i}(x) * A_{j_i}(y), \tag{13}$$

for all $i \in I$. If $f \subsetsim M^{(n)}$ is extensional w.r.t. $\overline{\mathcal{R}}_I$ then (5) is valid.

Here, we provide an alternative proof of this assertion independent of the above theorem.

Proof: Due to extensionality, we have

$$A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{y}) \leq R(\bar{x}, \bar{y}) = \bigvee_{\mathbf{j} \in J^n} A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{y}) \leq$$
$$\leq f(\bar{x}) \leftrightarrow f(\bar{y}), \text{ and hence}$$
$$A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{x}) * f(\bar{x}) \leq A_{\mathbf{j}}(\bar{x}) * (A_{\mathbf{j}}(\bar{y}) \to f(\bar{y})),$$
$$A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{x}) * f(\bar{x}) \leq A_{\mathbf{j}}(\bar{x}) * \bigwedge_{\bar{y} \in C} (A_{\mathbf{j}}(\bar{y}) \to f(\bar{y})),$$
$$f(\bar{x}) * \bigvee_{\mathbf{j} \in J^n} A_{\mathbf{j}}(\bar{x}) * A_{\mathbf{j}}(\bar{x}) \leq$$
$$\leq \bigvee_{\mathbf{j} \in J^n} (A_{\mathbf{j}}(\bar{x}) * \bigwedge_{\bar{y} \in C} (A_{\mathbf{j}}(\bar{y}) \to f(\bar{y}))) = f^{\downarrow}_{F,k}(\bar{x}),$$

for all $\bar{x} \in M^{(n)}$. Adding the left side of inequality (2), we receive the first required estimation (5). Analogously we proceed to prove the inequality for $f^{\downarrow}_{F,k}$. QED

Hence, to obtain the desired ε-approximation[4], it only remains to choose D w.r.t. $\mathcal{R}_I$ so that

$$(\forall \bar{x} \in C)(\varepsilon \leq \bigvee_{\bar{d} \in D} R(\bar{x}, \bar{d}) * R(\bar{d}, \bar{x})),$$

which completes the task.

Corollary 8 *Let $\varepsilon \in L, D \subseteq M^{(n)}$ and $f \subsetsim M^{(n)}$ be extensional w.r.t. $\mathcal{S}_I$ defined by (11). If for each $\bar{x} \in C$ there exists $d \in D :\ \varepsilon \leq R(\bar{x}, d) * R(d, \bar{x})$ then*

$$\varepsilon \leq \begin{array}{l} \mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \\ \mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}) \leftrightarrow f(\bar{x}), \end{array} \quad \forall \bar{x} \in C. \tag{14}$$

Corollary 9 *Let $\varepsilon \in L$ and $f \subsetsim M^{(n)}$ be extensional w.r.t. $\overline{\mathcal{R}}_I$ defined by (13). If for each $\bar{x} \in C$ there exists $\mathbf{j} \in J^n$: $\varepsilon \leq A_{\mathbf{j}}(x) * A_{\mathbf{j}}(x)$ then*

$$\varepsilon \leq f^{\downarrow(\uparrow)}_{F,k}(\bar{x}) \leftrightarrow f(\bar{x}),\ \forall \bar{x} \in C. \tag{15}$$

Corollary 10 *Let $L = [0, 1]$, $*$ be a continuous t-norm, $\varepsilon \in L$, M be a pre-compact (totally bounded) subset over a standard metric space, $R \subsetsim M$ be reflexive and uniformly continuous. If $f \subsetsim M$ is extensional w.r.t. R then for every $\varepsilon < 1$ there exists a finite $D \subseteq M$ such that (14) is valid.*

Whenever we assume each $R \in \mathcal{R}$ to be a $*$-equivalence, we obtain an estimation by means of the following pseudo-metrics:

$$\begin{aligned} d_n(\bar{x}, \bar{y}) &= \varphi^{-1} R_1(x_1, y_1) \oplus_\varphi \ldots \oplus_\varphi \varphi^{-1} R_n(x_n, y_n) \\ &= \bigoplus_{i \in I} \varphi^{-1} R_i(x_i, y_i),\ \forall \bar{x}, \bar{y} \in M^{(n)}, \\ d(x, y) &= \varphi^{-1}(x \leftrightarrow y),\ \forall x, y \in L \end{aligned}$$

where φ is an order-reversing bijection on L, $\oplus_\varphi$ is an operation defined by $x \oplus_\varphi y = \varphi^{-1}(\varphi(x) * \varphi(y))$.

Corollary 11 *Let each $R \in \mathcal{R}_I$ be a $*$-equivalence and $d_n : M^{(n)} \times M^{(n)} \to L, d : L \times L \to L$ be defined as above. If $f \subsetsim M^{(n)}$ is extensional w.r.t. $\mathcal{S}_I$ defined by (11) then f is 1-Lipschitz continuous w.r.t. (d, d_n), i.e.,*

$$d(f(\bar{x}), f(\bar{y})) \leq d_n(\bar{x}, \bar{y}),\ \textit{for all } \bar{x}, \bar{y} \in M^{(n)},$$

and

$$\begin{array}{l} d(\mathrm{DNF}^D_{\mathrm{CNF}^C_f}(\bar{x}), f(\bar{x})) \\ d(\mathrm{CNF}^D_{\mathrm{DNF}^C_f}(\bar{x}), f(\bar{x})) \end{array} \leq \bigwedge_{\bar{d} \in D} 2d_n(\bar{x}, \bar{d})$$
$$= \bigwedge_{\bar{d} \in D} d_n(\bar{x}, \bar{d}) \oplus d_n(\bar{d}, \bar{x}),\ \forall \bar{x} \in C. \tag{16}$$

[4]Notice that the choice of $\mathcal{R}_I$ is the limitation for ε as well as for D, e.g. D might become very huge if ε is close to $\mathbf{1}$.

ISBN: 978-989-95079-6-8

Let us consider a simple problem to illustrate the way of approximation by F-transforms.

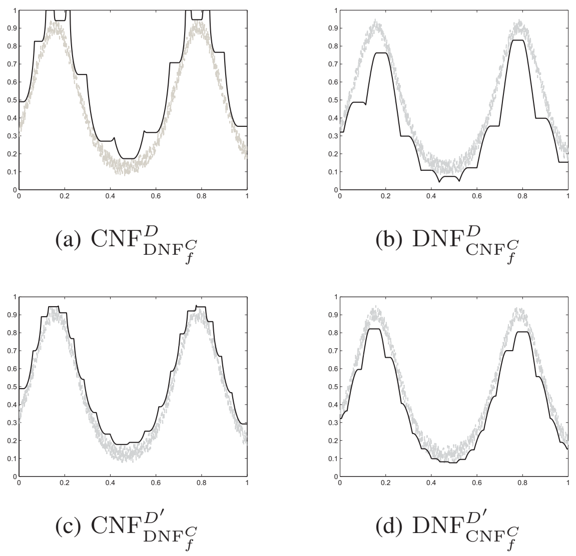
Example 12 *Let us take the following function*

$$f(x) = \frac{1}{3} \cdot e^{\sin(10x)} + Rand(x),\ \forall x \in M = [0,1],$$

where $Rand(x)$ is some additive random noise. Moreover, assume $*$ to be the t-norm generated by $g(x) = (1-x)^2$ and $\mathcal{L}$ be the associated residuated lattice. Take

$$R(x,y) = 1 \wedge (0 \vee (1.2 - 10 \cdot |x-y|)),$$

and create the approximation using 11 and 21 nodes and equidistant discretization $C = \{k \cdot 0.001 | \, k = 1, \dots, 999\}$. The resulting approximations are depicted on Fig. 1, where f is represented by grey line and inverse $F^{\uparrow(\downarrow)}$-transforms by black lines, set D contains 11 nodes and set D' contains 21.



(a) $\mathrm{CNF}^{D}_{\mathrm{DNF}^{C}_{f}}$ (b) $\mathrm{DNF}^{D}_{\mathrm{CNF}^{C}_{f}}$

(c) $\mathrm{CNF}^{D'}_{\mathrm{DNF}^{C}_{f}}$ (d) $\mathrm{DNF}^{D'}_{\mathrm{CNF}^{C}_{f}}$

Figure 1: Approximation of f (grey line) from Example 12 using F-transforms (black line) with 11 (set D) and 21 (set D') nodes.

4 Conclusions

We have shown that the only one requirement is needed to show (5):

New requirements
$\mathcal{L}$ be a residuated lattice
f be extensional w.r.t. $\{\bar{R}_i\}_{i \in I}$

While in [1], we were limited to the stronger algebraic structure, a special type of partition, $E \subseteq C$, additionally requiring normality, and extensionality of f (see Subsection 2.2). Notice that all the additional requirements follow from the procedure of replacement of fuzzy sets A_{i_i} by P_i based on the special set E (such relations can be found in [12]). We have found a way how to get around this construction and to reach such simplification.

Moreover, our approach enabled to express direct and inverse $\mathrm{F}^{\uparrow(\downarrow)}$-transforms using combinations of normal forms. This allowed us to show an optimality of choice of all coefficients of the direct $\mathrm{F}^{\uparrow(\downarrow)}$-transform in the inverse formula at once (see Theorem 5). Additionally, we gave some interesting corollaries that contributes only to the theory of normal forms and they cannot be translated into the theory of $\mathrm{F}^{\uparrow(\downarrow)}$-transforms, which shows that even though both theories have a lot in common, it is also important to study them separately.

Acknowledgment

We gratefully acknowledge partial support of the projects MSM6198898701 and 1M0572 of the MŠMT ČR.

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ISBN: 978-989-95079-6-8