Fuzzy differential equation with π-derivative

Y. Chalco-Cano[1] H. Román-Flores[2] M.D. Jiménez-Gamero[3]

1.Departamento de Matemática, Universidad de Tarapacá
Arica, Chile
2.Instituto de Alta Investigación, Universidad de Tarapacá
Arica, Chile
3.Departamento de Estadística e Investigación Operativa Facultad de Matemáticas, Universidad de Sevilla
Sevilla, Spain
Email: {ychalco,hroman}@uta.cl, dolores@us.es

Abstract— *The π-derivative for fuzzy function is considered. Consequently, using this π-derivative we study fuzzy differentuial equations. In particular, we build a solution for a fuzzy differential equation with the help of a system of ordinary differential equations which is generates of the π-derivative.*

Keywords— π-derivative for set valued functions, derivative for fuzzy functions, fuzzy differential equations.

1 Introduction

The spaces $\mathcal{K}$, of all closed and bounded intervals of $\mathbb{R}$ is not linear spaces since it do not contain inverse elements for the addition and therefore subtraction is not well defined. As a consequence, alternative formulations for subtraction have been suggested. One of those alternatives are the H-diference [3, 13] and by embedding the space $\mathcal{K}$ in a linear space [22].

The Radström's embedding theorem [22] tell us that there is a real normed linear space $\mathcal{B}$ and an isometric mapping $\pi : \mathcal{K} \to \mathcal{B}$. Then, taking advantage of this embedding theorem, a set-valued mappings F is π-differentiable at t_0 if $\pi \circ F$ is differentiable at t_0. In [3], we can find various properties of this derivative and its connection with other definitions of derivatives for set-valued mappings. On the other hand, in [20] each element of $\mathcal{B}$ can be represented in the form (x, δ). Thus, the π-derivative of $\pi \circ F$ is associated with the derivative of an ordered pair. In [9] the authors obtained a set $F_\pi^{'}(t) \in \mathcal{K}$, which is said π-derivative of F at t. Some properties and the conection with generalized derivative are studied.

The aim of this paper is introduce the π-derivative for fuzzy interval valued functions based in the generalization of π-derivative for set-valued mappings given in [9]. Consequently, using the π-derivative for fuzzy functions, we study fuzzy differential equations. In particular, we build a solution for a fuzzy differential equation with the help of a system of ordinary differential equations which is generates of the π-derivative.

2 Preliminaries

Let $\mathbb{R}$ be the 1-dimensional Euclidian space. We denote by $\mathcal{K}$ the family of all bounded intervals

$$\mathcal{K} = \{A = [a, b] \subset \mathbb{R} \mid A \neq \emptyset \text{ is bounded } \}$$

The Hausdorff metric H is defined by

$$H(A, B) = \max\{|a - c|, |b - d|\}.$$

where $A = [a, b]$ and $B = [c, d]$.

Also, by using the Minkowski sum between two sets, we defined the following operations on $\mathcal{K}$

$$A + B = \{a + b \mid a \in A,\ b \in B\} \text{ and } \lambda A = \{\lambda a \mid a \in A\}. \tag{1}$$

The spaces $\mathcal{K}$ is not linear spaces since it do not contain inverse elements and therefore subtraction is not well defined [1, 2].

We have, from (1), that $A - B = A + (-1)B$. Also, if $A = B + C$, then the Hukuhara difference of A and B is denoted by $A -_H B$ and it is equal to C. The Hukuhara difference of A and B is also called the geometrical difference between the sets A and B [27].

In [3] the authors introduce subtraction in $\mathcal{K}$ by using the Radström's embedding theorem [22] which tell us that there is a real normed linear space $\mathcal{B}$ and an isometry $\pi : \mathcal{K} \to \mathcal{B}$ such that $\pi(\mathcal{K})$ is a convex cone in $\mathcal{B}$.

To construct $\mathcal{B}$, we consider the following equivalence relation in the space $\mathcal{K} \times \mathcal{K}$

$$(A, B) \sim (C, D) \quad \text{if and only if } \ A + D = B + C.$$

If $\langle A, B\rangle$ is the equivalence class of the pair (A, B), then $\mathcal{B}$ is the quotient space $\mathcal{K} \times \mathcal{K}/\sim$. Now, in $\mathcal{B}$ we introduce the operations of addition and scalar multiplication by means

$$\langle A, B\rangle + \langle C, D\rangle = \langle A + C, B + D\rangle,$$

$$\lambda \langle A, B\rangle = \begin{cases} \langle \lambda A, \lambda B\rangle & \lambda \geq 0, \\ \langle |\lambda| B, |\lambda| A\rangle & \lambda < 0 \end{cases}$$

Then $\mathcal{B}$ is a linear space.

One defines the embedding $\pi : \mathcal{K} \to \mathcal{B}$ as follows

$$\pi(A) = \langle A, 0\rangle, \quad A \in \mathcal{K},$$

so that $\langle A, 0\rangle$ is the equivalence class $\{(A + D, D) \mid A, D \in \mathcal{K}\}$.

The metric and the norm in $\mathcal{B}$ are defined by

$$\rho(\langle A, B\rangle\ ,\ \langle C, D\rangle) = H(A + D\ ,\ C + B)$$

$$\|\langle A, B\rangle\| = \rho(\langle A, B\rangle\ ,\ \langle 0, 0\rangle)$$

If $A, B \in \mathcal{K}$, then the difference of the sets A and B is an alement of $\mathcal{B}$ equal to $\langle A, B\rangle$. Since $\mathcal{B}$ is a linear space, difference has all the properties of difference in linear spaces.

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In general, the difference of two sets in $\mathcal{K}$ is not necessarily an element of $\mathcal{K}$. On the other hand if the Hukuhara diference is well defined for two sets A, B in $\mathcal{K}$, then

$$\langle A, B\rangle = \langle A -_H B, 0\rangle .$$

3 π-derivative for set-valued mappings

In this Section we present the π-derivative for set-valued mappings. This concept has been studied by the authors in [9].

Elements of the space $\mathcal{K}$ are bounded and closed intervals, therefore each equivalence class can be represented in the form

$$\langle [x, x+\delta]\ ,\ 0\rangle\ ,\ \delta \geq 0 \quad \text{or} \quad \langle 0\ , [-x, -x-\delta]\rangle\ ,\ \delta < 0.$$

We will call this the canocical representation [20].

In fact, an arbitrary element $([a, b]\ ,\ [c, d])$ belongs to a class of the first kind if $b - a \geq d - c$ (then $x = a - c,\ \delta = (b - a) - (d - c)$) and to a class of the second kind if $b - a < d - c$ (then $x = a - c,\ \delta = (b - a) - (d - c)$).

For the canonical representation of an equivalence class we will use the notation (x, δ), which implies that

$$(x, \delta) = \begin{cases} \langle [x,\ x+\delta]\ ,\ 0\rangle & \text{if} \quad \delta \geq 0 \\ \langle 0,\ [-x,\ -x-\delta]\rangle & \text{if} \quad \delta < 0. \end{cases} \tag{2}$$

Given $A = [a, b] \in \mathcal{K}$ we have

$$\pi(A) = \pi([a, b]) = \langle [a,\ b]\ ,\ 0\rangle = \langle [x, x+\delta], 0\rangle\ ,$$

where $x = a$ and $\delta = b - a$.

Theorem 1 *([20]) The following results hold*

(a) $(x, \delta) = (y, \beta) \Longleftrightarrow x = y,\ \delta = \beta$*;*

(b) $(x, \delta) + (y, \beta) = (x + y, \delta + \beta)$*;*

(c) $a(x, \delta) = (ax, a\delta)$ *for each* $a \in \mathbb{R}$*.*

Theorem 2 *([20]) The equality*

$$D_\pi(x(t), \delta(t)) = (x^{'}(t), \delta^{'}(t))$$

holds, where $D_\pi(x(t), \delta(t))$ *is the* π*-derivative ([3]) of the pair* $(x(t), \delta(t))$ *and* $x^{'}$ *is the derivative of the real-valued function* $x(t)$*.*

Let $F : T \to \mathcal{K}$ be a set-valued mapping. If we denote by $F(t) = [f(t), g(t)]$, then

$$\pi(F(t)) = \langle [x(t), x(t) + \delta(t)], 0\rangle\ ,$$

where $x(t) = f(t)$ and $\delta(t) = g(t) - f(t)$.

From Theorem 2 $D_\pi(x(t), \delta(t)) = (x^{'}(t), \delta^{'}(t))$ and from canonical representation (2) we obtain the set $F^{'}_\pi(t) \in \mathcal{K}$ such that

$$F^{'}_\pi(t) = \begin{cases} \left[x^{'}(t), x^{'}(t) + \delta^{'}(t)\right] & \text{if} \quad \delta^{'}(t) \geq 0 \\ \left[x^{'}(t) + \delta^{'}(t), x^{'}(t)\right] & \text{if} \quad \delta^{'}(t) < 0. \end{cases} \tag{3}$$

We will say that $F^{'}_\pi(t_0)$ is the π-derivative of the set-valued mapping F at t_0.

Corollary 1 *([9]) Let* $F : T \to \mathcal{K}$ *be a set-valued mapping and we denote* $F(t) = [f(t), g(t)]$*. Then* F *is* π*-differentiable at* t_0 *if and only if* f *and* g *are differentiable functions at* t_0*.*

Example 1 *([9]) Consider the set-valued mapping* $F : (-1, 1) \to \mathcal{K}$ *defined by* $F(t) = \left[-t^2, e^{-t}\right]$*. If* $f(t) = -t^2$ *and* $g(t) = e^{-t}$*, we have that* f *and* g *are differentiable functions and* $f^{'}(t) = -2t$ *and* $g^{'}(t) = -e^{-t}$*. In this case* $x^{'}(t) = -t^2$ *and* $\delta^{'}(t) = -e^{-t} + 2t$*. Therefore, from (3) we have*

$$F^{'}_\pi(t) = \begin{cases} [-e^{-t}\ ,\ -2t] & \text{if} \quad t \in (-1, 0.35173] \\ [-2t\ ,\ -e^{-t}] & \text{if} \quad t \in (0.35173, 1). \end{cases}$$

Note that the π-derivative (3) is coincident with the generalized derivative introduced in [26].

4 π-derivative for fuzzy functions

Let $\mathcal{F}$ be the set of all fuzzy intervals with bounded α-level intervals. This means that if $u \in \mathcal{F}$ then the α-level set is a closed bounded interval which we denote by $[u]^\alpha$, for all $\alpha \in [0, 1]$.

Let T be a real interval. A mapping $X : T \to \mathcal{F}$ is called a fuzzy function. We denote

$$[X(t)]^\alpha = X_\alpha(t) = [f_\alpha(t), g_\alpha(t)],\ \ t \in T,\ \ 0 \leq \alpha \leq 1.$$

Note that X_α is a set-valued mapping for each $\alpha \in [0, 1]$. The π-derivative $X^{'}_\pi(t)$ of a fuzzy function X is defined by

$$[X^{'}_\pi(t)]^\alpha = X_{\alpha\pi}^{'}(t),\ \ 0 \leq \alpha \leq 1, \tag{4}$$

provided that is equation defines a fuzzy interval $X^{'}_\pi(t) \in \mathcal{F}$. Note that, if the family $\{X_{\alpha\pi}^{'}(t)\}$ satisfies the conditions of the Representation Theorem [10], then there exists the π-derivative $X_\pi(t)^{'}$ of the fuzzy function X.

Example 2 *Consider the fuzzy function* $X : (0, +\infty) \to \mathcal{F}_C$ *defined by*

$$X(t)(s) = \begin{cases} \frac{s}{t} + 1 & \text{if} \quad -t \leq s \leq 0 \\ -\frac{s}{t^2} + 1 & \text{if} \quad 0 \leq s \leq t^2 \\ 0 & \text{if} \quad s \notin [-t, t^2]. \end{cases}$$

Then, for all $\alpha \in [0, 1]$ *we have*

$$[X(t)]^\alpha = [f_\alpha(t), g_\alpha(t)] = [(\alpha - 1)t, (1 - \alpha)t^2],$$

and from (3)

$$X_{\alpha\pi}^{'}(t) = [(\alpha - 1), 2(1 - \alpha)t].$$

Now, the family $\{X_{\alpha\pi}^{'}(t)\}_{\alpha \in [0,1]}$ *satisfies the conditions from Representation Theorem [10] for each* $t > 0$*. Therefore, there exists the* π*-derivative* $X^{'}_\pi(t)$ *of the fuzzy function* X *for each* $t > 0$ *and*

$$\left[X^{'}_\pi(t)\right]^\alpha = [(\alpha - 1), 2(1 - \alpha)t].$$

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5 Fuzzy differential equation with π-derivative

We consider the initial value problem

$$X^{'}(t) = F(t, X(t)), \quad X(0) = X_0, \tag{5}$$

where $F : [0,T] \times \mathcal{F} \to \mathcal{F}$ is a continuous function and X_0 is a fuzzy interval.

Note that there are different interpretations for the problem (5). For example, in the problem (5), we can consider the H-derivative [15, 16, 17, 19, 23, 24, 25]; the Generalized derivative [4, 5, 6, 7, 8]. Also, the problem (5) can be rewrite as a family of differential inclusions [11, 12, 14]. In this section we will consider the π-derivative in the problem (5), i.e. we will study the following problem

$$X_{\pi}^{'}(t) = F(t, X(t)), \quad X(0) = X_0, \tag{6}$$

where $F : [0,T] \times \mathcal{F} \to \mathcal{F}$ is a continuous function and X_0 is a fuzzy interval. A solution of (6) is a continuous fuzzy function $X : T \to \mathcal{F}$ which verifies the equation (6) for each $t \in T$.

Taking in account the Theorem 1 and Theorem 2, we obtain a useful procedure to solve the fuzzy differential equation (6).

In fact, denote

$$[X(t)]^{\alpha} = [f_{\alpha}(t), g_{\alpha}(t)] \;, \qquad [X_0]^{\alpha} = [u_{\alpha}^0, v_{\alpha}^0]$$

and

$$[F(t,X(t))]^{\alpha} = [U_{\alpha}(t, f_{\alpha}(t), g_{\alpha}(t)) \;,\; V_{\alpha}(t, f_{\alpha}(t), g_{\alpha}(t))].$$

Then, using the canocical representation (Section 3), we obtain

$$\pi\left([X(t)]^{\alpha}\right) = (x_{\alpha}(t) \;,\; \delta_{\alpha}(t)) \;,\quad \pi\left([X_0]^{\alpha}\right) = \left(x_{\alpha}^0 \;,\; \delta_{\alpha}^0\right)$$

where $x_{\alpha}(t) = f_{\alpha}(t)$, $\delta_{\alpha}(t) = g_{\alpha}(t) - f_{\alpha}(t)$, $x_{\alpha}^0 = u_{\alpha}^0$ and $\delta_{\alpha}^0 = v_{\alpha}^0 - u_{\alpha}^0$. Also

$$\pi\left([F(t,X(t))]^{\alpha}\right) = \left(U_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t)), V_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t))\right)$$

Thus, we consider the following family of equations corresponding to the problem (6)

$$\begin{array}{l} D_{\pi}(x_{\alpha}(t), \delta_{\alpha}(t)) = \left(U_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t)), V_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t))\right) \\ (x_{\alpha}(0), \delta_{\alpha}(0)) = (x_{\alpha}^0, \delta_{\alpha}^0) \end{array} \tag{7}$$

with $\alpha \in [0,1]$.

Now, using the Theorem 1 and 2 and the problem (7), we obtain a solution from problem (6). For this, we proceed as follows:

(i) Solve the differential system

$$\begin{cases} x_{\alpha}^{'}(t) = U_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t)), & x_{\alpha}(0) = x_{\alpha}^0 \\ \delta_{\alpha}^{'}(t) = V_{\alpha}^1(t, x_{\alpha}(t), \delta_{\alpha}(t)), & \delta_{\alpha}(0) = \delta_{\alpha}^0 \end{cases}$$

for x_{α} and δ_{α};

(ii) from class $(x_{\alpha}(t), \delta_{\alpha}(t))$ we obtain the interval

$$X_{\alpha}(t) = \begin{cases} [x_{\alpha}(t), x_{\alpha}(t) + \delta_{\alpha}(t)] & \text{if} \quad \delta_{\alpha}(t) \geq 0 \\ [x_{\alpha}(t) + \delta_{\alpha}(t), x_{\alpha}(t)] & \text{if} \quad \delta_{\alpha}(t) < 0 \end{cases}$$

(iii) By using the Representation Theorem [10], we build a fuzzy solution $X(t)$ such that $[X(t)]^{\alpha} = X_{\alpha}(t)$, for all $\alpha \in [0,1]$.

Example 3 *Let us consider the fuzzy malthusian problem*

$$\begin{cases} X_{\pi}'(t) = -\lambda X(t) \\ X(0) = \; X_0, \end{cases} \tag{8}$$

where $\lambda > 0$ *and, as in [12], the initial condition* X_0 *is a symmetric triangular fuzzy number with support* $[-a, a]$. *That is,*

$$[X_0]^{\alpha} = [-a(1-\alpha), a(1-\alpha)].$$

For obtain a solution of (8), we have solve the following differential system

$$\begin{cases} x_{\alpha}^{'}(t) = -\lambda x_{\alpha}(t), & x_{\alpha}(0) = -a(1-\alpha) \\ \delta_{\alpha}^{'}(t) = -\lambda \delta_{\alpha}(t), & \delta_{\alpha}(0) = 2a(1-\alpha). \end{cases}$$

The solutions of this system are

$$x_{\alpha}(t) = -a(1-\alpha)e^{-\lambda t} \quad \textit{and} \quad \delta_{\alpha}(t) = 2a(1-\alpha)e^{-\lambda t},$$

where $\delta_{\alpha}(t) \geq 0$ *for all* $t > 0$. *Thus there exists a fuzzy solution from (8)* $X(t)$ *for all* $t > 0$, *such that*

$$\begin{array}{rcl} [X(t)]^{\alpha} & = & [x_{\alpha}(t), x_{\alpha}(t) + \delta_{\alpha}(t)] \\ & = & [-a(1-\alpha)e^{-\lambda t}, a(1-\alpha)e^{-\lambda t}]. \end{array}$$

6 Conclusions

In this paper we present the concept of π-derivative for fuzzy function using the canonical representation of a class equivalence of two intervals. Those classes of equivalence are obtained by embedding the space of closed and bounded intervals in a Banach space using the embedding theorem [22].

In a forthcoming paper we will study the properties of the π-derivative fuzzy and the connection between this approach and the π-derifferentiabiliy in the fuzzy context introduced in [21]. Also, we will study the connection with others concept of fuzzy derivatives. For example, the connection with the G-differentiability introduced in [4]. Note that this concept of π-derivative is equivalent to generalized derivative in the case of set-valued function [9].

In Section 5 we study fuzzy differential equations considering π-derivative. In particular we present a form for obtain a fuzzy solution from FDEs. This approach have some advantages in relation to others interpretation, for example, an advantage that have the fuzzy solutions obtained considering π-derivative seem to be more intuitive than other solution using H-derivative (Example 3). It is worthy to stress also that the interpretation using extension principle [7, 18] and fuzzy differential inclusion [11, 12, 14] has a disadvantage because we cannot speak about the fuzzy derivative. In this issue, in a forthcoming paper, we will study the existence of solution of a fuzzy differential equation considering π-derivative.

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