Phi-Calculus fuzzy arithmetic in control: Application to model based control

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Abstract*—This paper aims at applying a fuzzy arithmetic of intervals calculus and fuzzy quantities to automatic control. This one called φ–calculus fuzzy arithmetic is more practical than the extension principle one and α-cut based methods. It comes from a different representation of fuzzy numbers. The present paper follows up work in introducing. The present paper is interested in its use for a fuzzy internal model control scheme based.*

Keywords*—* Fuzzy arithmetic, fuzzy model inversion, fuzzy number, weighted fuzzy fusion, model based control.

1 Introduction

The fuzzy set theory elaborated by L.A. Zadeh [1] has been shown to be used in the characterization of fuzziness and/or uncertainty using a coherent mathematical model. Various applications came at hand, in particular in the fuzzy control area [2, 3]. The uncertain calculus was undergoing quite a boom these last years. Several works exhibit interesting results, often referring to intervals theory [4, 5] or arithmetic based on fuzzy numbers such as Triangular or Trapezoidal Fuzzy Numbers [6, 7]. In the fuzzy arithmetic case, the different approaches are generally based either on Zadeh's extension principle [1, 6], either on fuzzy relation [8], or finally use the α-cuts [9]. However, no general approach allowing common arithmetic operations to be used on fuzzy numbers is available. This work proposes the use of a so-called φ–calculus fuzzy arithmetic based on a "different" modeling of fuzzy numbers [10, 11]. For this algebra, the modeling of fuzzy numbers is considered through the distribution function instead of the classical membership function. The first part presents some generalities on this φ–calculus arithmetic.

One of the main interests of this algebra is to provide some nice properties for what is called "exact calculus" with fuzzy numbers. These properties can be used in order to invert a fuzzy model. Therefore, the second part presents a possible application in control to fuzzy internal model control scheme.

2 Fuzzy arithmetic: φ-calculus

In the literature, many modeling approaches of imprecision, which is involved in many applications and domains, use fuzzy numbers and fuzzy arithmetic. In many works, the methods have led to the development of various membership functions for representing fuzzy numbers. Fuzzy numbers are often represented in applications by LR fuzzy sets and in particular, triangular and trapezoidal fuzzy sets. There exist two approaches for fuzzy arithmetic, on one hand the Zadeh's extension principle, on the other hand, the alpha-cuts and intervals arithmetic. Our works concern the first case, using a new modeling for fuzzy numbers [11], and it allows including a large part of the results existing in the domain of arithmetic [4, 6, 8, 12].

2.1 Fuzzy numbers modeling

Usually, a fuzzy number is modeled by its membership function $\mu_{\tilde{a}}$ not null on a bounded set $Supp(\tilde{a}) \subset \mathbb{R}$. For example, a triangular fuzzy number (TFN) $\tilde{a}$ can be represented by the shorthand symbol (b, m, c) with $\mu_{\tilde{a}}(x) = 1$ for $x = m$ (mode) and the kernel defined by the interval $[m-b, m+c]$.

Herein, instead of the classical membership function, the modeling used for the representation of fuzzy numbers is based on the distribution function $\varphi_{\tilde{a}}$ defined by the following expression:

$$\varphi_{\tilde{a}}(x) = \frac{\int_{-\infty}^{x} \mu_a(t) \cdot dt}{\int_{-\infty}^{+\infty} \mu_a(t) \cdot dt} \quad (1)$$

Convergence of $\int_{-\infty}^{+\infty} \mu_a(t) \cdot dt$ (finite cardinality of $\tilde{a}$) is assumed by considering only membership functions on a compact support $I \subset \mathbb{R}$. Thus, the major interest is that the distribution function for all fuzzy number $\tilde{a}$ is always an increasing monotone function, from $Supp(\tilde{a})$ to $[0,1]$. Thus, an inverse function $\varphi_{\tilde{a}}^{-1}$ from $[0,1]$ to $Supp(\tilde{a}) = [\underline{a}, \overline{a}]$ can always be defined and its definition is very important for the operations. The set of fuzzy numbers represented by a distribution function is noted Φ.

It should be noticed that the distribution function for a singleton is also a singleton and the one for an interval is a line between its bounds.

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2.2 *Fuzzy realization and extension*

Similarly to defuzzification and fuzzification concepts, it is necessary to define a relation between a crisp value and a fuzzy number. Therefore, an application from Φ to $\mathbb{R}$, which associates a crisp number to a fuzzy number, is called *fuzzy realization*. Conversely, an application from $\mathbb{R}$ to Φ is called *fuzzy extension*.

The choice of a realization depends obviously on the application. The most frequently encountered are:

- The median realisation noted $R_{med}(\tilde{a})$, it associates to a distribution function $\varphi_{\tilde{a}}$ the number a_0 such as $\varphi_{\tilde{a}}(a_0)=0.5$. Through this median realization, the following equivalence relation $\mathfrak{R}$ in Φ can be defined: the set of the functions $\varphi_{\tilde{a}}$ such as $\varphi_{\tilde{a}}(a_0)=0.5$ defines the class of equivalence of $a_0 \in \mathbb{R}$. Every element of this class will be called "fuzzy a_0", and denoted $\tilde{a}_0$.
- The modal realisation noted $R_{\mathrm{mod}}(\tilde{a})$ is defined as: $a_0 = R_{\mathrm{mod}}(\tilde{a}) \Leftrightarrow \varphi_{\tilde{a}}(a_0) = \underset{x\in\tilde{a}}{Max}\left[\mu_{\tilde{a}}(x)\right]=1$.
- The mean realization noted $R_{mean}(\tilde{a})$ is defined as: $a_0 = R_{mean}(\tilde{a}) = \int_0^1 \varphi_{\tilde{a}}^{-1}(y).dy$.

2.3 *arithmetic operations*

Consider two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ using their distribution $\varphi_{\tilde{a}}$ and $\varphi_{\tilde{b}}$. For a fuzzy number $\tilde{a}$ whose support includes 0 let us also define its "negative" $\varphi_{\underline{a}}$ and "positive" $\varphi_{\overline{a}}$ parts as $\varphi_{\tilde{a}} = \varphi_{\underline{a}} + \varphi_{\overline{a}}$ with:

$$\varphi_{\underline{a}}(x)=\begin{cases}0 & \text{if } x<\underline{a}\\ \varphi_{\tilde{a}}(x) & \text{if } x\in[\underline{a},0]\\ 1 & \text{if } x>0\end{cases}, \qquad (2)$$

$$\varphi_{\overline{a}}(x)=\begin{cases}0 & \text{if } x<0\\ \varphi_{\tilde{a}}(x) & \text{if } x\in[0,\overline{a}]\\ 1 & \text{if } x>\overline{a}\end{cases} \qquad (3)$$

The classical arithmetic operations on fuzzy numbers (addition $\tilde{a}+\tilde{b}$, pseudo-opposite $-\tilde{a}$, subtraction $\tilde{a}-\tilde{b}$, multiplication $\tilde{a}\times\tilde{b}$, pseudo-inverse only for a non mixed-type fuzzy number $\tilde{b}^{-1}=1/\tilde{b}$ and division $\tilde{a}/\tilde{b}$) are defined hereinafter [9].

$$\begin{aligned}
&\varphi_{\tilde{a}+\tilde{b}}^{-1}(x)=\varphi_{\tilde{a}}^{-1}(x)+\varphi_{\tilde{b}}^{-1}(x)\\
&\varphi_{-\tilde{a}}(x)=1-\varphi_{\tilde{a}}(-x)\ \forall x\in\mathbb{R}\\
&\varphi_{\tilde{a}-\tilde{b}}^{-1}(x)=\varphi_{\tilde{a}}^{-1}(x)+\varphi_{-\tilde{b}}^{-1}(x)\\
&\varphi_{\tilde{a}\times\tilde{b}}^{-1}(x)=\varphi_{\underline{\tilde{a}\times\tilde{b}}}^{-1}(x)+\varphi_{\overline{\tilde{a}\times\tilde{b}}}^{-1}(x)\\
&\varphi_{\tilde{a}^{-1}}(x)=1-\varphi_{\tilde{a}}(1/x)\ \forall x\in\mathbb{R}^*\\
&\varphi_{\tilde{a}/\tilde{b}}^{-1}(x)=\varphi_{\tilde{a}\times\tilde{b}^{-1}}^{-1}(x)
\end{aligned} \qquad (4)$$

with $\varphi_{\underline{\tilde{a}\times\tilde{b}}}^{-1}=\min\left[\varphi_{\underline{a}}^{-1}\times\varphi_{\overline{b}}^{-1},\varphi_{\overline{a}}^{-1}\times\varphi_{\underline{b}}^{-1}\right]$ and

$\varphi_{\overline{\tilde{a}\times\tilde{b}}}^{-1}=\max\left[\varphi_{\underline{a}}^{-1}\times\varphi_{\underline{b}}^{-1},\varphi_{\overline{a}}^{-1}\times\varphi_{\overline{b}}^{-1}\right]$.

Let us notice that these four classical operations are compatible with the Moore's interval operations.

2.4 *Weighted fuzzy fusion operator*

This part deals with a fusion operator introduced within the framework of this Φ-calculus algebra. This operator is called the weighted fuzzy fusion (WFF) [13] and is useful for the next sections.

Consider two fuzzy numbers $\tilde{a},\tilde{b}\in\Phi$ and their distribution $\varphi_{\tilde{a}},\varphi_{\tilde{b}}: x\in\mathbb{R}\rightarrow y\in[0,1]$. The weighted fuzzy fusion (WFF) of the two fuzzy numbers is defined as:

$$\varphi_{WFF}(x)=\frac{p_{\tilde{a}}\cdot\varphi_{\tilde{a}}(x)+p_{\tilde{b}}\cdot\varphi_{\tilde{b}}(x)}{p_{\tilde{a}}+p_{\tilde{b}}},\quad \forall x\in\mathbb{R} \qquad (5)$$

with $p_{\tilde{a}},p_{\tilde{b}}\in\mathbb{R}$ and $p_{\tilde{a}}+p_{\tilde{b}}=1$.

3 Internal model control

3.1 *Principle of the structure*

As with any open-loop control scheme, the internal model control (IMC) structure applies only on stable systems with a minimum phase behavior, according to Fig. 1.

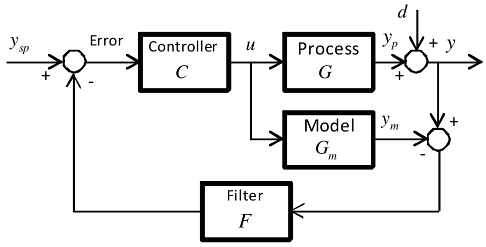

Figure 1: Internal model control scheme.

The filter F is introduced in order to filter out the measurement noise and to introduce some robustness in the loop. Its static gain is 1. Classically for linear models, it can be shown that the output equation y_k (Fig. 1) is given by:

$$y_k = F_{FB}(q^{-1})\cdot y_{sp_k} + S_{yd}(q^{-1})\cdot d_k \qquad (6)$$

with:

$$F_{FB}(q^{-1})=\frac{G(q^{-1})C(q^{-1})}{1+C(q^{-1})F(q^{-1})\left[G(q^{-1})-G_m(q^{-1})\right]}$$

$$S_{yd}(q^{-1})=\frac{1-G_m(q^{-1})C(q^{-1})F(q^{-1})}{1+C(q^{-1})F(q^{-1})\left[G(q^{-1})-G_m(q^{-1})\right]}$$

and the control law equation is given by:

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$$u_k = \frac{C\left(q^{-1}\right)}{1-C\left(q^{-1}\right)F\left(q^{-1}\right)G_m\left(q^{-1}\right)}\left(y_{c_k} - F\left(q^{-1}\right)y_k\right) \quad (7)$$

The minimum purpose of the internal model structure is to guarantee closed-loop stability and disturbance rejection, i.e. $F_{FB}(1)=1$ and $S_{yd}(1)=0$.

Recall also that the internal model control performs a kind of model simplification. Therefore its extension to fuzzy internal control needs the definition of an inverse model. The next part treats this issue.

3.2 Model inversion

Let us consider a stable time-invariant controllable SISO linear process model whose mathematical description is given in the form of recurrent equation:

$$y_k = -\sum_{i=1}^{N}\left(a_i \cdot y_{k-i} + b_i \cdot u_{k-i}\right) \quad (8)$$

where y_k and u_k correspond to the process output and input at sample k, a_i, b_i, $i \in \{1,\dots,N\}$, denote the model parameters.

Suppose now that these parameters are fuzzy numbers, $\tilde{a}_i, \tilde{b}_i$, $i \in \{1,\dots,N\}$ leading to:

$$y_k = -\sum_{i=1}^{N}\left(\tilde{a}_i \cdot y_{k-i} + \tilde{b}_i \cdot u_{k-i}\right) \quad (9)$$

For model inversion, the following question arises [14]:

knowing the description of uncertainties for the model, is it possible to synthesize a controller, based on the inverse model, able to maintain the model output within a tolerance envelope around the exact trajectory y_{sp}?

Which means:

$$\text{Model output} \in \left[y_{sp} - \Delta,\ y_{sp} + \Delta\right] \quad (10)$$

where Δ is the accepted tolerance around the nominal trajectory.

To answer to this question, this work will use an approach developed in [14]. Notice that the "fuzzy" part of the uncertainties is not really taken into account, i.e. this problem could be solved using interval techniques, for example [15, 16, 17]. Nevertheless, the arithmetic provided therein gives an "easy" way to solve non exact calculus by means of probability distribution.

To compute the control law u_k at sample k, the terms defined at previous samples $\{y_{k-1}, i=1,\dots,N\}$ and $\{u_{k-1}, i=1,\dots,N\}$ are known. As $b_1 \neq 0$, equation (6) can be rewritten as:

$$u_k = \frac{1}{b_1}\left(y_{k+1} + \sum_{i=1}^{N}\left(a_i \cdot y_{k-i+1}\right) - \sum_{i=2}^{N}\left(b_i \cdot u_{k-i+1}\right)\right) \quad (11)$$

Or, considering the unknown term at sample k, y_{k+1}, expression (9) can be rewritten as follow:

$$u_k = \frac{1}{b_1}\left(y_{k+1} + \psi\left(z(k)\right)\right) \quad (12)$$

with

$$\psi\left(z(k)\right) = \sum_{i=1}^{N} a_i \cdot y_{k-i+1} - \sum_{i=2}^{N} b_i \cdot u_{k-i+1} \text{ and}$$

$$z(k) = \left[y_k \ \dots \ y_{k-N+1} \ u_{k-1} \ \dots \ u_{k-N+1}\right]^T \in \mathbb{R}^{2N-1}.$$

For a linear model without time delay, if y_{k+1} is replaced with the desired reference trajectory the ideal result is: $y_{k+1} = y_{sp_k}$. Therefore, with a perfect compensation, the model output follows the desired trajectory with a pure time delay corresponding to one sample.

Going back to the question (cf. (10)), we want to guarantee that $y_{k+1} \in \left[\underline{y}_{k+1}, \overline{y}_{k+1}\right]$. With equations (11) and (12), for a fuzzy model (9), we have straightforwardly:

$$\tilde{u}_k = \frac{1}{\tilde{b}_1}\left(\tilde{y}_{k+1} + \tilde{\psi}\left(\tilde{z}(k)\right)\right) \quad (13)$$

$$\tilde{\psi}\left(\tilde{z}(k)\right) = \sum_{i=1}^{N} \tilde{a}_i \cdot \tilde{y}_{k-i+1} - \sum_{i=2}^{N} \tilde{b}_i \cdot \tilde{u}_{k-i+1} \text{ and}$$

$$\tilde{z}(k) = \left[\tilde{y}_k \ \dots \ \tilde{y}_{k-N+1} \ \tilde{u}_{k-1} \ \dots \ \tilde{u}_{k-N+1}\right]^T \in \mathbb{R}^{2N-1}.$$

Equation (13) gives a model inversion description using fuzzy numbers. The term $\tilde{y}_{k+1}$ is replaced with a desired trajectory defined by a fuzzy number $\tilde{y}_{sp_k}$ to obtain a causal control law. For example, if $\tilde{y}_{sp_k}$ is a triangular fuzzy number (TFN) written as:

$$\tilde{y}_{sp_k} = triple\left(y_{sp} - \Delta,\ y_{sp},\ y_{sp} + \Delta\right) \quad (14)$$

the control law is:

$$\tilde{u}_k = \frac{1}{\tilde{b}_1}\left(\tilde{y}_{sp_k} + \tilde{\psi}\left(\tilde{z}(k)\right)\right) \quad (15)$$

$$\tilde{\psi}\left(\tilde{z}(k)\right) = \sum_{i=1}^{N} \tilde{a}_i \cdot \tilde{y}_{sp_k-i+1} - \sum_{i=2}^{N} \tilde{b}_i \cdot \tilde{u}_{k-i+1} \text{ and}$$

$$\tilde{z}(k) = \left[\tilde{y}_{sp_k} \ \dots \ \tilde{y}_{sp_k-N+1} \ \tilde{u}_{k-1} \ \dots \ \tilde{u}_{k-N+1}\right]^T \in \mathbb{R}^{2N-1}.$$

Of course, the vector $\tilde{z}(k)$ depends on the control part meaning that all $\tilde{y}_i$ are replaced with desired set point $\tilde{y}_{sp_i}$ which corresponds to the schematic diagram of Fig. 2.

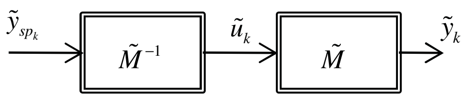

Figure 2: Fuzzy model inversion.

Naturally, perfect cancellation, i.e. $\tilde{y}_{k+1} = \tilde{y}_{sp_k}$ is not possible. Moreover using arithmetic of fuzzy numbers, or intervals or φ–calculus will introduce over-estimations. To illustrate this important point, an example issued from [14] is discussed hereinafter.

Let us consider the second-order:

$$\tilde{y}_k = -\tilde{a}_1 \cdot \tilde{y}_{k-1} - \tilde{a}_2 \cdot \tilde{y}_{k-2} + \tilde{b}_1 \cdot \tilde{u}_{k-1} + \tilde{b}_2 \cdot \tilde{u}_{k-2} \quad (16)$$

where $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$ and $\tilde{b}_2$ are fuzzy numbers defined as:

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$$\begin{aligned}\tilde{a}_1 &= triple(-0.6, -0.55, -0.5)\\ \tilde{a}_2 &= triple(0.05, 0.1, 0.15)\\ \tilde{b}_1 &= triple(0.6, 0.625, 0.65)\\ \tilde{b}_2 &= triple(0.15, 0.2, 0.25)\end{aligned} \tag{17}$$

The desired trajectory is:

$$y_{sp_k} = 0.5 \cdot \left(\sin\left(\frac{2\pi k}{50}\right) + \sin\left(\frac{2\pi k}{75}\right) \right) \tag{18}$$

and $\tilde{y}_{sp_k}$ is chosen as

$$\tilde{y}_{sp_k} = triple\left(y_{sp_k} - 1, y_{sp_k}, y_{sp_k} + 1 \right) \tag{19}$$

Using (15), the evolutions of the set point $\tilde{y}_{sp_k}$ envelopes (minimum $\underline{y}_{sp_k}$ and maximum $\overline{y}_{sp_k}$ of $Supp\left(\tilde{y}_{sp_k}\right)$) and the output model $\tilde{y}_k$ envelopes are illustrated in Fig. 3. Obviously, like interval calculus or α-cut fuzzy arithmetic, it clearly appears that the φ–calculus arithmetic generates overestimated results.

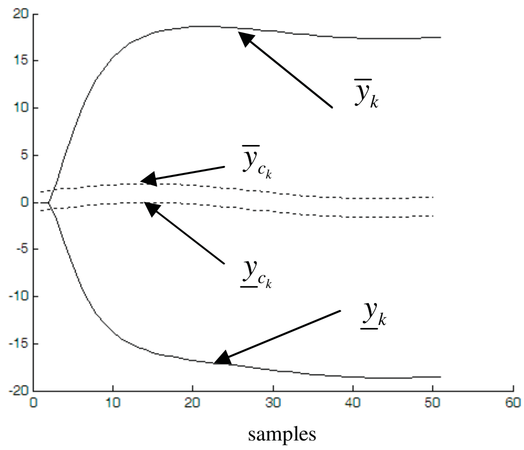

Figure 3: Set point $\tilde{y}_{sp_k}$ and output model $\tilde{y}_k$ envelopes.

We claim that this overestimation is not justified in such cases. Therefore, we need to introduce some extra constraints in order to take into account some knowledge and to reduce the pessimism of the results.

3.3 "Exact" inverse computation

Equation (15) corresponds to resolution of a fuzzy affine equation:

$$\tilde{b} \times \tilde{x} + \tilde{a} = \tilde{y} \tag{20}$$

The idea developed in [15] is to exactly solve this equation according to the two following steps.

Step1: Solve (20) with respect to $\tilde{d} = \tilde{b} \times \tilde{x}$.

The objective is to compute an "acceptable" solution $\tilde{d}$ of the equation $\tilde{d} + \tilde{a} = \tilde{y}$; i.e. the solution being exact when possible and approximated otherwise.

Notice that the exact solution can always be computed according to the arithmetic used:

$$\varphi_{\tilde{d}}^{-1}(u) = \varphi_{\tilde{y}}^{-1}(u) - \varphi_{\tilde{a}}^{-1}(u) \tag{21}$$

The problem arises when the result $\varphi_{\tilde{d}}^{-1}(u)$ is not an increasing function, resulting to $\tilde{d} \notin \Phi$. Nevertheless we can provide an approximate solution to the problem $\tilde{d}_{app} \in \Phi$.

The fact of proposing an alternative to exact calculus is the key idea. Contrary to what generally authors accept, i.e. no solution for interval arithmetic or α-cut fuzzy arithmetic cases [14], we provide an approximate solution based on the initial $\varphi_{\tilde{d}}^{-1}(u)$.

For that purpose, we propose to use the weighted fuzzy fusion (WFF) operator on the set of pairs $\left(\varphi_{\tilde{d}}^{-1}(u_i), u_i\right)$, $i \in \{1, \ldots, N\}$. It results in a re-ordering of the pairs that constructs an increasing function $\tilde{d}_{app}$, thus $\tilde{d}_{app} \in \Phi$ is a feasible solution to the problem.

Step2: Finding a solution to equation $\tilde{d} = \tilde{b} \times \tilde{x}$ with $0 \notin Supp\left(\tilde{b}\right)$. If $\tilde{b} > 0$ and $\tilde{d} > 0$, the exact calculus uses a point by point division:

$$\varphi_{\tilde{x}}^{-1}(u) = \varphi_{\tilde{d}}^{-1}(u) / \varphi_{\tilde{b}}^{-1}(u) \tag{22}$$

In the general case, we use a "positive-negative" decomposition (see section 2.3) for $\tilde{d}$ and $\tilde{x}$. We define:

$$\begin{cases} \varphi_{\tilde{d}}^{-1}(u) = \varphi_{\tilde{d}_-}^{-1}(u) + \varphi_{\tilde{d}_+}^{-1}(u) \\ \varphi_{\tilde{x}}^{-1}(u) = \varphi_{\tilde{x}_-}^{-1}(u) + \varphi_{\tilde{x}_+}^{-1}(u) \end{cases} \tag{23}$$

with $\varphi_{\tilde{d}_-}^{-1}(u)$, $\varphi_{\tilde{d}_+}^{-1}(u)$, $\varphi_{\tilde{x}_-}^{-1}(u)$ and $\varphi_{\tilde{x}_+}^{-1}(u)$ defined equations (3) and (4). At last, the exact solution is given by:

$$\begin{cases} if\ \tilde{b} > 0, \begin{cases} \tilde{x}_+ = \tilde{d}_+ / \tilde{b},\ \tilde{x}_- = -\left(\left(-\tilde{d}_-\right)/\tilde{b}\right), \\ \tilde{x} = \tilde{x}_+ + \tilde{x}_- \end{cases} \\ if\ \tilde{b} < 0, \begin{cases} \tilde{x}_+ = \tilde{d}_+ / \left(-\tilde{b}\right),\ \tilde{x}_- = -\left(\left(-\tilde{d}_-\right)/\left(-\tilde{b}\right)\right), \\ \tilde{x} = -\left(\tilde{x}_+ + \tilde{x}_-\right) \end{cases} \end{cases} \tag{24}$$

Once again the problem arises when the result $\varphi_{\tilde{x}}^{-1}(u)$ is not an increasing function, resulting to $\tilde{x} \notin \Phi$. Therefore approximate solution to the problem $\tilde{x}_{app} \in \Phi$ is generated in the same way using the WFF operator on the set of pairs $\left(\varphi_{\tilde{x}}^{-1}(u_i), u_i\right)$, $i \in \{1, \ldots, N\}$.

Consider an example to illustrate this second step with the fuzzy numbers $\tilde{b} = triple(1, 2, 2)$ and $\tilde{d} = triple(3, 3, 4)$. As shown Fig. 4(a), the exact solution $\tilde{x} \notin \Phi$, i.e. $\varphi_{\tilde{x}}^{-1}$ is not a monotone increasing function. Using the weighted fuzzy fusion (WFF) operator, an approximate solution $\tilde{x}_{app}$ is generated by re-ordering $\varphi_{\tilde{x}}^{-1}$. Fig. 4(b) shows the comparison between $\tilde{d}$ and $\tilde{b} \times \tilde{x}_{app}$. In a sense, the difference between these two fuzzy numbers exhibits the pessimism induced by the method when no exact solution in Φ is available. This pessimism can be shown comparing the supports of the numbers:

$$Supp\left(\tilde{d}\right) = [3, 4] \subset Supp\left(\tilde{b} \times \tilde{x}_{app}\right) = [1.87, 6].$$

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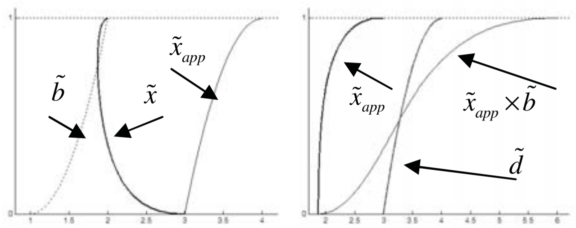
The next example demonstrates very clearly the interest of the complete algorithm used to find solutions to the problem $\tilde{b}\times\tilde{x}+\tilde{a}=\tilde{y}$. Consider again the inverse model of section 3.2 with the fuzzy numbers (17) and the desired trajectory (18), (19). Fig. 5 shows the same trial as for Fig. 3. For this test, we obtain the result considering, i.e. $\tilde{y}_{k+1}=\tilde{y}_{sp_k}$. This result indicates that an "exact solution" has been found at each time k.



a) « Exact » result b) φ–calculus result

Figure 4: Results with the numbers $\tilde{b}$ and $\tilde{d}$.

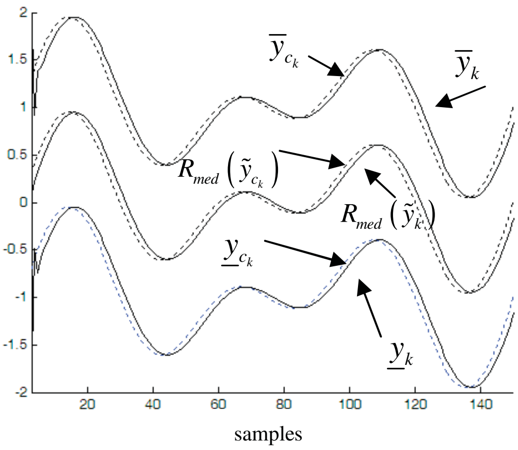


Figure 5: Set point $\tilde{y}_{sp_k}$ and output model $\tilde{y}_k$ envelopes.

In order to show the interest of the method, we will change the constraints level. To do so, the values of model uncertainties (17) are increased in order to deal with cases where an exact inversion is not always possible. New bounds for the fuzzy numbers (17) are defined. They correspond to the greatest possible values with pole-zero assignment located inside the unit circle – for evident stability purpose. Moreover, the fuzzy numbers are not more symmetric and correspond to:

$$\begin{aligned}\tilde{a}_1&=triple(-0.825,-0.4,0.3)\\ \tilde{a}_2&=triple(-0.175,0.25,0.95)\\ \tilde{b}_1&=triple(0.415,0.625,0.835)\\ \tilde{b}_2&=triple(-0.01,0.2,0.41)\end{aligned}\tag{25}$$

Fig. 6 repeats the same trial with the desired trajectory (18), (19). The figure exhibits that an exact simplification, i.e. $\tilde{y}_{k+1}\neq\tilde{y}_{sp_k}$ is not always possible. Contrary to have no solution in such cases, the proposed approximate solutions seem perfectly adapted.

With the help of this inverse model control strategy based on constraints, justified in the control framework, it is now possible to consider a fuzzy internal model control.

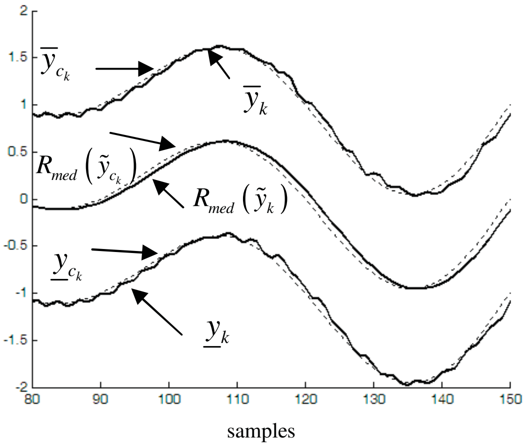


Figure 6: Set point $\tilde{y}_{sp_k}$ and output model $\tilde{y}_k$ envelopes.

3.4 Fuzzy internal model control

From Fig. 6 showing IMC scheme and different equations (6) and (7), it is possible to define several solutions.

First solution: Let $G_m(q^{-1})=q^{-r}G_m^+(q^{-1})$, $G_m^+(q^{-1})$ corresponds to function with an inverse (exactly proper and stable). We chose $C(q^{-1})=\dfrac{1}{G_m^+(q^{-1})}$ and propose the equivalent model given Fig. 7.

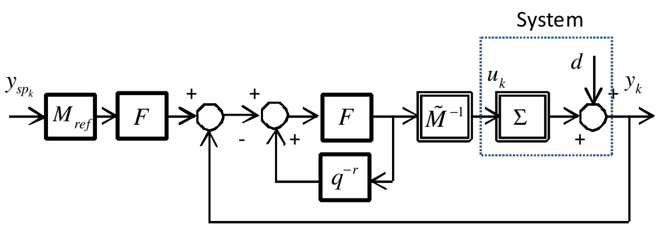


Figure 7: IMC1, equivalent scheme.

Indeed, equation (7) can be rewritten as follow:

$$u_k=\frac{1}{G_m^+(q^{-1})(1-q^{-r}\cdot F(q^{-1}))}\left(y_{c_k}-F(q^{-1})y_k\right)\tag{26}$$

By analogy, we have the fuzzy inverse model $\left[\tilde{G}_m^+(q^{-1})\right]^{-1}$ multiplied by linear transfer function $\dfrac{F(q^{-1})}{1-q^{-r}\cdot F(q^{-1})}$.

For every case, it is possible to add a model reference control M_{ref} in order to attenuate the control signal.

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3.5 Example

Consider a nominal second-order transfer function:

$$\frac{y(q^{-1})}{u(q^{-1})}=\frac{B(q^{-1})}{A(q^{-1})}=\frac{0.0233\cdot q^{-1}+0.0197\cdot q^{-2}}{1-1.5729\cdot q^{-1}+0.6037\cdot q^{-2}} \quad (27)$$

and the following definition for the fuzzy numbers:

$$\begin{aligned}\tilde{a}_1&=triple(-1.574,-1.573,-1.497)\\ \tilde{a}_2&=triple(0.602,0.603,0.652)\\ \tilde{b}_1&=triple(0.013,0.023,0.033)\\ \tilde{b}_2&=triple(0.01,0.02,0.03)\end{aligned} \quad (28)$$

The control structure used is the IMC1 Fig. 7 with a reference model defined as:

$$\frac{B_m(q^{-1})}{A_m(q^{-1})}=\frac{0,1784\cdot q^{-1}+0,1071\cdot q^{-2}}{1-0,9315\cdot q^{-1}+0,2169\cdot q^{-2}} \quad (29)$$

Fig. 8 shows the results for different set points. The first set-point leads to results around the nominal model (27) (represented by the value 2). Secondly, two successive variations are made (set-point value of 3 and set-point value of 4), lastly a return to nominal position.

During both successive changes the model is changed in order to reach the bounds on $\tilde{a}_1$ and $\tilde{b}_2$. Two disturbances have been also added at samples 900 and 2300. The responses show a good robustness according to the parametric variations of the model.

This work presents a first possible track to use fuzzy arithmetic for control purpose. Next steps would be to prove stability and evaluate the robustness of these approaches. We can think to several possibilities including Lyapunov approach that are insightfully used for Takagi-Sugeno models [18] or Kharitonov polynomials [19].

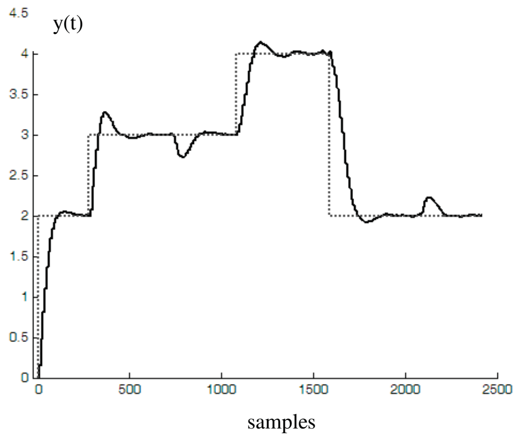

Figure 8: IMC1 results.

4 Conclusions

This paper attempts to show possible applications of the φ–calculus arithmetic for automatic control. As a first step, a new characterization of fuzzy numbers by the distribution function instead of the classical membership function has been presented. For this algebra, it is possible to use a set of arithmetic operators (addition, opposite and subtraction, multiplication, inverse and quotient) compatible with the classical algebra by using the median realization, or the Moore's calculus by using the fuzzy number support. A procedure allowing an "exact" calculus of an inverse model was proposed on the basis of φ–calculus. This approach was successfully implemented on internal model control structures.

Thus, the application of φ–calculus arithmetic to the stability of system seems to open some new prospects for other basic concepts (observation, stabilization…).

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