Comparison of two versions of the Ferrers property of fuzzy interval orders

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Abstract— *We focus on the Ferrers property of fuzzy preference relations. We study the connection between the Ferrers property and fuzzy interval orders. A crisp total interval order is characterized by the Ferrers property of its strict preference relation. Also, a crisp preference structure is a total interval order if and only if its large preference relation satisfies the Ferrers property. For fuzzy relations the Ferrers property admits two non equivalent expressions. Here we compare both conditions by means of completeness. We also study if they characterize a fuzzy total interval order.*

Keywords— completeness, Ferrers property, partial interval order, total interval order, fuzzy relation

1 Introduction

Preference structures are the basis of the preference modelling theory. They formalize the answers of a decision maker over a set of alternatives. Formally, they are triplets of binary relations. For each pair of alternatives, any decision maker provides one (and only one) of the following answers: preference (by one of the alternatives), indifference (between both alternatives), inability to compare (the alternatives). These three situations lead to the three relations that make up a preference structure. Associated to a preference structure we find a relation called large preference relation that characterizes the preference structure from which it comes. Those four relations are the basis of the preference modelling ([9]).

Any preference structure can contain non-coherent relations if the decision maker gives conflicting answers. Different properties were formalized to model coherence. The Ferrers property is one of the most important ones. It has been proved to be a more realist condition than transitivity since the Ferrers property of the large preference relation does not imply the transitivity of the associated indifference relation (a non-realistic condition for some authors). However it does imply the transitivity of the strict preference relation. In a preference structure without incomparability, the large preference relation satisfies the Ferrers property if and only if the associated strict preference relation does.

A class of coherent preference structures are (total) interval orders: the alternatives can be identified with intervals not necessarily of the same length on the real line. A crisp preference structure without incomparabilities is a (total) interval order if and only if its associated large preference relation or its strict preference relation satisfy the Ferrers property.

Since several years ago, crisp relations are considered not flexible enough to model human decisions. In order to model real life situations in a more accurate way, fuzzy relations were introduced [10] and they have received a wide attention (see, e.g., [2, 3, 4, 6]). The notion of fuzzy preference structure was a topic of debate for several years. In this work we present a similar study for the notion of fuzzy interval order. This notion admits many different equivalent definitions in the classical context. We focus on the characterizations based on the Ferrers property. We first discuss the Ferrers property for fuzzy relations. This property admits two different ways to be formalized in our wider context. We compare them and we study their connection to fuzzy interval orders. In this context, a t-norm is needed to define fuzzy interval orders and the Ferrers property. Although we aim at considering any operator in our study, we prove that only two particular families have an appropriate behavior. Thus, we pay special attention to those operators. Since interval order is an important concept for the analysis of temporal events, each of which occurs over some time span (for instance, the time spans over which animal species are found or the occurrence of styles of pottery in archaeological strata), this kind of events in a fuzzy environment could be a potential application area for this study.

This work is organized as follows. In Section 2 we recall crisp relations. In Section 3 we formalize the two versions of the Ferrers property for fuzzy relations and we study their relationship. Section 4 is devoted to the completeness property. For crisp relations every reflexive Ferrers relation is complete. We study the implication for fuzzy relations. Section 5 contains the results on the connection between the first type of Ferrers property and fuzzy interval orders. Section 6 includes an analogous study for the second version of the Ferrers property. Section 7 contains some conclusions.

2 Crisp relations

2.1 Crisp preference structures

Let us consider a set of alternatives A whose elements we want to compare. In preference modeling the comparison is always carried out by pairs. When a decision maker consider two of those alternatives, the answer is always among the following ones: preference for one of the alternatives, indifference among both possibilities, inability to compare them. In the classical setting, each one of the three previous situations is identified with an $A \times A \to \{0, 1\}$ application or relation. The first case is identified with the *strict preference relation*, denoted by P. We state $P(a, b) = 1$ if the decision maker prefers a to b and 0 otherwise. The second case is identified with the *indifference relation* denoted I. We assign $I(a, b) = 1$ if a and b are indifferent and $I(a, b) = 0$ if not. The *incomparability relation* is denoted J. If the alternatives a and b cannot be compared by the decision maker, $J(a, b) = 1$. Otherwise, $J(a, b) = 0$. P, I and J make up a preference structure. Let A^2 be the set of ordered pairs

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((a,b) and (b,a) are different pairs) of alternatives in A. The converse or transpose of Q is defined as $Q^t(a,b) = Q(b,a)$, its complement as $Q^c(a,b) = 1 - Q(a,b)$ and its dual as $Q^d = (Q^t)^c$. Every relation is also identified with a subset of A^2: $Q(a,b) = 1 \Leftrightarrow (a,b) \in Q$. One easily verifies that if we think of P, I and J as subsets of A^2, the quadruplet (P, P^t, I, J) establishes a particular partition of A^2. Formally [9]:

Definition 1 *A preference structure on A is a triplet (P, I, J) of relations on A that satisfy:*

(i) P is irreflexive, I is reflexive and J is irreflexive;

(ii) P is asymmetrical, I and J are symmetrical;

(iii) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$;

(iv) $P \cup P^t \cup I \cup J = A^2$.

Every preference structure is identified with a unique reflexive relation called *large preference relation* $R = P \cup I$. It connects a to b if the decision maker considers a preferred or indifferent to b. This relation leads back to the preference structure in the following way:

$$(P, I, J) = (R \cap R^d, R \cap R^t, R^c \cap R^d)\,. \tag{1}$$

We say that a relation R is *complete* if aRb or bRa for all a, b in the set of alternatives A. In [8] it was proven that R is complete if and only if the associated J does not connect any pair of alternatives (a,b),

$$R \text{ complete} \quad \Leftrightarrow \quad J = \emptyset.$$

A relation R is a Ferrers relation (see among others [8, 9]) if for any four alternatives a, b, c, d in A, it holds that if aRb and cRd then aRd or cRb. Another way of expressing this property is the following one: if aRb and cRd and $c \not R b$ then aRd. This property is also called biorder by some authors.

Let us remark also that the Ferrers and the completeness properties are connected.

Lemma 1 *Every reflexive Ferrers relation is complete.*

Let us denote the composition of two binary relations Q_1 and Q_2 by $Q_1 \circ Q_2$. That is, $a(Q_1 \circ Q_2)b$ if and only if there exists c such that $aQ_1c \wedge cQ_2b$. Then the equivalent compositional definition of Ferrers property is $R \circ R^d \circ R \subseteq R$.

2.2 Crisp interval orders

Interval orders are one of the most important types of "coherent" preference structures. Formally, we distinguish between partial and total interval order, depending on the possibility or impossibility of incomparable elements.

Definition 2 *A* partial interval order *is a preference structure (P, I, J) such that $P \circ I \circ P \subseteq P$. In the particular case $J = \emptyset$, (P, I, J) is said to be a* total interval order.

As commented in the introduction, this original definition admits different ways of being expressed. The following properties (among others) were shown to be equivalent in [8]:

Proposition 1 *Let $(P, I, \emptyset)$ be a preference structure without incomparability and R its large preference relation. The following conditions are equivalent.*

a) (P, I, J) is an interval order.

b) P satisfies the Ferrers property.

c) R satisfies the Ferrers property.

These definitions are equivalent only if the decision maker is able to compare all the pairs of alternatives, i. e., if $J = \emptyset$, equivalently if R is complete. If we do not impose this condition, we can still consider the previous properties and study their relationship. In what remains of this section we carry out this study.
It is very easy to prove that whenever R satisfies the Ferrers property, P satisfies the Ferrers property and $P \circ I \circ P \subseteq P$. It also holds that for any preference structure (P, I, J), if P satisfies the Ferrers property then $P \circ I \circ P \subseteq P$, but the converse implications do not hold.

The following proposition summarizes the relationship among the conditions of Proposition 1 when we consider any preference structure.

Proposition 2 *Let (P, I, J) be a (crisp) preference structure and let R be its large preference relation. Then,*

i) (P, I, J) is a total interval order and R satisfies the Ferrers property are equivalent properties.

ii) (P, I, J) is a total interval order implies that P satisfies the Ferrers property. In addition to this, if P satisfies the Ferrers property then $P \circ I \circ P \subseteq P$. The converse implications to the ones showed in this item do not hold.

Propositions 1 and 2 can be outlined as we show in Figure 1, where the missing implications do not hold.

$$\begin{array}{ccc} (P, I, J) \text{ total int. ord.} & \Leftrightarrow & R \text{ Ferrers} \\ & \Downarrow \ \Updownarrow (J = \emptyset) & \\ & P \text{ Ferrers} & \\ & \Downarrow \ \Updownarrow (J = \emptyset) & \\ & (P, I, J) \text{ partial int. ord.} & \end{array}$$

Figure 1: Connection among the characterizations of a crisp interval order.

3 Fuzzy preference structures

All the relations involved in a (crisp) preference structure are crisp. Then they are not always appropriate to model human decisions. They do not include intensity of preference. The lack of flexibility in crisp set theory led to introduce fuzzy relations. They allow the decision maker to encode degrees of preference rather than just its absence or presence. Fuzzy relations were introduced as a natural extension of the concept of crisp binary relation. They have been widely studied. For a complete review about fuzzy relations see [6].

Formally, a fuzzy relation is defined as a function Q from the cartesian product $A \times A$ into the interval $[0, 1]$. For every pair $(a, b) \in A \times A$, the value $Q(a, b)$ shows the degree of truth of the fact that aQb in the crisp sense. The complementary, transpose (or converse) and dual of the binary relation Q are defined for any $(a, b) \in A \times A$ as $Q^c(a, b) = 1 - Q(a, b)$, $Q^t(a, b) = Q(b, a)$ and $Q^d(a, b) = 1 - Q(b, a)$.

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The intersection of fuzzy relations is usually defined pointwisely based on some *t-norm*. We recall that the binary operator $T : [0,1] \times [0,1] \to [0,1]$ is a *triangular norm* or *t-norm* for short if it is commutative, associative, monotone and has 1 as neutral element. The three most important t-norms are the minimum operator $T_{\mathbf{M}}(x,y) = \min(x,y)$, the algebraic product $T_{\mathbf{P}}(x,y) = xy$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. The minimum operator is the greatest t-norm; the smallest t-norm is the drastic product $T_{\mathbf{D}}$ defined by

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0 & \text{if } \max(x,y) < 1\,, \\ \min(x,y) & \text{otherwise}\,. \end{cases}$$

On the other hand, we say that a value $x \in (0,1)$ is a zero-divisor of a t-norm T if there exists a value $y \in (0,1)$ such that $T(x,y) = 0$. The minimum and the product are t-norms that do not admit zero-divisors, while $T_{\mathbf{L}}$ and $T_{\mathbf{D}}$ do admit them.
An important family of t-norms that admit zero-divisors are the rotation invariant t-norms. A *rotation invariant t-norm* is a t-norm T that verifies:

$$T(x,y) \le z \;\Leftrightarrow\; T(x,1-z) \le 1-y, \quad \forall x,y,z \in [0,1].$$

The rotation invariance property for T means that the part of the space $[0,1]^3$ which is exactly below the graph of T remains invariant under an order three transformation. This transformation is indeed a rotation of $[0,1]^3$ with angle $2\pi/3$ around the axis that passes through the points $(0,0,1)$ and $(1,1,0)$. Rotation invariant t-norms satisfy in particular that $T(x,y) > 0 \Leftrightarrow x+y > 1$, i.e. the lower-left triangle of the unit square constitutes the zero-divisors of T. This excludes t-norms without zero-divisors (such as $T_{\mathbf{P}}$), nilpotent t-norms other than $T_{\mathbf{L}}$ and the drastic product $T_{\mathbf{D}}$.
As extreme members of the family we find the Łukasiewicz t-norm $T_{\mathbf{L}}$ and the nilpotent minimum $T_{\mathbf{nM}}$. This t-norm is defined as:

$$T_{\mathbf{nM}}(x,y) = \begin{cases} 0 & \text{if } x+y \le 1\,, \\ \min(x,y) & \text{otherwise}\,. \end{cases}$$

Similarly to the intersection, the union of fuzzy relations is based on a *t-conorm*, *i.e.* an increasing, commutative and associative binary operation on $[0,1]$ with neutral element 0. t-norms and t-conorms come in dual pairs: to any t-norm T there corresponds a t-conorm S through the relationship

$$S(x,y) = 1 - T(1-x, 1-y).$$

For the main three t-norms this yields the maximum operator $S_{\mathbf{M}}(x,y) = \max(x,y)$, the probabilistic sum $S_{\mathbf{P}}(x,y) = x+y-xy$ and the Łukasiewicz t-conorm (bounded sum) $S_{\mathbf{L}}(x,y) = \min(x+y,1)$. For more information on t-norms and t-conorms, we refer to [7].

The definition of a fuzzy preference structure has been discussed for many years [6]. The *assignment principle*, expressing that for any pair of alternatives (a,b) the decision maker is allowed to assign at least one of the degrees $P(a,b)$, $P(b,a)$, $I(a,b)$ and $J(a,b)$ freely in the unit interval, leads to a graduation of Definition 1 with intersection based on the Łukasiewicz t-norm and union based on the Łukasiewicz t-conorm. This definition admits the same short formulation as the classical one: a triplet (P,I,J) of fuzzy relations on A is an *additive fuzzy preference structure* on A if and only if I is reflexive ($I(a,a) = 1$ for any $a \in A$) and symmetrical, and for any $(a,b) \in A^2$ it holds that $P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1$. This expression justifies the adjective *additive*. Note that P is irreflexive, and that J is irreflexive and symmetrical.

Another difficult point has been how to construct such a structure from a reflexive fuzzy relation. The most recent and most successful approach is that of De Baets and Fodor based on (indifference) *generators* [1].

Definition 3 *A generator i is a commutative $[0,1]^2 \to [0,1]$ mapping that satisfies $T_{\mathbf{L}} \le i \le T_{\mathbf{M}}$.*

Note that a generator always has neutral element 1. For any reflexive fuzzy relation R on A it holds that the triplet (P,I,J) of fuzzy relations on A defined by:

$$\begin{aligned} P(a,b) &= R(a,b) - i(R(a,b),R(b,a))\,, \\ I(a,b) &= i(R(a,b),R(b,a))\,, \\ J(a,b) &= i(R(a,b),R(b,a)) - (R(a,b)+R(b,a)-1)\,, \end{aligned}$$

is an additive fuzzy preference structure on A such that R is the union by the Łukasiewicz t-conorm of P and I, this is, $R(a,b) = P(a,b) + I(a,b)$. The fuzzy relation R is again called the *large preference relation*.
A family of generators that present important properties is the family of Frank t-norms (see, e.g., [6]). In the following section we will see that a particular Frank t-norm is a specially good generator: the Łukasiewicz t-norm.

Definition 4 *Given a t-norm T and a t-conorm S, a fuzzy relation Q is a type 1 T-S-*Ferrers *relation if*

$$T(Q(a,b),Q(c,d)) \le S(Q(a,d),Q(c,b)),\ \forall\, a,b,c,d, \in A.$$

The t-norm T and the t-conorm S are not necessarily connected. However, given T, the most usual t-conorm considered is the dual of T. In this case, the t-conorm is not specified and we just talk about type 1 T-Ferrers property. Along this work, this will be the setting: we will always consider as t-conorm the dual of the fixed t-norm.

Definition 5 *Given a t-norm T, a fuzzy relation Q is a* type 2 T-Ferrers *relation if*

$$T(Q(a,b),Q^d(b,c),Q(c,d)) \le Q(a,d),\ \forall\, a,b,c,d \in A.$$

The composition of two fuzzy relations Q_1 and Q_2 defined on A also involves t-norms. Given a t-norm T, the T-composition of Q_1 and Q_2 is the fuzzy relation $Q_1 \circ_T Q_2$ defined as follows

$$(Q_1 \circ_T Q_2)(a,b) = \sup\{T(Q_1(a,c),Q_2(c,b)) | c \in A\}.$$

With this notation we can also say that a fuzzy relation Q is a type 2 T-Ferrers relation if it holds that $Q \circ_T Q^d \circ_T Q \subseteq Q$.

Definition 6 [2] *An additive fuzzy preference structure is a T-*partial interval order *if it holds that*

$$P \circ_T I \circ_T P \subseteq P\,,$$

or, equivalently, $T(T(P(a,b),I(b,c)),P(c,d)) \le P(a,d)$, $\forall\, a,b,c,d \in A$. *It is a T-*total interval order *if it is a T-partial interval order and $J = \emptyset$.*

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Clearly, if $T_1 \leq T_2$, every type 1 T_2-Ferrers relation is also a type 1 T_1-Ferrers relation, every type 2 T_2-Ferrers relation is also a type 2 T_1-Ferrers relation and every T_2-interval order is also a T_1-interval order.

The equivalence between the notions of type 1 and type 2 T-Ferrers relations has already been studied in some particular cases ([5]). The following theorem further generalizes these results and characterizes all t-norms for which both notions coincide.

Theorem 1 *Consider a t-norm T. Then the following statements are equivalent:*

(i) *Any type 1 T-Ferrers relation is type 2 T-Ferrers.*

(ii) *Any type 2 T-Ferrers relation is type 1 T-Ferrers.*

(iii) *T is rotation-invariant.*

It follows from this theorem that in particular, for any reflexive relation R and for any strict preference relation P, the equivalence between the type 1 T-Ferrers property and the type 2 T-Ferrers property holds if and only if T is a rotation invariant t-norm. This justifies the importance we give to this family of t-norms in this work.

The theorem above does not apply to the minimum t-norm. However it is an important operator and the greatest t-norm. What can we say about it? Since $T_{\mathbf{nM}} \leq T_{\mathbf{M}}$, it follows immediately that any type 1 $T_{\mathbf{M}}$-Ferrers relation is a type 2 $T_{\mathbf{nM}}$-Ferrers relation and that any type 2 $T_{\mathbf{M}}$-Ferrers relation satisfies the type 1 $T_{\mathbf{nM}}$-Ferrers property. This result can be strengthened considerably.

Proposition 3

(i) *$T_{\mathbf{nM}}$ is the largest t-norm T such that any type 1 $T_{\mathbf{M}}$-Ferrers relation is also type 2 T-Ferrers.*

(ii) *$T_{\mathbf{nM}}$ is the largest t-norm T such that any type 2 $T_{\mathbf{M}}$-Ferrers relation is also type 1 T-Ferrers.*

4 Completeness of the large preference relation

We have seen that for crisp relations the completeness of the large preference relation follows from the Ferrers property.
In this section we consider both types of the Ferrers property. We investigate the kind of completeness exhibited by a fuzzy reflexive relation satisfying one of the two properties.

Definition 7 *Let us consider a t-conorm S and a fuzzy relation Q defined on a set of alternatives A. Q is S-complete if it satisfies*

$$S(Q(a,b), Q(b,a)) = 1, \qquad \forall\, a, b \in A.$$

The two most important notions are based on the minimum and the Łukasiewicz t-norms. Given a relation Q we say that it is

- *Strongly complete:* if $Q(a,b) = 1$ or $Q(b,a) = 1, \forall a, b \in A$.
- *weakly complete:* if $Q(a,b) + Q(b,a) \geq 1, \forall a, b \in A$.

The weakest S-completeness condition is based on $S_{\mathbf{D}}$. Given Q, it is $S_{\mathbf{D}}$-complete if $\max(Q(a,b), Q(b,a)) > 0$ for all a, b in A.

No one of the "fuzzy notions" of completeness of R is equivalent to the condition $J = \emptyset$, but we can establish the following equivalence.

Lemma 2 [4] *Let R be a reflexive fuzzy relation and let J be the incomparability relation associated to R by means of any generator i. Then the following equivalence holds*

$$J = \emptyset \Leftrightarrow \begin{cases} R \text{ is weakly complete} \\ i|_S = T_{\mathbf{L}} \end{cases}$$

where $S = \{(u,v) \in [0,1]^2 : \exists (x,y) \in A^2$ with $R(x,y) = u, R(y,x) = v\}$.

Let us recall that when $i = T_{\mathbf{L}}$, the fuzzy preference structure obtained from a weakly complete reflexive relation R is the triplet $(P_{\mathbf{L}}, I_{\mathbf{L}}, J_{\mathbf{L}}) = (R^d, \min(R, R^t), \emptyset)$. We would like to stress the fact that the strict preference relation becomes the dual of the large preference relation.

We next focus on the completeness assured for reflexive type 1 and type 2 T-Ferrers relations.

Proposition 4 *Let T be a t-norm and S its dual t-conorm. Every type 1 T-Ferrers reflexive relation is S-complete.*

As an immediate consequence we get the following: Every type 1 T-Ferrers reflexive relation satisfies the weakest kind of completeness ($S_{\mathbf{D}}$-completeness).

Since strong and weak completeness are in fact $S_{\mathbf{M}}$-completeness and $S_{\mathbf{L}}$-completeness, it also follows from Proposition 4 that every type 1 $T_{\mathbf{L}}$-Ferrers relation is weakly complete and every type 1 $T_{\mathbf{M}}$-Ferrers relation is strongly complete. We can say more.

Theorem 2 *Consider a t-norm T. Every reflexive type 1 T-Ferrers relation is weakly complete if and only if $T(x,y) > 0$ for any pair $(x,y) \in [0,1]^2$ such that $x + y > 1$.*

This theorem applies to all rotation-invariant t-norms. In particular, it applies to $T_{\mathbf{L}}$, and hence to any $T \geq T_{\mathbf{L}}$. Theorem 2 also applies to t-norms without zero-divisors, although in that case the following much stronger result holds.

Theorem 3 *Any reflexive type 1 T-Ferrers relation is strongly complete if and only if the t-norm T has no zero-divisors.*

This theorem applies in particular to $T_{\mathbf{M}}$.

Next we repeat the same study for reflexive type 2 T-Ferrers relations.

Theorem 4 *Any reflexive type 2 T-Ferrers relation is weakly complete.*

A first comparison between types 1 and 2 of the T-Ferrers property is in favour of the type 2 definition as the previous theorem holds for any t-norm T. However, our enthusiasm is tempered by the following negative result.

Proposition 5 *Not all reflexive type 2 $T_{\mathbf{M}}$-Ferrers relations are strongly complete.*

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Despite this negative result, t-norms without zero-divisors have good properties.

Theorem 5 *Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability and let R be its associated large preference relation. Let T be a t-norm. The following assertions are equivalent.*
(i) *If $(P, I, \emptyset)$ is a T-total interval order, then R is strongly complete.*
(ii) *If P is type 1 T-Ferrers, then R is strongly complete.*
(iii) *If P is type 2 T-Ferrers, then R is strongly complete.*
(iv) *If R is type 1 T-Ferrers, then R is strongly complete.*
(v) *T does not admit zero-divisors.*

The results presented in this section show that there are two important families of t-norms in the study of the Ferrers property: rotation invariant t-norms and t-norms without zero-divisors. In the two following sections we pay special attention to those two families [8].

5 Fuzzy interval orders and the type 1 Ferrers property

In this section we study the implications of Figure 1 that remain true for fuzzy relations when the Ferrers property of a crisp relation is generalized by means of the type 1 Ferrers property.

Lemma 3 [2] *Not every $T_{\mathbf{M}}$-total interval order verifies that its strict preference relation is type 1 $T_{\mathbf{M}}$-Ferrers. Not every $T_{\mathbf{L}}$-total interval order verifies that its strict preference relation is type 1 $T_{\mathbf{L}}$-Ferrers.*

Therefore, we cannot assure that for every total interval order its strict preference relation satisfies the type 1 Ferrers property.

Let us focus on the connection between the large and the strict preference relations.

Let us recall first of all that the results we present in this contribution concern a fixed generator $i = T_{\mathbf{L}}$, as it has shown an appropriate behavior (see Lemma 2).

Theorem 6 *The following statements are equivalent:*

(i) The strict preference relation obtained from a type 1 T-Ferrers reflexive relation also satisfies the type 1 T-Ferrers property.

(ii) The t-norm T satisfies $T(x, y) > 0$ for any pair $(x, y) \in [0, 1]^2$ such that $x + y > 1$.

In particular, the previous result can be applied to rotation-invariant t-norms and t-norms without zero-divisors.

The result presented in Theorem 6 can be strengthened under completeness conditions.

Theorem 7 *Consider a fuzzy reflexive relation R with corresponding fuzzy preference structure $(P_{\mathbf{L}}, I_{\mathbf{L}}, J_{\mathbf{L}})$ generated by means of $i = T_{\mathbf{L}}$. For any t-norm T, the following statements are equivalent:*

- *R weakly complete and type 1 T-Ferrers*
- *$J_{\mathbf{L}} = \emptyset$ and $P_{\mathbf{L}}$ type 1 T-Ferrers*

Proposition 6 *Let (P, I, J) be an additive fuzzy preference structure and let T be a rotation invariant t-norm. If P is a type 1 T-Ferrers relation, then (P, I, J) is a T-partial interval order.*

The previous implication cannot be extended to t-norms without zero-divisors. In [2] it was proven that not every type 1 $T_{\mathbf{M}}$-Ferrers reflexive relation satisfies that the fuzzy preference structure obtained from it (by the Łukasiewicz t-norm) is a $T_{\mathbf{M}}$-total interval order. It follows from here and Theorem 6 that the implication of Proposition 6 does not hold for the family of t-norms without zero-divisors.

Now we focus on the connection between a total interval order and the type 1 T-Ferrers property of the associated large preference relation.

Corollary 1 *Let T denote the minimum or the Łukasiewicz t-norms. Not every T-total interval order verifies that its associated large preference relation satisfies the type 1 T-Ferrers property.*

The converse implication neither holds for $T_{\mathbf{M}}$ [2]. However the implication does hold for rotation invariant t-norms.

Proposition 7 *Let T be a rotation invariant t-norm. The additive fuzzy preference structure obtained (by the Łukasiewicz t-norm) from a type 1 T-Ferrers reflexive relation is a T-total interval order.*

In Figures 2 and 3 we summarize the results obtained for the type 1 Ferrers property both for the family of rotation invariant t-norms and for the family of t-norms without zero divisors. We have proven that the implications missing do not hold.

(P, I, J) T-total int. ord. $\Leftarrow$ R type 1 T-Ferrers
$(J = \emptyset)\Nwarrow$ $\Swarrow$ $\not\Nearrow (J = \emptyset)$
P type 1 T-Ferrers
$\Downarrow$
(P, I, J) T-partial int. ord.

Figure 2: Connection among the type 1 T-Ferrers property and the notion of interval order for T a rotation invariant t-norm.

(P, I, J) T-total int. ord. R type 1 T-Ferrers
$(J = \emptyset)$ $\Swarrow$ $\not\Nearrow (J = \emptyset)$
P type 1 T-Ferrers

(P, I, J) T-partial int. ord.

Figure 3: Connection among the type 1 T-Ferrers property and the notion of interval order for T a t-norm without zero-divisors.

6 Fuzzy interval orders and the type 2 Ferrers property

We already know the behaviour of the type 1 Ferrers property when dealing with interval orders. Now we focus on the type 2 Ferrers property.

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Proposition 8 *Let T be a t-norm without zero-divisors and let $(P, I, \emptyset)$ be a T-total interval order, then P satisfies the type 2 T-Ferrers property.*

However the implication cannot be extended to any t-norm.

Proposition 9 *Let T be a t-norm such that $T(x, 1-x) = 0$ for some $x \in (0, 1)$. Not every T-total interval order satisfies that its strict preference relation is type 2 T-Ferrers.*

A particular type of the t-norms described in the previous proposition are the rotation invariant t-norms.

Next we study how the type 2 Ferrers property propagates between R and P.

Theorem 8 *Consider a t-norm T. Then the following statements are equivalent:*

(i) *For any reflexive relation R, if R satisfies the type 2 T-Ferrers property, then $J_{\mathbf{L}} = \emptyset$ and $P_{\mathbf{L}}$ satisfies the type 2 T-Ferrers property.*

(ii) *For any reflexive relation R, if $J_{\mathbf{L}} = \emptyset$ and $P_{\mathbf{L}}$ satisfies the type 2 T-Ferrers property, then R satisfies the type 2 T-Ferrers property.*

(iii) *T is rotation-invariant.*

We now study the relationship between the type 2 T-Ferrers property of P and a T-partial interval order.

Proposition 10 *Let (P, I, J) be an additive fuzzy preference structure and let T be a t-norm. If P is type 2 T-Ferrers then (P, I, J) is a T-partial interval order.*

It follows from the crisp case that the converse implication does not hold for any t-norm.

The characterization of total interval orders based on the Ferrers property of the associated large preference relation gets lost for fuzzy relations when considering the type 2 T-Ferrers condition.

Proposition 11 *It does not hold that for every $T_{\mathbf{M}}$-total interval order the associated large preference relation is type 2 $T_{\mathbf{M}}$-Ferrers. It neither holds that for any type 2 $T_{\mathbf{M}}$-Ferrers reflexive relation the associated by the Łukasiewicz t-norm preference structure is a $T_{\mathbf{M}}$-total interval order.*

Proposition 12 *For any type 2 $T_{\mathbf{L}}$-Ferrers reflexive relation R the associated fuzzy preference structure $(P_{\mathbf{L}}, I_{\mathbf{L}}, J_{\mathbf{L}})$ is a $T_{\mathbf{L}}$-total interval order.*
The converse implication cannot be assured for any type 2 $T_{\mathbf{L}}$-total interval order.

We draw in Figures 4 and 5 the results obtained for the type 2 Ferrers property.

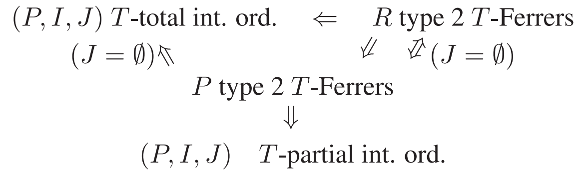

Figure 4: Connection between the type 2 T-Ferrers property and interval orders for T a rotation invariant t-norm.

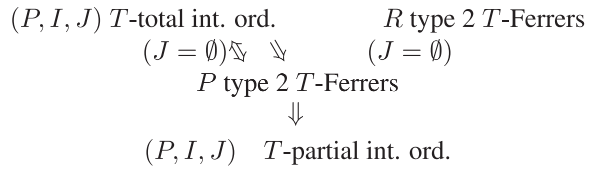

Figure 5: Connection between the type 2 T-Ferrers property and interval orders for T a t-norm without zero-divisors.

7 Conclusions

We have studied the connection between the fuzzy Ferrers property and the definition of fuzzy interval orders. Since the Ferrers property admits two different ways to be formalized for fuzzy relations, we have compared those two versions by its connection to fuzzy interval orders. We have seen that we cannot identify a best definition since each property behaves better in different cases. We have seen that both properties are equivalent only when dealing with rotation invariant t-norms. For these operators, Ferrers property is a stronger condition than the notion of total interval order. For t-norms without zero-divisors, no one of them is good. The type 2 condition seems to assure the equivalence between total interval orders and the Ferrers property of the associated strict preference relation. However, type 1 behaves better if we want to propagate the Ferrers property between the large and the strict preference relations.

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