Numerical Solution of Interval Differential Equations with generalized Hukuhara differentiability

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Abstract— *In the present paper we propose novel numerical methods for interval differential equations with generalized Hukuhara differentiability. The algorithms proposed here are based on characterization theorems of interval differential equations with ODEs. Usng the characterizations we can translate an interval problem into two ODE systems. We observe that an interval problem may have several solutions under the interpretation considered. Several examples are presented and discussed.*

Keywords— Generalized Hukuhara Differentiability, Interval Differential Equations, Interval Analysis.

1 Introduction

Interval Analysis was introduced as an attempt to handle interval (nonstatistical) uncertainty that appears in our mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis is the book of R. Moore, [10]. Since then, Reliable Computing, Validated Numerics and Interval problems with Differential Equations are discussed in several monographs and research papers ([11], [2]).

Another major approach to a set of similar problems is that of differential inclusions and multivalued analysis ([1], [5]). This approach is also able to deal with discontinuous dynamical systems which do not fully fit into the Interval Analysis topic.

Hukuhara derivative of a set-valued mapping was first introduced by Hukuhara and it has been studied in several works. The paper of Hukuhara was the starting point for the topic of Set Differential Equations and later also for Fuzzy Differential Equations [8], [12], [6]. Recently, several works as e.g.[9], have brought back into the attention of the nonlinear analysis community, the topics of set differential equations and the Hukuhara derivative.

Recently a generalization of the Hukuhara difference for compact convex sets (gH-difference) was presented in [13] and the strongly and weakly generalized (Hukuhara) differentiability concepts have been proposed in [3]. Then, in [14] a combination (gH-derivative) was proposed and it was shown to be a useful concept in the interval setting. Our approach to interval differential equations is different from the approaches based on differential inclusions or interval analysis, however we can see that there is some connection among these approaches and the presented one, some connections with differential inclusions we discuss here.

After recalling some results of [14], we propose a numerical solution method for IDEs. Also, a numerical method for differential inclusions is discussed and several examples are proposed.

2 Generalized Hukuhara difference

Consider the space $\mathbb{I}$ of real intervals. Given two elements $A, B \in \mathbb{I}$ and $k \in \mathbb{R}$, the usual interval arithmetic operations, i.e. Minkowski addition and scalar multiplication, are defined by $A+B = \{a+b | a \in A, b \in B\}$ and $kA = \{ka | a \in A\}$. It is well known that addition is associative and commutative and with neutral element $\{0\}$. If $k = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a | a \in A\}$ but, in general, $A + (-A) \neq \{0\}$, i.e. the opposite of A is not the inverse of A in Minkowski addition (unless $A = \{a\}$ is a singleton). A first implication of this fact is that, in general, additive simplification is not valid, i.e. $(A + C = B + C) \not\Rightarrow A = B$ or $(A + B) - B \neq A$ (the Minkowski difference is $A - B = A + (-1)B$).

To partially overcome this situation, the Hukuhara H-difference has been introduced as a set C for which $A \ominus B = C \iff A = B + C$ and an important property of $\ominus$ is that $A \ominus A = \{0\}\ \forall A \in \mathcal{K}_C^n$ and $(A+B) \ominus B = A, \forall A, B \in \mathcal{K}_C^n$.

The H-difference is unique, but it does not always exist (a necessary condition for $A \ominus B$ to exist is that A contains a translate $\{c\} + B$ of B).

A generalization of the Hukuhara difference proposed in [13] aims to overcome this situation.

Definition 1 *([13]) The generalized Hukuhara difference of two sets $A, B \in \mathbb{I}$ (gH-difference for short) is defined as follows*

$$A \ominus_g B = C \iff \begin{cases} & (a) \quad A = B + C \\ or & (b) \quad B = A + (-1)C \end{cases} \tag{1}$$

The following property ensures the existence of gH-differences.

Proposition 1 *([13]) The gH-difference $C = A \ominus_g B$ of two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ always exists and*

$$[a^-, a^+] \ominus_g [b^-, b^+] = [c^-, c^+] \tag{2}$$
$$\textit{with}$$
$$c^- = \min\{a^- - b^-, a^+ - b^+\}$$
$$c^+ = \max\{a^- - b^-, a^+ - b^+\}.$$

Conditions (a) and (b) in (1) are satisfied simultaneously if and only if the two intervals have the same length and $c^- = c^+$.

3 Differentiation of interval valued functions

There exist several alternative definitions for the derivative of an interval-valued function. The first concept is a partic-

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ularization of the fuzzy concepts presented in [3] to the interval case, based on the generalized fuzzy differentiability concept, more general than one-sided derivatives, using the usual Hukuhara difference $\ominus$.

Definition 2 *([3]) Let $f\ :]a,b[\to \mathbb{I}$ and $x_0\ \in]a,b[$. We say that f is strongly generalized (Hukuhara) differentiable at x_0, if there exists an element $f'(x_0)\in\mathbb{I}$, such that, for all $h>0$ sufficiently small,*

(i) $\exists f(x_0+h)\ominus f(x_0)$, $f(x_0)\ominus f(x_0-h)$ and

$$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h}=$$
$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0),$$

or (ii) $\exists f(x_0)\ominus f(x_0+h)$, $f(x_0-h)\ominus f(x_0)$ and

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=$$
$$\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0),$$

or (iii) $\exists f(x_0+h)\ominus f(x_0)$, $f(x_0-h)\ominus f(x_0)$ and

$$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h}=$$
$$\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0),$$

or (iv) $\exists f(x_0)\ominus f(x_0+h)$, $f(x_0)\ominus f(x_0-h)$ and

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=$$
$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0).$$

Based on the gH-difference in [14] the following definition was proposed.

Definition 3 *Let $x_0\in]a,b[$ and h be such that $x_0+h\in]a,b[$, then the gH-derivative of a function $f\ :]a,b[\to\mathbb{I}$ at x_0 is defined as*

$$f'(x_0)=\lim_{h\to 0}\frac{1}{h}[f(x_0+h)\ominus_g f(x_0)].\qquad(3)$$

If $f'(x_0)\in\mathbb{I}$ satisfying (3) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

The next result is a characterization of the derivative concept proposed in (3) and it was proved in [14].

Theorem 2 *Let $f\ :\ [a,b]\ \longrightarrow\ \mathbb{I}$ be such that $f(x)=[f^-(x),f^+(x)]$. The function $f(x)$ is gH-differentiable if and only if $f^-(x)$ and $f^+(x)$ are differentiable real-valued functions and*

$$f'(x)=[\min\{\left(f^-\right)'(x),\left(f^+\right)'(x)\},\qquad(4)$$
$$\max\{\left(f^-\right)'(x),\left(f^+\right)'(x)\}].\qquad(5)$$

The following definition will be helpful in the discussion of the numerical methods for IDEs.

Definition 4 *Let $f:[a,b]\longrightarrow\mathbb{I}$ be gH-differentiable at $x_0\in]a,b[$. We say that f is (i)-gH-differentiable at x_0 if*

$$(i.)\qquad f'(x_0)=[\left(f^-\right)'(x_0),\left(f^+\right)'(x_0)]\qquad(6)$$

and that f is (ii)-gH-differentiable at x_0 if

$$(ii.)\qquad f'(x_0)=[\left(f^+\right)'(x_0),\left(f^-\right)'(x_0)].\qquad(7)$$

It is an interesting problem to see how the switch between the two cases (i.) and (ii.) in Definition 4 can occur.

Definition 5 *We say that a point $x_0\ \in]a,b[$ is an l-critical point of f if it is a critical point for the length function $len(f(x))=f^+(x)-f^-(x)$.*

Definition 6 *We say that a point $x_0\ \in]a,b[$ is a switching point for the differentiability of f, if in any neighborhood V of x_0 there exist points $x_1<x_0<x_2$ such that*

type I) at x_1 (6) holds while (7) does not hold and at x_2 (7) holds and (6) does not hold, or

type II) at x_1 (7) holds while (6) does not hold and at x_2 (6) holds and (7) does not hold.

First, let us observe that any switching point is also an *l-critical* point. Indeed, if x_0 is a switching point then $[\left(f^-\right)'(x_0),\left(f^+\right)'(x_0)]=[\left(f^+\right)'(x_0),\left(f^-\right)'(x_0)]$ and so $\left(f^+\right)'(x_0)=\left(f^-\right)'(x_0)$ and $len(f(x_0))'=0$. Clearly, not all *l-critical* points are also switching points.

The relation which exists between the two differentiability concepts was given in [14].

Theorem 3 *Let $f\ :]a,b[\to\ \mathbb{I}$ be a function $f(x)=[f^-(x),f^+(x)]$. The following affirmations are equivalent:*

(1) f is GH-differentiable

(2) f is gH-differentiable and the set of switching points is finite.

If the number of l-critical points is finite then the number of switching points is also finite. So, from now on we will use the GH-differentiability concept or equivalently the gH-differentiability with a finite number of switching points.

4 Interval differential equations

In this section we consider an interval valued differential equation

$$y'=f(x,y)\ ,\ y(x_0)=y_0\qquad(8)$$

where

$$f:[a,b]\times\mathbb{I}\longrightarrow\mathbb{I}$$

with $f(x,y)=[f^-(x,y),f^+(x,y)]$ for $y\in\mathbb{I}$

$$y=[y^-,y^+],\ y_0=[y_0^-,y_0^+].$$

We consider only GH-differentiable solutions, i.e. such that there exists $\delta>0$ such that there are no switching points in $[x_0,x_0+\delta]$.

An existence result for the solutions of an IDE was obtained in [14]. Let us consider $\overline{B}(y_0,q)\subset\mathbb{I}$ be a closed ball with center y_0 and radius q.

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Theorem 4 *Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $y_0 \in \mathbb{I}$ nontrivial and $f : R_0 \to \mathbb{I}$ be continuous, nontrivial (i.e., y nontrivial interval gives $f(x, y)$ a nontrivial interval). If f satisfies the Lipschitz condition $D(f(x, y), f(x, z)) \le L \cdot D(y, z)$, $\forall(x, y), (x, z) \in R_0$ then the interval problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

has two unique solutions $y^i, y^{ii} : [x_0, x_0 + r] \to \overline{B}(y_0, q)$ and the successive iterations in

$$y_0(x) = y_0$$

$$y_{n+1}(x) \ominus_g y_0 = \int_{x_0}^{x} f(t, y_n(t))dt,$$

or more precisely,

$$y^i_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y^i_n(t))dt \qquad i.e.$$

$$(a)\begin{cases} y^{i-}_{n+1}(x) = y^-_0 + \int_{x_0}^{x} f^-(t, y^i_n(t))dt \\ y^{i+}_{n+1}(x) = y^+_0 + \int_{x_0}^{x} f^+(t, y^i_n(t))dt \end{cases}$$

and

$$y_0 = y^{ii}_{n+1}(x) - \int_{x_0}^{x} f(t, y^{ii}_n(t))dt \qquad i.e.$$

$$(b)\begin{cases} y^{ii-}_{n+1}(x) = y^-_0 + \int_{x_0}^{x} f^+(t, y^{ii}_n(t))dt \\ y^{ii+}_{n+1}(x) = y^+_0 + \int_{x_0}^{x} f^-(t, y^{ii}_n(t))dt \end{cases}$$

converge to these two solutions y^i and y^{ii} respectively.

Considering the interval problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \tag{9}$$

we set

$$\begin{cases} \varphi^-(x, y^-, y^+) = f^-(x, y) \\ \varphi^+(x, y^-, y^+) = f^+(x, y) \end{cases} \tag{10}$$

with $\varphi^- \le \varphi^+$ defined on a subset of $\mathbb{R}^3$.

Finally, we obtain two situations:

ODE(a): $y'^-(x) \le y'^+(x)$; the differential equations are

$$\begin{cases} y'^-(x) = \varphi^-(x, y^-(x), y^+(x)) \\ y'^+(x) = \varphi^+(x, y^-(x), y^+(x)) \\ \text{s.t.} \\ y^-(x_0) = y^-_0 \\ y^+(x_0) = y^+_0 \end{cases} \tag{11}$$

and if φ^- depends only on y^- and φ^+ depends only on y^+, the two equations are independent.

ODE(b): $y'^-(x) \ge y'^+(x)$; the differential equations are

$$\begin{cases} y'^-(x) = \varphi^+(x, y^-(x), y^+(x)) \\ y'^+(x) = \varphi^-(x, y^-(x), y^+(x)) \\ \text{s.t.} \\ y^-(x_0) = y^-_0 \\ y^+(x_0) = y^+_0 \end{cases} \tag{12}$$

and if φ^+ depends only on y^- and φ^- depends only on y^+, the two equations are independent.

Now we present a characterization result obtained in [14]:

Theorem 5 *Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $y_0 \in \mathbb{I}$ nontrivial and $f : R_0 \to \mathbb{I}$ be nontrivial and continuous. If f satisfies the Lipschitz condition $D(f(x, y), f(x, z)) \le L \cdot D(y, z)$, $\forall(x, y), (x, z) \in R_0$ then the interval problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \tag{13}$$

is equivalent to the union of the ODEs (11) and (12) on some interval $[x_0, x_0 + q]$. Here, by equivalent, we understand that $y = [y^-, y^+] : [x_0, x_0 + q] \to \mathbb{I}$ is a solution of 13 if and only if $(y^-, y^+) : [x_0, x_0 + q] \to \mathbb{R}^2$ is a solution of one of the problems (11) or (12).

Remark 1 *We can write the IDE (13) in terms of the midpoint representations of y, y' and $f(x, y)$. Let $y = \langle \widehat{y}, \overline{y} \rangle$, $y_0 = \langle \widehat{y}_0, \overline{y}_0 \rangle$, $f(x, y) = \left\langle \widehat{f}(x, y), \overline{f}(x, y) \right\rangle$; then $y' = \langle \widehat{y}', |\overline{y}'| \rangle$ and (13) becomes*

$$\begin{cases} \widehat{y}' = \widehat{f}(x,y) \\ |\overline{y}'| = \overline{f}(x,y) \\ \widehat{y}(x_0) = \widehat{y}_0 \\ \overline{y}(x_0) = \overline{y}_0; \end{cases} \tag{14}$$

they represent a system of differential algebraic equations (DAEs) and can be investigated by the help of the corresponding theory.

All the preceding results hold for general interval-valued functions so possibly they depend on interval parameters, i.e. interval extensions of real-valued functions of the form $f(x, y, p_1, ..., p_n)$ with $p_i \in P_i \in \mathbb{I}$.

Let us analyze the case when $f(x, y)$ is the interval extension of a real valued function $f(x, p)$, i.e.

$$f(x, y) = [\inf_{p\in y} f(x, p), \sup_{p\in y} f(x, p)]. \tag{15}$$

Let us suppose further that f is monotonic with respect to p.

1) if $f(x, p)$ is increasing with respect to p,

$$case(a) : y'^-(x) \le y'^+(x)$$
$$\begin{cases} y'^-(x) = f(x, y^-(x)) \\ y'^+(x) = f(x, y^+(x)), \\ \text{s.t. } y^-(x_0) = y^-_0, \\ y^+(x_0) = y^+_0 \end{cases} \tag{16}$$

$$case(b) : y'^-(x) \ge y'^+(x)$$
$$\begin{cases} y'^+(x) = f(x, y^-(x)) \\ y'^-(x) = f(x, y^+(x)), \\ \text{s.t. } y^-(x_0) = y^-_0, \\ y^+(x_0) = y^+_0 \end{cases} \tag{17}$$

2) if $f(x, p)$ is decreasing with respect to p

$$case(a) : y'^-(x) \le y'^+(x)$$
$$\begin{cases} y'^-(x) = f(x, y^+(x)) \\ y'^+(x) = f(x, y^-(x)), \\ \text{s.t. } y^-(x_0) = y^-_0, \\ y^+(x_0) = y^+_0 \end{cases} \tag{18}$$

$$case(b) : y'^-(x) \ge y'^+(x)$$
$$\begin{cases} y'^+(x) = f(x, y^+(x)) \\ y'^-(x) = f(x, y^-(x)), \\ \text{s.t. } y^-(x_0) = y^-_0, \\ y^+(x_0) = y^+_0 \end{cases} \tag{19}$$

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Also, in this case, a very interesting result from [4] connects these cases to differential inclusions. In the followings we formulate a particularization of the result of Chalco-Cano and Roman-Flores. Namely we have:

Theorem 6 *[4] Let $f(x, p)$ be a monotonic function with respect to $p \in \mathbb{R}$ and let $y_0 \in \mathbb{I}$ nontrivial interval.*

a) If $f(x, y)$ is the interval extension of f and f is an increasing function, then the solution in case (a) of (16) and the attainable set of the differential inclusion $y' = f(x, y)$, $y(x_0) \in y_0$ coincide on some interval $[x_0, x_0 + q]$.

b) If $f(x, y)$ is the interval extension of f and f is a decreasing function, then the solution in case (b) of (18) and the attainable set of the differential inclusion $y' = f(x, y)$, $y(x_0) \in y_0$ coincide on some interval $[x_0, x_0 + q]$.

5 Solution methods for interval differential equations

From the results and discussion of previous sections, the interval differential equation concept presented in this paper does not coincide with the concept of a differential inclusion. It is a different, new approach to model interval uncertainty in dynamical systems. It is related (as it is seen from the above theorems) to differential inclusions but we do have in our case more than one solution. The existence of several solutions can be an advantage when we search for solutions with specific properties like e.g., periodic, almost periodic, asymptotically stable, etc. Also, it can be very useful when we have unknown correlations between the parameters. In those situations, the uncertainty about the correlation introduces supplementary uncertainty in our system, so the existence of several solutions appears to be natural in this case.

The above characterization theorems, together with the existence results, lead easily to a numerical algorithm to solve interval differential equations.

5.1 General description of the solution methods

First, let us remark that a switch between the cases (i) and (ii) of gH-differentiability is possible if and only if $y'(x)$ is a singleton as in fact possible switch-points are l-critical points, i.e., critical points of the length of $y(x)$. Let us remark that if at a point $y(x_0)$ is a singleton for some x_0, then this point is enforcing a switch to the case (i), because, according to the existence result in Theorem 4, the second solution does not exist in this case. All other l-critical points make possible a switch from (i) to (ii) case or vice versa, but they do not enforce a switch, so at each l-critical point x_0 with $y(x_0)$ nontrivial interval, two new local solutions arise. One is (i)-differentiable, the other is (ii)-differentiable.

These remarks, together with the characterization Theorem 5, lead to the following general approach to numerically solve interval differential equations. We solve basically ODEs (11) and (12) on subintervals of the time domain having the initial value y_0 updated at all the possible critical points. In this way on a bounded time interval we obtain a finite number of solutions. For the solution of (11) and (12) by the characterization theorem above, any efficient numerical method for ODEs can be used. This is an advantage of the method presented here, as we do not need to reinvent numerical methods for interval differential equations. Instead we can use the classical ones on the ODE translations of the interval differential equations.

This algorithm generates a tree structure for the solutions of the IDE. All the nodes of the tree will be critical points except the terminal nodes. All the branches represent local solutions of (i) or (ii) kind between two nodes. This is illustrated in Figure 1. Each node represents an l-critical point with a switching and each branch corresponds to one of the cases (i) or (ii) of differentiability.

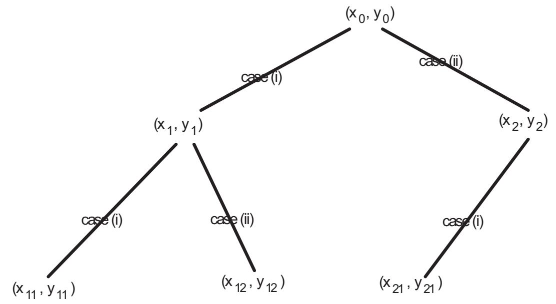

Figure 1. The structure of the tree of local solutions.

Our algorithm which generates all the solutions on a bounded interval is as follows.

Algorithm 1. (Find solutions of an interval differential equation - IDE)

Let us consider y_0 be any interval.

- Step 1. If y_0 is a singleton then we solve (11) and we obtain solution y_1.
- Step 2. Else we solve both (11) and (12) and we obtain solutions y_1, y_2.
- Step 3. Let $x_1 = \inf\{x > x_0 : y_1'(x) \in \mathbb{R}\}$ and $x_2 = \inf\{x > x_0 : y_2'(x) \in \mathbb{R}$ or $y_2(x) \in \mathbb{R}\}$ be the nearest l-critical point. (Let us remark here that since the algorithm applies to general interval-valued functions, it is possible that $y_2(x) \in \mathbb{R}$ however $y_2'(x) \notin \mathbb{R}$)
- Step 4. We insert the solutions which are found in a tree structure: y_1 in the left branch and y_2 in the right branch (the root will be simply (x_0, y_0)).
- Step 5. If $x_1 < X$ (a preset maximum value for x) then let $x_0 = x_1$ and $y_0 = y_1(x_1)$ and go to step 1.
- Step 6. If $x_2 < X$ then let $x_0 = x_2$ and $y_0 = y_2(x_2)$ and go to step 1.
- Step 7. Else Return.
- Step 8. Using a standard backtracking algorithm we generate all the solutions from the tree structure generated above.

The presented algorithm will generate a finite number of solutions on the interval $[x_0, X]$ provided that there are a finite number of critical points. Later we can extract those solutions which are closely reflecting the phenomenon which we have to model.

An approach to interval and fuzzy differential equations has been suggested in [7], based on differential inclusions.

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When $f(x, y)$ is the interval extension of a continuous function $f(x, p)$, $p \in \mathbb{R}$ and if we are interested in finding the *attainable set* for the differential inclusions $y' = f(x, y)$, $y(x_0) \in y_0$, $x \in [x_0, X]$, then we have a simpler algorithm based on the characterization Theorem 5 and the results shown in Theorem 6.

Let us consider y_0 be any interval. Let $f(x, y)$ be the interval extension of a real function $f(x, p)$. We use the same notation for both functions, and from the context we can identify them.

Algorithm 2. (Find the *attainable set* of a differential inclusion)

- Step 1. We find the points where the real function f changes its monotonicity w.r.t. p (if f is differentiable then we solve $\frac{\partial f}{\partial p} = 0$ and we find the critical points where monotonicity w.r.t. the second variable is changed). Let (x_1, y_1) be a critical point such that in $]x_0, x_1[$ there are no other critical points.

- Step 2. If f is increasing w.r.t. p on $[x_0, x_1] \times [y_0, y_1]$ we solve (11)

- Step 3. Else if f is decreasing w.r.t. p on $[x_0, x_1] \times [y_0, y_1]$ we solve (12)

- Step 4. If $x_1 < X$ then let $x_0 = x_1$ and $y_0 = y(x_1)$ and go to step 2.

- Else Stop.

This algorithm leads to the unique solution (*attainable set*) of the differential inclusion $y' = f(x, y)$, $y(0) \in y_0$.

It is easy to see that the proposed methods are very efficient from the numerical point of view, since for the local solutions we can use any standard algorithm.

5.2 Examples

The above algorithms were implemented in MATLAB. We have used MATLAB's standard ODE solver *ode45*, which is based on a Runge-Kutta (4,5) formula. Surely, any other solver could be used. Let us remark here that the critical points were in all the cases *a priori* determined. The critical points cannot be easily found on the run. The problem is that if we detect a critical point, due to the machine precision the algorithm finds further points which are close to the correct critical point as critical ones. Also, if we have a solution in the case (ii) with length decreasing asymptotically to zero, due to the machine precision the program detects them as critical points.

We will start with a simple example, which is easy to be solved analytically and we can compare the analytical solution to the numerical solution.

5.2.1 Example 1

Let us consider the interval differential equation

$$\begin{cases} y' = -y + [1, 2]x \\ y(0) = [0, 1] \end{cases} x \in [0, 4]. \tag{20}$$

We denote $y = [u, v]$, where u, v are real-valued functions. The systems (11) and (12) are respectively

$$\begin{cases} u' = -v + x \\ v' = -u + 2x \\ u(0) = 0 \\ v(0) = 1 \end{cases} \text{and} \begin{cases} u' = -u + 2x \\ v' = -v + x \\ u(0) = 0 \\ v(0) = 1 \end{cases} \tag{21}$$

This equation (20) has exactly two solutions. One of them starts with the case (i) of differentiability

$$y_1(x) = [2x - e^x + 2e^{-x} - 1, x + e^x + 2e^{-x} - 2]$$

and there are no critical points in the trajectory.

If we start with case (ii) then we have a critical point of type II at $x = 1$ (i.e., $y(1)$ is a singleton). In this case we have to switch to case (i) of differentiability since the (ii)-differentiable solution does not exist. We obtain

$$u(x) = \begin{cases} 2x + 2e^{-x} - 2 & \text{if } 0 \le x \le 1 \\ 2x - e^{x-1} + 2e^{-x} - 1 & \text{if } 1 \le x \end{cases}$$

$$v(x) = \begin{cases} x + 2e^{-x} - 1 & \text{if } 0 \le x \le 1 \\ x + e^{x-1} + 2e^{-x} - 2 & \text{if } 1 \le x \end{cases}$$

The analytic solution and the numerical solutions obtained by the proposed algorithms are shown in Figs 2. and 3. respectively. We can see that the proposed method is very accurate. The error is controlled by the precision of *ode45* algorithm in MATLAB, and it is less than 10^{-6}.

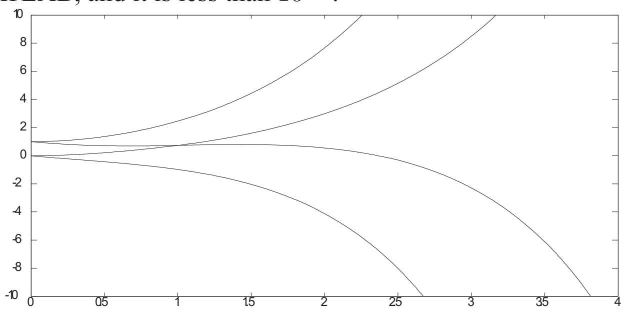

Figure 2. Analytic (exact) solutions of the IDE (20)

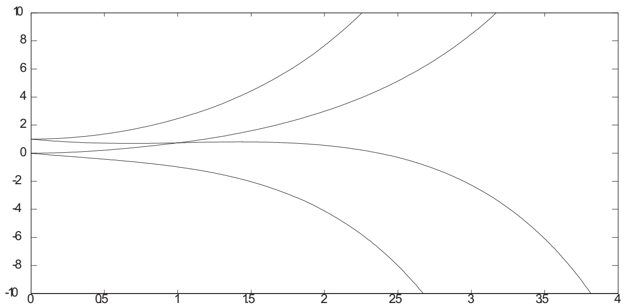

Figure 3. Numerical solutions of (20) by the proposed *Algorithm 1*.

5.2.2 Example 2

The structure of solution of an interval DE can be well illustrated on the problem

$$\begin{cases} y' = y \sin x + 2x, \\ y(0) = [1, 3] \end{cases} \tag{22}$$

According to the characterization theorems it can be written as

$$\begin{cases} u'(x) = u(x)\sin(x) + 2x \\ v'(x) = v(x)\sin(x) + 2x \end{cases}$$

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or

$$\begin{cases} u'(x) = v(x)\sin(x) + 2x \\ v'(x) = u(x)\sin(x) + 2x \end{cases}.$$

The *l-critical points* are at $x = k\pi$, $k \in \mathbb{Z}$. If we solve the problem in the interval $[0, X]$ with $X < 2\pi$ we have only one *l-critical point* in the interval. Then we may have *four* solutions, as illustrated in Figure 4.

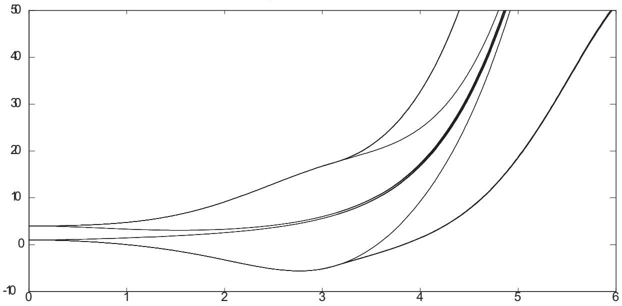

Figure 4. The four solutions of IDE (22) on the interval $[0, 6]$

If we consider time intervals containing more critical points, we may have more solutions. In Figure 5. we illustrate the solutions found by *Algorithm 1* to IDE (22) for $x \in [0, 11]$.

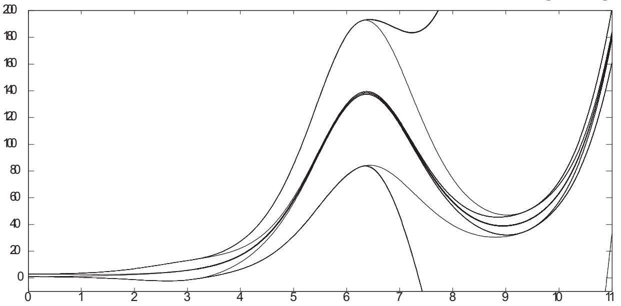

Figure 5. Existence of several solutions for IDE (22) on $[0, 11]$

5.2.3 Example 3

Regarding the second algorithm for solving differential inclusions when the function $f(x, y)$ is the interval extension of a crisp function we present the following problem

$$\begin{cases} y' = y \sin x + 2x, x \in [0, 6] \\ y(0) \in [1, 2]. \end{cases} \quad (23)$$

We observe that the function $y \sin x$ changes monotonicity at the points $k\pi$, $k \in \mathbb{Z}$. Also, we observe that in the interval $[0, \pi]$ it is increasing w.r.t. y. Using the *Algorithm 2* proposed for differential inclusions we obtain the solution presented in Figure 6.

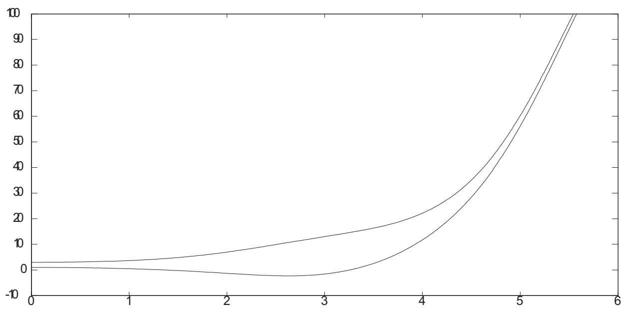

Figure 6. Solution of differential inclusion (23) using the proposed *Algorithm 2*

Remark: It is interesting to remark that the solution (attainable set) of the differential inclusion (23) is obtained by solving one of the ordinary differential equations (11) or (12) in steps 2. and 3. derived from the use of the generalized Hukuhara derivative.

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