Fuzzy Transforms, Korovkin Theorems and the Durrmeyer Operator

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Abstract— *In the present paper a Korovkin-type theorem is proposed for the approximation operators defined by the inverse F-transforms. These results allow us to choose between a variety of shapes to be used as atoms of the fuzzy partitions used within the F-transform's framework. In this way we can enlarge considerably the class of F-transforms proposed recently by I. Perfilieva. The new fuzzy partitions are shown to include, for example the Bernstein basis polynomials. The F-transform considered with the Bernstein basis polynomials as atoms of a fuzzy partition, is shown to be the classical Durrmeyer operator.*

Keywords— F-transform, Korovkin Theorems, Bernstein Polynomial, Durrmeyer operator.

1 Introduction

The F-transform was proposed by I. Perfilieva in [4] and studied in several papers [5], [7], etc. As starting point for the F-transforms, one considers first a fuzzy partition [6]. In the present paper we enlarge the class of F-transforms by considering arbitrary shapes and not necessarily small support for the atoms of the fuzzy partition used in the F-transform as an approximation method. To study the possibility of such extension we exploit the classical Korovkin theorems, [1] which are particularized in the present paper for the F-transform. The Korovkin Theorems are very important results in classical Approximation Theory, stating that the convergence of a sequence of positive linear approximation operators on a finite set of test functions implies the convergence of the operator for any function. There are two versions of the Korovkin type results. A qualitative and a quantitative version. The quantitative versions provide error estimates, beyond proving the convergence results of qualitative type. We will adopt the quantitative approach. Further-on in the present paper we consider the F-transforms based on Bernstein basis polynomials. It turns out that the F-transform that uses Bernstein basis polynomials as atoms for the fuzzy partition under discussion, leads to the classical Durrmeyer operators [2]. In this way, the theory of fuzzy approximation and especially approximation by inverse F-transforms can be embedded and studied, as a refined part of the classical theory of approximation.

In the present paper a fuzzy partition is defined as a finite sequence of fuzzy sets $A_i : [a,b] \to [0,1]$, $i = 1,...,k$, such that $\sum_{i=1}^{k} A_i(x) = 1$ for any $x \in [a,b]$. The following is a generalized version of the Fuzzy Transform (F-transform) proposed in [4], since the condition regarding the small support of the atoms of the fuzzy partitions occurring in the F-transform is released.

According to [4], the F-transform is given by

$$f_i = \frac{\int_a^b A_i(x) f(x)dx}{\int_a^b A_i(x)dx}, i = 1,...,k \tag{1}$$

and the discrete F-transform is given by

$$f_i = \frac{\sum_{j=1}^{n} A_i(x_j) f(x_j)}{\sum_{j=1}^{n} A_i(x_j)}, i = 1,...,k \tag{2}$$

where $x_j \in [a,b]$, $j = 1,...,n$ are given data. The inverse discrete F-transform is

$$F(x) = \sum_{i=1}^{k} A_i(x) f_i. \tag{3}$$

We denote by $C(K)$ the space of continuous functions $f : K \to \mathbb{R}$ with K a compact metric space. Then the space $C(K)$ is a Banach space with the uniform norm

$$\|f\| = \sup_{x\in K} |f(x)|.$$

If $K = [a,b]$ then we use the notation

$$C[a,b] = \{f : [a,b] \to \mathbb{R}, f \text{ continuous}\}.$$

We will consider inverse F-transform as an approximation operator. Let $\mathcal{L}(C(K))$ denote the space of linear operators of the form $T : C(K) \to C(K)$, $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, $\forall \alpha, \beta \in \mathbb{R}$ and $f, g \in C(K)$. An operator is said to be positive if $T(f(x)) \geq 0\ \forall x \in K$ whenever $f(x) \geq 0, \forall x \in K$. In the present paper positive linear operators play a very important role.

The inverse F-transform will be regarded in the present paper as a sequence of positive linear operators. Indeed, let us consider $F_{n,k} : C[a,b] \to C[a,b]$, given by $F_{n,k}(f)(x) = F(x)$, where $F(x)$ is given by (3) and f_i given by (1) or (2). Then, it is easy to see that $F_{n,k}$ is a linear operator, i.e., $F_{n,k} \in \mathcal{L}(C[a,b])$. Moreover, if we assume $f(x) \geq 0$ $\forall x \in [a,b]$, we can see that both the continuous and discrete F-transforms $f_i, i = 1,...,k$ satisfy $f_i \geq 0$, and as a consequence the inverse F-transform $F_{n,k}(f)(x) = F(x) \geq 0$ and as a conclusion $F_{n,k}$ is a positive linear operator.

In order to obtain an error estimate and the uniform convergency of the F-transforms that we intend to study, seen as approximation operators via a particularization of the Korovkin Theory related to positive linear operators. For this aim we need the modulus of continuity. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then $\omega(f,\cdot) : [0,\infty) \to [0,\infty)$, defined by

$$\omega(f,\delta) = \bigvee \{|f(x) - f(y)|; x, y \in [a,b],\ |x-y| \leq \delta\}$$

ISBN: 978-989-95079-6-8

is called the first order modulus of smoothness (modulus of continuity) of f.

2 Korovkin type theorem for the F-transform

Let us recall Korovkin's Theorem in what follows. We can see that it is directly applicable for the F-transform, since the F-transform is a positive linear operator, however, we will deduce later particularizations of the Korovkin results for the F-transform to better understand the F-transform method and to be able to provide error estimates for them.

Theorem 1 *(Korovkin, [3]) Let $L_n \in \mathcal{L}(C[a,b])$, $n = 1,2,...$ be a sequence of positive linear operators. Then there exists a finite set of test functions S, such that $\lim_{n\to\infty} L_n(s) = s$ for any $s \in S$ implies $\lim_{n\to\infty} L_n(f) = f$, for any $f \in C[a,b]$. The finite set S is called a Korovkin set.*

If we consider a sequence of positive linear operators on $C[0,1]$, then a Korovkin set (it is not unique) consists of $e_0, e_1, e_2,$ where $e_i(x) = x^i$, $i = 0,1,2$. If we consider $\mathcal{L}(C[-\pi,\pi])$ a Korovkin set consists of e.g., $e_0(x) = 1$, $e_1(x) = \sin x$ and $e_2(x) = \cos x$.

The adaptation of Korovkin Theorem for the F-transform is as follows.

Theorem 2 *(Korovkin Theorem for the discrete inverse F-transform) Let $F_{n,k} \in \mathcal{L}(C[a,b])$, $n,k = 1,2,...$ be a sequence of inverse F-transforms. The following general error estimate holds true for any $\delta > 0$:*

$$\|F_{n,k}(f) - f\|$$

$$\leq \left(1 + \frac{1}{\delta}\sqrt{e_2 - 2e_1 F_{n,k}(e_1) + F_{n,k}(e_2)}\right)\omega(f,\delta),$$

where $e_1(x) = x, e_2(x) = x^2$.

Proof. The inverse discrete F-transform can be expressed as

$$F_{n,k}(f)(x) = \sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)f(x_j)}{\sum_{j=1}^{n} A_i(x_j)}.$$

Let us observe that for the inverse F-transform we have $F_{n,k}(e_0) = e_0$, where $e_0(x) = 1\ \forall x \in [a,b]$, so $f(x)$ can be written as

$$f(x) = \sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)f(x)}{\sum_{j=1}^{n} A_i(x_j)}.$$

Then we get

$$|F(x) - f(x)| \leq \sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)\,|f(x) - f(x_j)|}{\sum_{j=1}^{n} A_i(x_j)}.$$

Further, using the properties of the modulus of continuity, by standard reasoning we have

$$|F(x) - f(x)|$$

$$\leq \left(1 + \frac{1}{\delta}\sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)|x - x_j|}{\sum_{j=1}^{n} A_i(x_j)}\right)\omega(f,\delta),$$

for any fixed $\delta > 0$. The error is controlled by the ratio

$$R_{n,k}(x) = \sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)|x - x_j|}{\sum_{j=1}^{n} A_i(x_j)}.$$

Using Cauchy-Schwarz inequality we have

$$\sum_{j=1}^{n} A_i(x_j)|x - x_i| \leq \left(\sum_{j=1}^{n} A_i(x_j)|x - x_j|^2\right)^{\frac{1}{2}}$$

$$\cdot\left(\sum_{j=1}^{n} A_i(x_j)\right)^{\frac{1}{2}},$$

and we obtain

$$R_{n,k}(x) \leq \sum_{i=1}^{k} A_i(x)\frac{\left(\sum_{j=1}^{n} A_i(x_j)|x - x_j|^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} A_i(x_j)\right)^{\frac{1}{2}}}$$

$$= \sum_{i=1}^{k} \sqrt{A_i(x)}\frac{\left(A_i(x)\sum_{j=1}^{n} A_i(x_j)|x - x_j|^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} A_i(x_j)\right)^{\frac{1}{2}}}.$$

Using Cauchy-Schwarz inequality again we obtain

$$\sum_{i=1}^{k} \sqrt{A_i(x)}\frac{\left(A_i(x)\sum_{j=1}^{n} A_i(x_j)|x - x_j|^2\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n} A_i(x_j)\right)^{\frac{1}{2}}}$$

$$\leq \left(\sum_{i=1}^{k} A_i(x)\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k}\frac{A_i(x)\sum_{j=1}^{n} A_i(x_j)|x - x_j|^2}{\sum_{j=1}^{n} A_i(x_j)}\right)^{\frac{1}{2}}.$$

Taking into account that $A_i(x)$ is a fuzzy partition we have

$$R_{n,k}(x) \leq \left(\sum_{i=1}^{k} A_i(x)\frac{\sum_{j=1}^{n} A_i(x_j)(x^2 - 2xx_j + x_j^2)}{\sum_{j=1}^{n} A_i(x_j)}\right)^{\frac{1}{2}}$$

$$= \left(x^2 - 2xF_{n,k}(e_1)(x) + F_{n,k}(e_2)(x)\right)^{\frac{1}{2}}$$

which completes the proof. ■

Corollary 3 *Let $F_{n,k} \in \mathcal{L}(C[a,b])$, $n,k = 1,2,...$ be a sequence of inverse F-transforms. If $\lim_{n,k\to\infty} F_{n,k}(e_i) = e_i$, $i = 1,2$, then $\lim_{n\to\infty} L_n(f) = f$, for any $f \in C[a,b]$.*

For the continuous case we have a similar result.

Theorem 4 *(Korovkin Theorem for the inverse F-transform) Let $F_k \in \mathcal{L}(C[a,b])$, $k = 1,2,...$ be a sequence of inverse F-transforms. The following general error estimate holds true for any $\delta > 0$:*

$$\|F_k(f) - f\| \leq \left(1 + \frac{1}{\delta}\sqrt{e_2 - 2e_1 F_k(e_1) + F_k(e_2)}\right)\omega(f,\delta),$$

where $e_1(x) = x, e_2(x) = x^2$.

Corollary 5 *Let $F_k \in \mathcal{L}(C[a,b])$, $k = 1,2,...$ be a sequence of continuous inverse F-transforms. If $\lim_{k\to\infty} F_k(e_i) = e_i$, $i = 1,2$, then $\lim_{n\to\infty} L_n(f) = f$, for any $f \in C[a,b]$.*

ISBN: 978-989-95079-6-8

We observe that the inverse F-transform can be expressed as

$$F_k(f)(x) = \sum_{i=1}^{k} A_i(x) \frac{\int_a^b A_i(x) f(x)dx}{\int_a^b A_i(x)dx}$$

so, similar reasoning to the previous Theorem can be followed.

The Korovkin type Theorem above shows that if we are able to control $\sqrt{e_2 - 2e_1 F_{n,k}(e_1) + F_{n,k}(e_2)}$ for the discrete case, or $\sqrt{e_2 - 2e_1 F_k(e_1) + F_k(e_2)}$ for the continuous case, for a given sequence of fuzzy partitions A_i, $i = 1, ..., k$, then we are able to control the error for any continuous function.

3 Example

Using the Korovkin-type result shown in the previous section we consider the F-transforms, with Bernstein basis polynomials used as a fuzzy partition this time. The proposed construction leads to the well-known Durrmeyer operator. Let us recall that the Bernstein basis polynomials are

$$p_{k,i}(x) = \binom{k}{i} x^i (1-x)^{k-i},$$

$k = 1, 2, ...$ $i = 0, 1, ..., k$ $x \in [0, 1]$.

The inverse F-transform is given by

$$F_k(f)(x) = \sum_{i=0}^{k} p_{k,i}(x) \frac{\int_0^1 p_{k,i}(x) f(x)dx}{\int_0^1 p_{k,i}(x)dx},$$

that is the classical Durrmeyer operator [2]. Let us calculate $F_k(e_1)$ and $F_k(e_2)$ in this case. It is well known that $p'_{k,.i}(x) = k(p_{k-1,i-1}(x) - p_{k-1,i}(x))$. Integrating we obtain $\int_0^1 p_{k-1,i-1}(x)dx = \int_0^1 p_{k-1,i}(x)dx$, i.e., for fixed k integrals of all Bernstein basis polynomials are the same. Since $\sum_{i=0}^{k} p_{n,i}(x) = 1$ we get $\int_0^1 p_{k,i}(x)dx = \frac{1}{k+1}$. By direct calculation

$$\int_0^1 p_{k,i}(x) x dx = \int_0^1 \binom{k}{i} x^{i+1}(1-x)^{k-i} dx$$

$$= \frac{i+1}{k+1} \int_0^1 p_{k+1,i+1} dx = \frac{i+1}{(k+1)(k+2)}.$$

Also,

$$\int_0^1 p_{k,i}(x) x^2 dx = \int_0^1 \binom{k}{i} x^{i+2}(1-x)^{k-i} dx$$

$$= \frac{i+1}{k+1} \frac{i+2}{k+2} \int_0^1 p_{k+2,i+2} dx = \frac{(i+1)(i+2)}{(k+1)(k+2)(k+3)}.$$

Then we obtain

$$f_i(e_1) = \frac{\int_0^1 p_{k,i}(x) x dx}{\int_0^1 p_{k,i}(x)dx}$$

$$= \frac{i+1}{k+2} \sim \mathcal{O}\left(\frac{i}{k}\right)$$

and

$$f_i(e_2) = \frac{\int_0^1 p_{k,i}(x) x^2 dx}{\int_0^1 p_{k,i}(x)dx}$$

$$= \frac{(i+1)(i+2)}{(k+2)(k+3)} \sim \mathcal{O}\left(\frac{i^2}{k^2}\right).$$

$$F_k(e_1)(x) \sim \sum_{i=1}^{k} p_{k,i}(x) \frac{i}{k} = B_k(e_1)(x) = x,$$

where B_k denotes the usual Bernstein operator

$$B_k(f)(x) = \sum_{i=1}^{k} p_{k,i}(x) f\left(\frac{i}{k}\right)$$

and it is well known that $B_k(e_1)(x) = x$. We also have

$$F_k(e_2)(x) \sim \sum_{i=1}^{k} p_{k,i}(x) \left(\frac{i}{k}\right)^2$$

$$= B_k(e_2)(x) = x^2 + \frac{x(1-x)}{k},$$

and finally we obtain: $\sqrt{e_2 - 2e_1 F_k(e_1) + F_k(e_2)} = \sqrt{\frac{x(1-x)}{k}}$ and by the above Korovkin type results we have

$$\|F_k(f) - f\| \le C\omega\left(f, \sqrt{\frac{x(1-x)}{k}}\right).$$

As a conclusion we observe that the F-transform is a generalization of the Durrmeyer operators. For the discrete F-transform with Bernstein basis polynomials a similar result can be proven.

4 Conclusion

Korovkin-type theorems for the discrete and continuous inverse F-transforms regarded as positive linear approximation operators were provided. As an application we have studied the F-transform with Bernstein basis polynomials as atoms of the underlying fuzzy partition. It turns out that the F-transform with Bernstein basis polynomials is the classical Durrmeyer operator. Surely one can imagine different fuzzy partitions as well, so the F-transforms can be seen as generalizations of the Durrmeyer operator.

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ISBN: 978-989-95079-6-8