Decomposition of the transitivity for additive fuzzy preference structures

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Abstract— *Transitivity is a very important property in order to provide coherence to a preference relation. Usually, t-norms are considered to define the transitivity of fuzzy relations. In this paper we deal with conjunctors, a wider family than t-norm, to define the transitivity. This more general definition allows to impove the results found in the literature. We characterize the behaviour with respect to transitivity of the strict preference and indifference relations of any fuzzy preference structure associated to any large preference relation. Those two characterizations provide very general expressions. We also obtain easier expressions in some particular cases.*

Keywords— Fuzzy preference relation, transitivity, conjunctor, indifference generator.

1 Introduction

Decision making is present in many situations in life. Preference models are an essential part of the design phase of a decision making process. The departure point in preference modelling is the comparison by pairs of possible alternatives. If this point of the process lacks of coherence, the whole process makes no sense. One of the most important property introduced to ensure coherence in the two-by-two comparison is transitivity. In this paper we focus on this property.

Given its relevance in preference modelling, transitivity has been widely studied (see [3, 13, 14], among others). In classical or crisp preference modelling there exists a basic relation R, called *large preference relation* that allows to compare a pair of possibilities: given the alternatives a and b, $(a, b) \in R$ expresses that alternative a is considered to be at least as good as alternative b. From R, three relations can be defined: the strict preference relation P, the indifference relation I and the incomparability relation J. It is well known that the transitivity of R is completely characterized by the transitivity of I and P and two additional relational inequalities involving I and P; in case of completeness, the transitivity of R is only characterized by the transitivity of I and P [2].

All those relations are crisp and therefore, they are not always appropriate to model human decisions. The lack of flexibility in crisp set theory lead to introduce fuzzy sets in preference modeling and to study the concept of fuzzy preference structure [8, 18, 24]; see [6] for a historical account of its development. Transitivity is traditionally defined for fuzzy relations by means of t-norms. In this context, the transitivity of the (fuzzy) large preference relation R has been characterized in a similar way as for crisp relations, in a particular case: when R is strongly complete [12]. But strong completeness is a quite restrictive condition. Subsequent works have treated more general preference relations [3, 4]. Those authors center their study on the translation of a fixed transitivity from a more general (fuzzy) large preference relation R to different associated indifference and strict preference relations (I and P respectively). We have continued along this line but we have treated the problem from a totally different viewpoint. On the one hand, we work with conjunctors, a wider family of operators than t-norms, so we handle a very general notion of transitivity. On the other hand, we do not study the preservation of the transitivity when decomposing R, but we explore the strongest transitivity we can assure I and P satisfy. In a previous work [15], we have fenced in the transitivity of I and P according to the transitivity R satisfies. In this work we go further: we provide an explicit expression for the transitivity that any I (respectively any P) satisfies. We prove that no stronger transitivity is fulfilled by all I (respectively for all P). Transitivity is a fundamental assumption in some decision-making models, like rational model (founded in quantitative disciplines like economics, mathematics and statistics) or political model (primarily based on the disciplines of political science, philosophy, psychology and sociology). Without this property, it is not possible to obtain a coherent order of the alternatives and therefore the choice of the best alternative is more complicate or even impossible. Thus, the study made in this work is a first step to develop a consistent theory to order the alternatives and therefore to provide a decision, when the comparison between alternatives can be graded.

The work is structured in six sections. In Section 2 we recall the most relevant concepts concerning additive fuzzy preference structures. In Section 3 we introduce conjunctors and we discuss some properties they satisfy and that will be useful in next sections. In Section 4 we include the general result we have obtained for the transitivity of the indifference relation I and we discuss the appearance of the general expression for the three most important particular cases. In Section 5 we characterize the transitivity of the strict preference relation P in a very general context. We also prove that the general expression gets much simplified for the most important particular cases. In Section 6 we briefly address some conclusions and future work.

2 Preference structures

2.1 Crisp preference structures

Let us consider a decision maker who is given a set of alternatives A. Let us suppose that this person compares the alternatives two by two. Given two alternatives, the decision maker

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can act in one of the following three ways: (i) he/she clearly prefers one to the other; (ii) the two alternatives are indifferent to him/her; (iii) he/she is unable to compare the two alternatives. According to these cases, three binary relations can be defined on A: the strict preference relation P, the indifference relation I and the incomparability relation J. Thus, for any $(a,b) \in A^2$, we classify:

$(a,b) \in P \quad \Leftrightarrow \quad$ he/she prefers a to b;

$(a,b) \in I \quad \Leftrightarrow \quad$ a and b are indifferent to him/her;

$(a,b) \in J \quad \Leftrightarrow \quad$ he/she is unable to compare a and b.

We recall that for a binary relation Q on A, its converse is defined as $Q^t = \{(b,a) \mid (a,b) \in Q\}$, its complement as $Q^c = \{(a,b) \mid (a,b) \notin Q\}$ and its dual as $Q^d = (Q^t)^c$. If we consider the set A^2 ordered, *i.e.* assuming (a,b) and (b,a) as different pairs, one easily verifies that the triplet (P,I,J) and P^t establish a particular partition of A^2 [21].

Definition 1 *A preference structure on A is a triplet (P,I,J) of binary relations on A that satisfy:*

(i) P is irreflexive, I is reflexive and J is irreflexive;

(ii) P is asymmetrical, I and J are symmetrical;

(iii) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$;

(iv) $P \cup P^t \cup I \cup J = A^2$.

Every preference structure has associated a reflexive relation that completely characterizes this structure. A preference structure (P,I,J) on A is characterized by the reflexive binary relation $R = P \cup I$, its large preference relation, in the following way:

$$(P,I,J) = (R \cap R^d, R \cap R^t, R^c \cap R^d). \tag{1}$$

Conversely, for any reflexive binary relation R on A, the triplet (P,I,J) constructed in this way from R is a preference structure on A such that $R = P \cup I$. As R is the union of the strict preference and the indifference, $(a,b) \in R$ means that a is at least as good as b.

Given a binary relation Q on A, we say that Q is transitive if $(aQb \wedge bQc) \Rightarrow aQc$, for any $(a,b,c) \in A^3$. Given two binary relations Q_1 and Q_2 on A, the composition is a binary relation denoted $Q_1 \circ Q_2$ such that for any $(a,b) \in A^2$ $a\,(Q_1 \circ Q_2)\,b \Leftrightarrow \exists c / aQ_1c \wedge cQ_2b$. Then, it is clear that Q is transitive if and only if $Q \circ Q \subseteq Q$. The transitivity of the large preference relation R can be characterized as follows [2].

Theorem 1 *For any reflexive relation R with corresponding preference structure (P,I,J) it holds that*

$$R \circ R \subseteq R \Leftrightarrow (P \circ P \subseteq P \wedge I \circ I \subseteq I \wedge P \circ I \subseteq P \wedge I \circ P \subseteq P).$$

In case R is complete, *i.e.* $R \cup R^t = A^2$, this characterization can be simplified as follows. Note that the completeness of R is equivalent to establish that any two elements are comparable, that is, $J = \emptyset$.

Theorem 2 *For any complete reflexive relation R with corresponding preference structure $(P,I,\emptyset)$ it holds that*

$$R \circ R \subseteq R \Leftrightarrow (P \circ P \subseteq P \wedge I \circ I \subseteq I).$$

Next we recall an important characterization of a preference structure. Let us consider for every relation its characteristic mapping *i.e.* $Q(a,b) = 1 \Leftrightarrow aQb$. Definition 1 can be written in the following minimal way [11]: I is reflexive and symmetrical, and for any $(a,b) \in A^2$:

$$P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1.$$

Classical preference structures can therefore also be considered as Boolean preference structures, employing 1 and 0 for describing presence or absence of strict preferences, indifferences and incomparabilities.

2.2 *Additive fuzzy preference structures*

In the classical model, relations only express presence or absence of relationship, while fuzzy relations capture the nuances of human choices. In fuzzy preference modelling, strict preference, indifference and incomparability are a matter of degree. These degrees can take any value between 0 and 1 and fuzzy relations are used for capturing them (for a complete review about fuzzy relations see [18]).

The intersection of fuzzy relations is usually defined pointwisely based on some t-norm, *i.e.* an increasing, commutative and associative binary operation on $[0,1]$ with neutral element 1. The three most important t-norms are the minimum operator $T_{\mathbf{M}}(x,y) = \min\{x,y\}$, the algebraic product $T_{\mathbf{P}}(x,y) = xy$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max\{x+y-1, 0\}$. The minimum operator is the greatest t-norm; the smallest t-norm is the drastic product defined by

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min\{x,y\} & \text{, if } \max\{x,y\} = 1\,, \\ 0 & \text{, otherwise.} \end{cases}$$

The above t-norms can be ordered (usual ordering of functions) as follows: $T_{\mathbf{D}} \le T_{\mathbf{L}} \le T_{\mathbf{P}} \le T_{\mathbf{M}}$. Similarly, the union of fuzzy relations is based on a t-conorm, *i.e.* a non-decreasing, commutative and associative binary operation on $[0,1]$ with neutral element 0. T-norms and t-conorms come in dual pairs: to any t-norm T there corresponds a t-conorm S through the relationship $S(x,y) = 1 - T(1-x, 1-y)$. For the above three t-norms this yields the maximum operator $S_{\mathbf{M}}(x,y) = \max\{x,y\}$, the probabilistic sum $S_{\mathbf{P}}(x,y) = x + y - xy$ and the Łukasiewicz t-conorm (bounded sum) $S_{\mathbf{L}}(x,y) = \min\{x+y, 1\}$. For more information on t-norms and t-conorms, we refer to [19]. Along this paper we use the notations for t-norms presented there.

The definition of a fuzzy preference structure has been a topic of debate during several years (see e.g. [18, 23, 24]). Accepting the *assignment principle* — for any pair of alternatives (a,b) the decision maker is allowed to assign at least one of the degrees $P(a,b)$, $P(b,a)$, $I(a,b)$ and $J(a,b)$ freely in the unit interval — has finally led to a graduation of Definition 1 with the intersection based on the Łukasiewicz t-norm and the union based on the Łukasiewicz t-conorm. Interestingly, a minimal definition is identical to the classical one if we replace ordinary by fuzzy binary relations: a triplet (P,I,J) of fuzzy binary relations on A is a fuzzy preference structure on A if and only if I is reflexive and symmetrical, and for any $(a,b) \in A^2$:

$$P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1,$$

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where $P^t(a,b) = P(b,a)$. This identity explains the name *additive fuzzy preference structures*.

Another topic of controversy has been how to construct such a structure from a reflexive fuzzy relation. Alsina [1] proved a kind of impossibility theorem showing that a construction based on a single t-norm is unfeasible. As a reaction, Fodor and Roubens adopted an axiomatic approach [18]. The most recent and most successful approach is that of De Baets and Fodor based on (indifference) generators [8].

Definition 2 *A generator i is a commutative $[0,1]^2 \to [0,1]$ mapping bounded by the Łukasiewicz t-norm $T_{\mathbf{L}}$, and the minimum operator $T_{\mathbf{M}}$: $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$.*

Note that the definition of a generator does not speak of monotonicity and therefore they are not necessarily t-norms, albeit having neutral element 1. For any reflexive fuzzy relation R on A it holds that the triplet (P, I, J) of fuzzy binary relations on A defined by:

$$\begin{aligned} P(a,b) &= R(a,b) - i(R(a,b), R(b,a))\,, \\ I(a,b) &= i(R(a,b), R(b,a))\,, \\ J(a,b) &= i(R(a,b), R(b,a)) - (R(a,b) + R(b,a) - 1)\,, \end{aligned}$$

is an additive fuzzy preference structure on A such that $R = P \cup_{S_{\mathbf{L}}} I$ *i.e.* $R(a,b) = P(a,b) + I(a,b)$.

Popular generators (see e.g. [18]) are the Frank t-norms. For the sake of completeness, we recall that the Frank t-norms are given by

$$T^{\mathbf{F}}_\lambda(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{, if } \lambda = 0\,, \\ T_{\mathbf{P}}(x,y) & \text{, if } \lambda = 1\,, \\ T_{\mathbf{L}}(x,y) & \text{, if } \lambda = \infty\,, \\ \log_\lambda(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}) & \text{, otherwise.} \end{cases}$$

We also recall that for any $\lambda \in [0,\infty]$ and for any $(x,y) \in [0,1]^2$,

$$T^{\mathbf{F}}_{1/\lambda}(x,y) = x - T^{\mathbf{F}}_\lambda(x, 1-y).$$

3 Conjunctors

3.1 Generalizing T-transitivity

The usual way of defining the transitivity of a fuzzy relation is by means of a t-norm. Recall that a fuzzy relation Q on A is T-transitive if $T(Q(a,b), Q(b,c)) \leq Q(a,c)$ for any $(a,b,c) \in A^3$. However, such a framework is too restrictive in the setting of fuzzy preference modelling. On the one hand, even when the large preference relation R is T-transitive w.r.t. a t-norm T, the transitivity of the generated P and I cannot always be expressed w.r.t. a t-norm [14, 16, 15]. On the other hand, the results we will present in the following sections also hold when R itself is transitive w.r.t. a more general operator. From a fuzzy preference modelling point of view, it is not that surprising that the class of t-norms is too restrictive, as a similar conclusion was drawn when identifying suitable generators, as briefly explained in the previous section.

As it was discussed in [15, 16], suitable operators for defining transitivity for fuzzy relations are conjunctors:

Definition 3 *A binary operation $f : [0,1]^2 \to [0,1]$ is called a conjunctor if*

1. *it coincides with the Boolean conjunction on $\{0,1\}^2$;*
2. *it is increasing in each variable.*

Given a conjunctor f, we say that a fuzzy relation Q defined on A is f-transitive if for any $(a,b,c) \in A^3$,

$$f(Q(a,b), Q(b,c)) \leq Q(a,c).$$

It is clear that the definition of conjunctor largely extends the notion of t-norm. However, conjunctors and generators are not connected. The smallest conjunctor c_S and greatest conjunctor c_G are given by

$$c_S(x,y) = \begin{cases} 0 & \text{, if } \min\{x,y\} < 1\,, \\ 1 & \text{, otherwise}\,, \end{cases}$$

and

$$c_G(x,y) = \begin{cases} 0 & \text{, if } \min\{x,y\} = 0\,, \\ 1 & \text{, otherwise.} \end{cases}$$

As it is logical, $c_S \leq T_{\mathbf{D}} \leq T_{\mathbf{M}} \leq c_G$. Moreover, for two conjunctors $f \leq g$, it obviously holds that g-transitivity implies f-transitivity.

Defining the composition $Q_1 \circ_f Q_2$ of two fuzzy relations Q_1 and Q_2 w.r.t. a conjunctor f by

$$Q_1 \circ_f Q_2(a,c) = \sup_b f(Q_1(a,b), Q_2(b,c))\,,$$

still allows us to use the shorthand $Q \circ_f Q \subseteq Q$ to denote f-transitivity.

If we restrict our study for reflexive fuzzy relations, such as a large preference relations, the set of all the conjunctors that allow us to define transitivity is a proper subset of the set of all the conjunctors. Thus, the upper bound of this subset is $T_{\mathbf{M}}$ instead of c_G.

3.2 Dominance and bisymmetry

Dominance is a well-known relation for t-norms (see e.g. [19]) and its usefulness has been demonstrated several times (see e.g. [9, 22]). It can be generalized to conjunctors without any problem.

Definition 4 *A conjunctor f_1 is said to dominate a conjunctor f_2, denoted $f_1 \gg f_2$, if for any $(x,y,z,t) \in [0,1]^4$ it holds that*

$$f_1(f_2(x,y), f_2(z,t)) \geq f_2(f_1(x,z), f_1(y,t)).$$

But conjunctors do not verify the same properties as t-norms do with respect to this property. For instance, not every conjunctor dominates itself (see [15]). Dominance and the classical order of binary operators are not related for conjunctors neither. The minimum for example (which is not the greatest conjunctor), dominates any other conjunctor (see [15]).

The notion of self-dominance of conjunctors is obviously equivalent to another well-known property: bisymmetry (see e.g. [19]).

Definition 5 *A conjunctor f is said to be bisymmetric if for any $(x,y,z,t) \in [0,1]^4$ it holds that*

$$f(f(x,y), f(z,t)) = f(f(x,z), f(y,t)).$$

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3.3 Implications

Given a t-norm T, an implication (also called R-implication or T-residuum) based on T is defined (see e.g. [18, 19]) as follows:

$$I_T(x,y) = \sup\{z \mid T(x,z) \le y\}.$$

This concept aims at generalizing the notion of Boolean implication. The definition usually concerns only left continuous t-norms. When T is left-continuous it holds that $T(x,z) \le y \Leftrightarrow z \le I_T(x,y)$.

Given a t-conorm S, a co-implication based on S is defined (see again e.g. [18, 19]) as follows:

$$J_S(x,y) = \inf\{z \mid S(x,z) \ge y\}.$$

When S is right-continuous it holds that $S(x,z) \ge y \Leftrightarrow z \ge J_S(x,y)$.

Following these ideas we can introduce two operators associated not only for a t-norm or a t-conorm, but to any commutative conjunctor.

Definition 6 *Given a commutative conjunctor $f : [0,1]^2 \to [0,1]$, we define the following operator from $[0,1]^2$ into $[0,1]$:*

$$I_f(x,y) = \sup\{z \mid f(x,z) \le y\},$$

and

$$J_f(x,y) = \inf\{z \mid f(x,z) \ge y\}.$$

Let us notice that I_f is an implication, but J_f is not a co-implication.

Lemma 1 *Given a commutative conjunctor f and its associated operators I_f and J_f, we have that*

1. *I_f and J_f are decreasing in their first arguments and increasing in their second arguments.*
2. *If f is left continuous then*
$$f(x,z) \le y \quad \Leftrightarrow \quad z \le I_f(x,y).$$
3. *If f is upper bounded by the minimum t-norm, then*
$$x \le y \quad \Rightarrow \quad I_f(x,y) = 1.$$
Moreover, if f is left-continuous and has 1 as neutral element, then we have the equivalence:
$$x \le y \quad \Leftrightarrow \quad I_f(x,y) = 1.$$

4 Transitivity of I

In this section we consider indifference relations. As it was shown in Subsection 2.2, for a generator i, the symmetric component of a large preference relation is obtained as $I = i(R, R^t)$. Here we study the transitivity we can derive for this relation I when we fix the transitivity of R defined by a conjunctor h.

We begin recalling some upper and lower bounds for the transitivity of I. We know from [15] that I is at least c_S-transitive when R is h-transitive for any conjunctor h. We also showed in that paper that the transitivity we can assure for I is defined by a conjunctor upper bounded both by the conjunctor h and by the generator i that builds I from R. These are upper and lower bounds. We next provide a characterization for the transitivity of I.

Theorem 3 *Let i be an increasing generator and h a conjunctor. For any reflexive fuzzy relation R with corresponding indifference relation I generated by means of i, it holds that*

$$R \text{ is } h\text{-transitive} \quad \Rightarrow \quad I \text{ is } f_h^i\text{-transitive},$$

where the conjunctor f_h^i is:

$$f_h^i(x,y) = \inf_{\substack{1 \ge u \ge x \\ 1 \ge v \ge y}} \left(i\left(h(u,v), h\left(J_i(v,y), J_i(u,x)\right)\right)\right).$$

Moreover, if i is continuous, the previous one is the strongest possible implication.

An interesting problem is to know when the transitivity of R is inherited by I, that is, to know when, departing from an h-transitive R we can assure that I is also h-transitive. We have already answered that question in [15]:

Theorem 4 *Let i be an increasing generator and let h be a commutative conjunctor upper bounded by the minimum t-norm. The associated conjunctor f_h^i is equal to h if and only if i dominates h.*

In particular this result could be applied to $i = T_{\mathbf{M}}$. Thus, when the indifference relation is obtained from the reflexive relation R by the minimum t-norm, $I = \min\{R, R^t\}$, it satisfies the same transitivity as R does.

Not only dominance allows us to obtain some general result, also the usual order among conjunctor, as it is showed in the following corollary.

Corollary 1 *For any increasing and bisymmetric generator i and any conjunctor h upper bounded by the minimum t-norm, if $i \le h$, then $f_h^i = i$.*

Since every t-norm is increasing and bisymmetric, this result can be applied in particular to any generator that is a t-norm. In that case, transitivity of R w.r.t. that t-norm is preserved.

We will now consider Theorem 4 and Corollary 1 to study the particular case of a t-norm of the Frank family as generator.

Corollary 2 *Let $T_\lambda^{\mathbf{F}}$ be a t-norm of the Frank family, for any $\lambda \in [0,\infty]$. It holds that $f_{T_{\mathbf{P}}}^{T_{\mathbf{M}}} = T_{\mathbf{P}}$, $f_{T_{\mathbf{L}}}^{T_\lambda^{\mathbf{F}}} = T_{\mathbf{L}}$ and $f_h^{T_\lambda^{\mathbf{F}}} = T_\lambda^{\mathbf{F}}$ for any $h \ge T_\lambda^{\mathbf{F}}$.*

Remark 1 *Combining the results in Corollary 2 leads to the following table, where the entries are the conjunctors f_h^i.*

$h \setminus i$	$T_{\mathbf{L}}$	$T_{\mathbf{P}}$	$T_{\mathbf{M}}$
$T_{\mathbf{L}}$	$T_{\mathbf{L}}$	$T_{\mathbf{L}}$	$T_{\mathbf{L}}$
$T_{\mathbf{P}}$	$T_{\mathbf{L}}$	$T_{\mathbf{P}}$	$T_{\mathbf{P}}$
$T_{\mathbf{M}}$	$T_{\mathbf{L}}$	$T_{\mathbf{P}}$	$T_{\mathbf{M}}$

5 Transitivity of P

In this section we focus on the strict preference relation P and the transitivity it can satisfy once the transitivity of R is fixed and defined by a conjunctor h.

Let us recall that the strict preference relation P is obtained from R by means of the generator i as follows: $P = R - i(R, R^t)$.

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We showed in [15] that the transitivity that can be assured for the strict preference relation associated to any h-transitive large preference relation R is upper bounded by the conjunctor h, i.e., we cannot assure for P a stronger transitivity than h-transitivity. Next we provide not a bound but the explicit expression for the transitivity of P.

Theorem 5 *Let i be an increasing 1-Lipschitz generator and h a commutative conjunctor with neutral element 1. For any reflexive fuzzy relation R with corresponding strict preference relation P generated by means of i, it holds that*

$$R \text{ is } h\text{-transitive} \quad \Rightarrow \quad P \text{ is } g_h^i\text{-transitive},$$

where the conjunctor g_h^i is:

$$g_h^i(x,y) = \inf_{\substack{1 \geq u \geq x \\ 1 \geq v \geq y}} (h(u,v) - i(h(u,v), \min\{I_h(v, I_i(u,u-x)), I_h(u, I_i(v,v-y))\})).$$

Moreover, the previous one is the strongest possible implication.

According to our result from [15], g_h^i should always be not greater than h. Again, we will focus our attention on the cases the transitivity of R is totally inherited by P.

Proposition 1 *Let h be a rotation-invariant conjunctor, i.e., for all $(x,y,z) \in [0,1]^3$ it holds that*

$$h(x,y) \leq z \quad \Leftrightarrow \quad h(y,1-z) \leq 1-x.$$

If $i = T_{\mathbf{L}}$, then $g_h^{T_{\mathbf{L}}} = h$.

Two well-known rotation-invariant t-norms are $T_{\mathbf{L}}$ and the minimum nilpotent $T_{\mathbf{nM}}$ defined by $T_{\mathbf{nM}}(x,y) = \min\{x,y\}\chi_{\{(x,y)|x+y>1\}}$, where χ_B denotes the characteristic function of any set B. Therefore, from Proposition 1, $g_{T_{\mathbf{L}}}^{T_{\mathbf{L}}} = T_{\mathbf{L}}$ and $g_{T_{\mathbf{nM}}}^{T_{\mathbf{L}}} = T_{\mathbf{nM}}$. This last conjunctor was directly obtained in [16].

If $i = T_{\mathbf{M}}$, only some special conjuntors satisfy the equality $g_h^{T_{\mathbf{M}}} = h$. The minimum t-norm is one of those t-norms. The following theorem characterizes all the conjunctors that verify the previous equality.

Proposition 2 *Let h be a commutative conjunctor with neutral element 1. Then h satisfies the equality $g_h^{T_{\mathbf{M}}} = h$ if and only if h is of the form*

$$h_k^d(x,y) = \begin{cases} 0 & \text{if } max(x,y) < d \\ k \cdot min(x,y) & \text{if } max(x,y) = d \\ min(x,y) & \text{otherwise} \end{cases}$$

where $k \in \{0,1\}$ and $d \in [0,1) \cup \{k\}$.

For the particular case when the conjunctor defining the transitivity of R and the indifference generator i are the same continuous t-norm the thorny general expression obtained in Theorem 5 gets much simpler.

Theorem 6 *Let T be a continuous t-norm. For any reflexive fuzzy relation R with corresponding strict preference relation P generated by means of T, it holds that*

$$R \text{ is } T\text{-transitive} \quad \Rightarrow \quad P \text{ is } g_T^T\text{-transitive},$$

where

$$g_T^T(x,y) = \inf_{0 \leq \alpha \leq \min\{1-x,1-y\}} \max\{T(x+\alpha, y+\alpha) - \alpha, 0\}$$

and this is the strongest possible implication.

In [5] the operator g_T^T was studied in depth not only for a t-norm T but in general for any binary aggregation operator A. In that general case, it was denoted as $\mathcal{D}[A]$. We can find there that for the Frank family of t-norms the expression can be simplified:

Proposition 3 *In case $i = T_\lambda^{\mathbf{F}} = h$ for $\lambda \in [0,\infty]$, it holds that*

$$g_{T_\lambda^{\mathbf{F}}}^{T_\lambda^{\mathbf{F}}}(x,y) = \begin{cases} T_\lambda^{\mathbf{F}}(x,y), & \text{if } x+y>1, \\ S_M(T_\lambda^{\mathbf{F}}(\frac{1+x-y}{2}, \frac{1+y-x}{2}) - \frac{1-x-y}{2}, 0), & \text{otherwise}. \end{cases}$$

In particular, it holds that $g_{T_{\mathbf{L}}}^{T_{\mathbf{L}}} = T_{\mathbf{L}}$ (also obtained as a consequence of Proposition 1), $g_{T_{\mathbf{M}}}^{T_{\mathbf{M}}} = T_{\mathbf{M}}$ (directly obtained in [13]) and

$$g_{T_{\mathbf{P}}}^{T_{\mathbf{P}}}(x,y) = \begin{cases} T_{\mathbf{P}}(x,y) - \left(\frac{T_{\mathbf{L}}(1-x,1-y)}{2}\right)^2, & \text{if } \sqrt{x}+\sqrt{y}>1 \\ 0, & \text{otherwise.} \end{cases}$$

Other particular cases involving the three most important t-norms, i.e. concerning $h \in \{T_{\mathbf{L}}, T_{\mathbf{P}}, T_{\mathbf{M}}\}$ and $i \in \{T_{\mathbf{L}}, T_{\mathbf{P}}, T_{\mathbf{M}}\}$ are presented in the following propositions.

Proposition 4 *If $i = T_\lambda^{\mathbf{F}}$ for $\lambda \in [0,\infty]$ and $h = T_{\mathbf{L}}$, it holds that*

$$g_{T_{\mathbf{L}}}^{T_\lambda^{\mathbf{F}}}(x,y) = T_{1/\lambda}^{\mathbf{F}}(T_{\mathbf{L}}(x,y), S_{\mathbf{M}}(x,y)).$$

Proposition 5 *If $i = T_\lambda^{\mathbf{F}}$ and $h = T_{\mathbf{M}}$, it holds, for any $\lambda \in [0,\infty]$, that*

$$g_{T_{\mathbf{M}}}^{T_\lambda^{\mathbf{F}}}(x,y) = T_{\mathbf{nM}}^{\varphi_{1/\lambda}}(x,y)$$

where φ_λ is the automorphism of the interval $[0,\infty]$ defined as follows

$$\varphi_\lambda(x) = \begin{cases} x & \text{if } \lambda = 0, \\ \sqrt{x} & \text{if } \lambda = 1, \\ \frac{x+1}{2}\chi_{(0,1]}(x) & \text{if } \lambda = \infty, \\ \log_\lambda\left(\sqrt{\frac{\lambda^x - 1}{\lambda - 1}}(\lambda-1)+1\right) & \text{otherwise}, \end{cases}$$

and $T_{\mathbf{nM}}^{\varphi_{1/\lambda}}$ is the transformation by the automorphism $\varphi_{1/\lambda}$ of the t-norm $T_{\mathbf{nM}}$. Moreover, it holds that $g_{T_{\mathbf{M}}}^{T_\lambda^{\mathbf{F}}}$ is a t-norm.

As a consequence of this proposition, we obtain that $g_{T_{\mathbf{M}}}^{T_{\mathbf{L}}} = T_{\mathbf{nM}}$ and

$$g_{T_{\mathbf{M}}}^{T_{\mathbf{P}}}(x,y) = T_{\mathbf{nM}}^{\varphi_1}(x,y) = \begin{cases} \min\{x,y\} & \text{if } \sqrt{x}+\sqrt{y}>1, \\ 0 & \text{otherwise}. \end{cases}$$

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In order to complete the study of all the combinations of the three most important t-norms, two cases are missing, namely $g^{T_{\mathbf{L}}}_{T_{\mathbf{P}}}$ and $g^{T_{\mathbf{M}}}_{T_{\mathbf{P}}}$, which will be revisited in Propositions 6 and 7.

Proposition 6 *It holds that*

$$g^{T_{\mathbf{L}}}_{T_{\mathbf{P}}}(x,y) = T_{\mathbf{M}}\left(T_{\mathbf{P}}(x,y), \frac{T_{\mathbf{L}}(x,y)}{T_{\mathbf{M}}(x,y)}\right)\cdot \chi_{(0,1]}(T_{\mathbf{M}}(x,y)).$$

When $T_{\mathbf{L}}$ is replaced by $T_{\mathbf{M}}$ as generator, the expression of $g^{i}_{T_{\mathbf{P}}}$ gets much more complicated.

Proposition 7 *It holds that*

$$g^{T_{\mathbf{M}}}_{T_{\mathbf{P}}}(x,y) = \max\{\min_{\substack{1\geq u\geq x\\ 1\geq v\geq y}} \left[uv - \min\left\{\tfrac{u-x}{v}, \tfrac{v-y}{u}\right\}\right], 0\}\cdot \chi_{(0,1]^2}(x,y).$$

Remark 2 *Combining the results in Propositions 1, 3, 4, 5, 6 and 7 leads to the following table, where the entries are the conjunctors* g^{i}_{h}.

$h \setminus i$	$T_{\mathbf{L}}$	$T_{\mathbf{P}}$	$T_{\mathbf{M}}$
$T_{\mathbf{L}}$	$T_{\mathbf{L}}$	$T_{\mathbf{P}}(T_{\mathbf{L}}, S_{\mathbf{M}})$	$T_{\mathbf{L}}(T_{\mathbf{L}}, S_{\mathbf{M}})$
$T_{\mathbf{P}}$	$g^{T_{\mathbf{L}}}_{T_{\mathbf{P}}}$	$g^{T_{\mathbf{P}}}_{T_{\mathbf{P}}}$	$g^{T_{\mathbf{M}}}_{T_{\mathbf{P}}}$
$T_{\mathbf{M}}$	$T_{\mathbf{nM}}$	$T^{\varphi_1}_{\mathbf{nM}}$	$T_{\mathbf{M}}$

6 Conclusions

This paper combines very general results with propositions concerning the most relevant particular cases. The general results can be applied to any conjunctor employed to define the transitivity of a large preference relation and any generator used to decompose that relation. The specific results concern the most important particular cases: those for which the conjunctors and the generators are the most important t-norms. The most general theorems we have introduced close the study of the transitivity that can satisfy the symmetric and asymmetric components of a (large) preference relation since those theorems involve (almost) any conjunctor and any generator we can use. The results concerning more particular operators provide more easy-to-use expressions for the most usual conjunctors and generators, i.e. for some t-norms. Only the last example leads to an unwieldy expression.

Despite the ugly general formulae obtained in Theorems 3 and 5, we have already proven that those conjunctors satisfy interesting properties for some particular cases. In future works we would like to study in depth the general expressions and the properties they satisfy.

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