Continuity and additivity of the trapezoidal approximation preserving the expected interval operator

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Abstract— *The nearest trapezoidal fuzzy number to a fuzzy number, with respect to a well-known metric and preserving the expected interval, was determined in recent articles. In the present paper the properties of additivity and continuity of the trapezoidal approximation operator are studied.*

Keywords— Additivity, Approximation, Continuity, Fuzzy number, Trapezoidal fuzzy number.

1 Introduction

In the papers [1] and [10] the problem to find the nearest (with respect to a well-known metric) trapezoidal fuzzy number $T(A)$ to a fuzzy number A such that $EI(T(A)) = EI(A)$, where $EI(B)$ denotes the expected interval of a fuzzy number B, was completely solved. Algorithms for computing the proper approximations are proposed in [6]. The properties of translation and scale invariance, identity, nearness criterion, expected interval invariance, order invariance, correlation invariance, uncertainty invariance, value and ambiguity invariance were considered in [1]. The properties of additivity and continuity are studied in the present paper.

In Section 3 we prove, by a simple example, that the additivity is not satisfied. The main result in this section contains a property of partial additivity. In the paper [5] the property of continuity of the approximations of fuzzy numbers is considered of extreme importance in applications, especially in fuzzy control systems, where it is sometimes indicated as robustness. The authors of the paper [2] pointed out that the method to study the continuity of the trapezoidal approximation without condition in [11] is not applicable in the case of trapezoidal approximation preserving the expected interval. In Section 4 we prove that the discussed approximation operator has the property to be Lipschitz, such that it is continuous.

2 Preliminaries

A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with the membership function μ_A which is normal, fuzzy convex, upper semicontinuous and supp A is bounded, where supp $A = cl\{x \in \mathbb{R} : \mu_A(x) > 0\}$ and cl is the closure operator.

Every λ-cut, $\lambda \in]0,1]$ of a fuzzy number A is a closed interval $A_\lambda = [A^-(\lambda), A^+(\lambda)]$, where

$$A^-(\lambda) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \lambda\},$$
$$A^+(\lambda) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \lambda\}.$$

We denote

$$A_0 = \left[A^-(0), A^+(0)\right] = \text{supp } A.$$

For two arbitrary fuzzy numbers A, B

$$A_\lambda = \left[A^-(\lambda), A^+(\lambda)\right]$$

and

$$B_\lambda = \left[B^-(\lambda), B^+(\lambda)\right]$$

the addition $A + B$ is introduced by

$$(A+B)_\lambda = \left[A^-(\lambda) + B^-(\lambda), A^+(\lambda) + B^+(\lambda)\right]$$

and the quantity

$$D^2(A,B) = \int_0^1 \left(A^-(\lambda) - B^-(\lambda)\right)^2 d\lambda + \int_0^1 \left(A^+(\lambda) - B^+(\lambda)\right)^2 d\lambda \quad (1)$$

gives a distance between A and B (see, e.g., [4]). The expected interval $EI(A)$ of a fuzzy number $A, A_\lambda = [A^-(\lambda), A^+(\lambda)]$, is defined by (see [3], [7])

$$EI(A) = \left[\int_0^1 A^-(\lambda)\, d\lambda, \int_0^1 A^+(\lambda)\, d\lambda\right].$$

An often used fuzzy number is the trapezoidal fuzzy number, completely characterized by four real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$, denoted by $T = (t_1, t_2, t_3, t_4)$, with

$$T^-(\lambda) = t_1 + (t_2 - t_1)\lambda,$$
$$T^+(\lambda) = t_4 - (t_4 - t_3)\lambda, \lambda \in [0,1].$$

Sometimes (see [10]) a trapezoidal fuzzy numbers is denoted by $T = (l, u; x, y)$, with $l, u, x, y \in \mathbb{R}$ such that $x, y \geq 0, x + y \leq 2(u - l)$,

$$T^-(\lambda) = l + x\left(\lambda - \frac{1}{2}\right),$$
$$T^+(\lambda) = u - y\left(\lambda - \frac{1}{2}\right), \lambda \in [0,1].$$

It is immediate that

$$l = \frac{t_1 + t_2}{2}, \quad (2)$$
$$u = \frac{t_3 + t_4}{2}, \quad (3)$$
$$x = t_2 - t_1, \quad (4)$$
$$y = t_4 - t_3, \quad (5)$$

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the expected interval of a trapezoidal fuzzy number represented in this way is the real interval $[l,u]$ and the distance between $T = (l,u;x,y)$ and $T' = (l',u';x',y')$ becomes ([10])

$$D^2(T,T') = (l-l')^2 + (u-u')^2 + \frac{1}{12}(x-x')^2 + \frac{1}{12}(y-y')^2. \quad (6)$$

We denote by $F(\mathbb{R})$ the set of all fuzzy numbers, by $F^T(\mathbb{R})$ the set of all trapezoidal fuzzy numbers and by $T(A)$ the nearest (with respect to the metric D) trapezoidal fuzzy number to fuzzy number A preserving the expected interval of A. To express in a simplified form the approximation operator $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ we denote

$$I(A) = I = \int_0^1 A^-(\lambda)d\lambda, \quad (7)$$

$$S(A) = S = \int_0^1 A^+(\lambda)d\lambda, \quad (8)$$

$$L(A) = L = \int_0^1 \lambda A^-(\lambda)d\lambda \quad (9)$$

and

$$U(A) = U = \int_0^1 \lambda A^+(\lambda)d\lambda, \quad (10)$$

for every fuzzy number $A, A_\lambda = [A^-(\lambda), A^+(\lambda)], \lambda \in [0,1]$ and we consider the following sets of fuzzy numbers

$$\Omega_1 = \{A \in F(\mathbb{R}) : 2I + S - 3L - 3U > 0\},$$
$$\Omega_2 = \{A \in F(\mathbb{R}) : -I - 2S + 3L + 3U > 0\},$$
$$\Omega_3 = \{A \in F(\mathbb{R}) : -I + S + 3L - 3U \leq 0\},$$
$$\Omega_4 = \Omega_1^c \cap \Omega_2^c \cap \Omega_3^c.$$

The set $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ is a partition of $F(\mathbb{R})$ (see the proof of Theorem 7 in [1]).

The following result was proved in [1] and [10].

Theorem 1 *(i) If $A \in \Omega_1$ then*

$$T(A) = T_1(A) = (I, I, I, 2S - I);$$

(ii) If $A \in \Omega_2$ then

$$T(A) = T_2(A) = (2I - S, S, S, S);$$

(iii) If $A \in \Omega_3$ then

$$T(A) = T_3(A) = (4I - 6L, -2I + 6L, -2S + 6U, 4S - 6U);$$

(iv) If $A \in \Omega_4$ then

$$T(A) = T_4(A) = (3I + S - 3L - 3U, -I - S + 3L + 3U, -I - S + 3L + 3U, I + 3S - 3L - 3U).$$

Four different operators $T_i, i \in \{1,2,3,4\}$ give us the nearest trapezoidal approximation preserving the expected interval: T_1 and T_2 lead to triangular fuzzy numbers with right side only or left side only, respectively, T_3 produces proper trapezoidal fuzzy numbers and T_4 produces triangular fuzzy numbers.

3 Additivity

The approximation operator $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ given in Theorem 1 is scale invariant (see [1], Theorem 12, (ii)) and invariant to translations (see [1], Theorem 12, (i)), that is

$$T(\alpha A) = \alpha T(A)$$

and

$$T(A + z) = T(A) + z,$$

for every $A \in F(\mathbb{R}), z \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. In this section we point out a partial result of additivity and prove, be a simple example, that the operator T is not additive.

Example 2 *Let us consider the fuzzy numbers A and B given by*

$$A^-(\lambda) = 1 + \sqrt{\lambda},$$
$$A^+(\lambda) = 30 - 27\sqrt{\lambda},$$
$$B^-(\lambda) = -1 + \sqrt{\lambda},$$
$$B^+(\lambda) = 1 - \sqrt{\lambda}, \lambda \in [0,1].$$

Then (see [1], Example 10 and Example 11) $A \in \Omega_1$,

$$T(A) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{67}{3}\right)$$

$B \in \Omega_4$,

$$T(B) = \left(-\frac{2}{3}, 0, 0, \frac{2}{3}\right),$$

therefore

$$T(A) + T(B) = \left(1, \frac{5}{3}, \frac{5}{3}, 23\right).$$

Because

$$(A+B)^-(\lambda) = 2\sqrt{\lambda},$$
$$(A+B)^+(\lambda) = 31 - 28\sqrt{\lambda}, \lambda \in [0,1],$$

we obtain

$$I(A+B) = \frac{4}{3},$$
$$S(A+B) = \frac{37}{3},$$
$$L(A+B) = \frac{4}{5}$$

and

$$U(A+B) = \frac{43}{10}.$$

We get $A + B \in \Omega_4$ and

$$T(A+B) = \left(\frac{31}{30}, \frac{49}{30}, \frac{49}{30}, \frac{691}{30}\right)$$

which implies

$$T(A+B) \neq T(A) + T(B).$$

Theorem 3 *If $A, B \in \Omega_i$ $(i \in \{1,2,3,4\})$ then*

$$T(A+B) = T(A) + T(B).$$

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Proof. The equalities

$$I(A+B)=I(A)+I(B),$$
$$S(A+B)=S(A)+S(B),$$
$$L(A+B)=L(A)+L(B),$$
$$U(A+B)=U(A)+U(B),A,B\in F(\mathbb{R})$$

are immediate. Then $A,B\in\Omega_i$ ($i\in\{1,2,3,4\}$) implies $A+B\in\Omega_i$ and $T(A)+T(B)=T(A+B)$ in every case $(i)-(iv)$ in Theorem 1. ∎

4 Continuity

An extended trapezoidal fuzzy number ([10]) is an order pair of polynomial functions of degree less than or equal to 1. An extended trapezoidal fuzzy number may be not a fuzzy number, but the distance between two extended trapezoidal fuzzy numbers is similarly defined as in (1) or (6). The extended trapezoidal approximation $T_e(A)=(l_e,u_e;x_e,y_e)$ of a fuzzy number A is defined ([10]) as the extended trapezoidal fuzzy number which minimizes the distance $D(A,B)$, where B is an extended trapezoidal fuzzy number, and preserves the expected interval of A. The real numbers x_e and y_e are non-negative (see [11]). In [5], the authors proved that $T_e(A)=T_3(A)$.

Let us denote by $T(A)=(l_{0,}u_0;x_0,y_0)$ the nearest (with respect to the metric D) trapezoidal fuzzy number to fuzzy number A, preserving the expected interval, given in Theorem 1, in the form proposed in [10] (see also Section 2). We have ([10])

$$l_0=l_e=\int_0^1 A^-(\lambda)d\lambda \tag{11}$$

$$u_0=u_e=\int_0^1 A^+(\lambda)d\lambda. \tag{12}$$

Let us consider in the Euclidean space $\mathbb{R}^2$ (it is a finite dimensional Hilbert space) the points $A_e(x_e,y_e),A_0(x_0,y_0)$ and the set

$$M=\{(x,y)\in\mathbb{R}^2:x\geq 0,y\geq 0,x+y\leq 2u_e-2l_e\}$$

which is a closed convex subset of $\mathbb{R}^2$. Then $P_M(A_e)$ is the unique element in M (see e.g. [8], Theorem 4.10, p. 79) which minimizes the Euclidean distance in $\mathbb{R}^2,D_E(A_e,P)$, where $P\in M$. Taking into account the remarks in [10] we get

$$A_0=P_M(A_e). \tag{13}$$

We present the main result of this section.

Theorem 4 *The nearest trapezoidal approximation operator preserving the expected interval $T:F(\mathbb{R})\rightarrow F^T(\mathbb{R})$ is continuous.*

Proof. Let us consider two fuzzy numbers A and B,

$$A_\lambda=\left[A^-(\lambda),A^+(\lambda)\right],$$
$$B_\lambda=\left[B^-(\lambda),B^+(\lambda)\right],\lambda\in[0,1],$$

$T_e(A)=(l_e,u_e;x_e,y_e),T_e(B)=(l'_e,u'_e;x'_e,y'_e)$ the extended trapezoidal approximations of A and B and $T(A)=(l_{0,}u_0;x_0,y_0),T(B)=(l'_0,u'_0;x'_0,y'_0)$ the trapezoidal approximations preserving the expected interval of A and B. The relations (11) and (12) imply

$$D^2(T(A),T(B))=(l_0-l'_0)^2+(u_0-u'_0)^2$$
$$+\frac{1}{12}(x_0-x'_0)^2+\frac{1}{12}(y_0-y'_0)^2$$
$$=(l_e-l'_e)^2+(u_e-u'_e)^2$$
$$+\frac{1}{12}(x_0-x'_0)^2+\frac{1}{12}(y_0-y'_0)^2$$

Because (see [9], Proposition 4.4)

$$D(T_e(A),T_e(B))\leq D(A,B) \tag{14}$$

we obtain

$$D^2(T(A),T(B))\leq D^2(A,B)$$
$$+\frac{1}{12}(x_0-x'_0)^2+\frac{1}{12}(y_0-y'_0)^2$$

or

$$D^2(T(A),T(B))\leq D^2(A,B)+\frac{1}{12}D_E^2(A_0,B_0), \tag{15}$$

where $D_E(A_0,B_0)$ denotes the Euclidean distance between $A_0(x_0,y_0)$ and $B_0(x'_0,y'_0)$. Let us assume (contrariwise the proof is similar)

$$2u'_e-2l'_e\geq 2u_e-2l_e.$$

We consider

$$M_A=\{(x,y)\in\mathbb{R}^2:x\geq 0,y\geq 0,x+y\leq 2u_e-2l_e\},$$
$$M_B=\{(x,y)\in\mathbb{R}^2:x\geq 0,y\geq 0,x+y\leq 2u'_e-2l'_e\}$$

and

$$C(2u_e-2l_e,0),$$
$$C'(0,2u_e-2l_e),$$
$$G(2u'_e-2l'_e,0),$$
$$G'(0,2u'_e-2l'_e),$$

the points which define the closed convex sets M_A and M_B in the Euclidean space $\mathbb{R}^2$. The pairs $(l_e,u_e),(l'_e,u'_e)$ being already fixed ((11), (12)), we denote $A_e(x_e,y_e)$ and $B_e(x'_e,y'_e)$ the points in $\mathbb{R}^2$ which represent the others components of the extended trapezoidal approximations of A and B, respectively. It is known that $x_e,y_e,x'_e,y'_e\geq 0$.

According to (13) we get

$$A_0=P_{M_A}(A_e)$$
$$B_0=P_{M_B}(B_e)$$

We denote by B_1 the projection of B_0 on the convex set M_A, that is the unique element in M_A which minimizes $D_E(B_0,Q)$, where $Q\in M_A$. We prove that B_1 is the

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projection of B_e on the set M_A, that is $B_1 \in M_A$ and $\min_{R \in M_A} D_E(B_e, R) = D_E(B_e, B_1)$, and

$$D_E(B_1, B_0) \le D_E(C, G) = D_E(C', G').$$

Indeed, if $x'_e + y'_e \le 2(u_e - l_e)$ then $B_e \in M_A$. Because $M_A \subseteq M_B$ we get

$$B_e \in M_B,$$
$$B_0 = B_1 = B_e$$

and

$$D_E(B_1, B_0) = 0 \le D_E(C, G).$$

Contrariwise, if $x'_e + y'_e > 2(u_e - l_e)$ then the following situations are possible:

(i) $x'_e - y'_e > 2(u'_e - l'_e)$. Then

$$B_0 = G,$$
$$P_{M_A}(B_e) = C$$

and

$$B_1 = P_{M_A}(B_0) = P_{M_A}(G) = C.$$

In addition,

$$D_E(B_1, B_0) = D_E(C, G).$$

(ii) $2(u_e - l_e) \le x'_e - y'_e \le 2(u'_e - l'_e)$. Then

$$B_1 = P_{M_A}(B_e) = P_{M_A}(B_0) = C$$

and

$$D_E(B_1, B_0) = D_E(C, B_0) \le D_E(C, G).$$

(iii) $-2(u_e - l_e) < x'_e - y'_e < 2(u_e - l_e)$. Because B_eB_0 is perpendicular on GG', B_eB_1 is perpendicular on CC' and GG', CC' are parallel we get

$$B_1 = P_{M_A}(B_e).$$

In addition,

$$D_E(B_1, B_0) < D_E(C, G).$$

(iv) $-2(u'_e - l'_e) \le x'_e - y'_e \le -2(u_e - l_e)$. Then

$$B_1 = P_{M_A}(B_e) = P_{M_A}(B_0) = C'$$

and

$$D_E(B_1, B_0) = D_E(C', B_0) \le D_E(C', G').$$

(v) $x'_e - y'_e < -2(u'_e - l'_e)$. Then

$$B_0 = G',$$
$$P_{M_A}(B_e) = C'$$

and

$$B_1 = P_{M_A}(B_0) = P_{M_A}(G') = C'.$$

In addition,

$$D_E(B_1, B_0) = D_E(C', G').$$

We have

$$\begin{aligned}
&D_E^2(C, G) \\
&= (2u'_e - 2l'_e - 2u_e + 2l_e)^2 \\
&= 4\left[(u'_e - u_e) - (l'_e - l_e)\right]^2 \\
&= 4\left[\int_0^1 \left(B^+(\lambda) - A^+(\lambda)\right) d\lambda \right. \\
&\left. - \int_0^1 \left(B^-(\lambda) - A^-(\lambda)\right) d\lambda\right]^2
\end{aligned}$$

$$\begin{aligned}
&\le 8\left[\int_0^1 \left(B^+(\lambda) - A^+(\lambda)\right) d\lambda\right]^2 \qquad (16)\\
&+ 8\left[\int_0^1 \left(B^-(\lambda) - A^-(\lambda)\right) d\lambda\right]^2 \\
&\le 8\int_0^1 \left(B^+(\lambda) - A^+(\lambda)\right)^2 d\lambda \qquad (17)\\
&+ 8\int_0^1 \left(B^-(\lambda) - A^-(\lambda)\right)^2 d\lambda
\end{aligned}$$

therefore

$$D_E^2(B_1, B_0) \le 8D^2(A, B).$$

Because M_A is a closed convex subset of R^2 we obtain (see [11], Appendix C)

$$D_E(P_{M_A}(A_e), P_{M_A}(B_e)) \le D_E(A_e, B_e)$$

that is

$$D_E(A_0, B_1) \le D_E(A_e, B_e).$$

We get

$$\begin{aligned}
D_E(A_0, B_0) &\le D_E(A_0, B_1) + D_E(B_1, B_0) \\
&\le D_E(A_e, B_e) + 2\sqrt{2}D(A, B).
\end{aligned}$$

From (14) we obtain

$$D_E^2(A_e, B_e) \le 12D^2(A, B), \qquad (18)$$

therefore

$$D_E(A_0, B_0) \le 2(\sqrt{2} + \sqrt{3})D(A, B). \qquad (19)$$

The inequalities (15) and (19) imply

$$D(T(A), T(B)) \le \sqrt{1 + \frac{(\sqrt{2} + \sqrt{3})^2}{3}} D(A, B) \qquad (20)$$

and the proof is complete. ■

If $D(A, B) = 0$ then (20) becomes equality.

If

$$D(T(A), T(B)) = \sqrt{1 + \frac{(\sqrt{2} + \sqrt{3})^2}{3}} D(A, B)$$

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then (16)-(18) must be equalities (contrariwise, inequality (20) becomes strict). Equality in (16) implies

$$\int_0^1 \left(B^+(\lambda) - A^+(\lambda)\right) d\lambda + \int_0^1 \left(B^-(\lambda) - A^-(\lambda)\right) d\lambda = 0 \qquad (21)$$

and equality in (17) implies

$$B^+(\lambda) - A^+(\lambda) = k^+, \text{ a.e. } \lambda \in [0,1]$$

and

$$B^-(\lambda) - A^-(\lambda) = k^-, \text{ a.e. } \lambda \in [0,1],$$

where $k^+, k^- \in \mathbb{R}$. Substituting in (21) we get $k^+ + k^- = 0$, therefore

$$B^+(\lambda) = A^+(\lambda) + k, \text{ a.e. } \lambda \in [0,1]$$

and

$$B^-(\lambda) = A^-(\lambda) - k, \text{ a.e. } \lambda \in [0,1],$$

where $k \in \mathbb{R}$. According to (7)-(10) we obtain

$$I(B) = I(A) - k,$$
$$S(B) = S(A) + k,$$
$$L(B) = L(A) - \frac{k}{2},$$

and

$$U(B) = U(A) + \frac{k}{2}.$$

Taking into account Theorem 1 (iii) and (4), (5),

$$x_e^{'} = -6I(B) + 12L(B) = -6I(A) + 12L(A) = x_e,$$

$$y_e^{'} = 6S(B) - 12U(B) = 6S(A) - 12U(A) = y_e.$$

We get $A_e = B_e$ and equality in (18) implies $D(A,B) = 0$.

We conclude that equality in (20) holds if and only if $D(A,B) = 0$.

5 Conclusion

Many approximation methods for fuzzy numbers were proposed in the last years. Because the quality of approximation is important, a list of criteria that the approximation operator should or just can possess has been given in [5]. Continuity and additivity of nearest trapezoidal approximation operator which preserves the expected interval are studied in the present paper. Because the property of scale invariance was already proved in [1], the property of partial additivity, given in Theorem 3, assures the linearity of the trapezoidal approximation operator, when the fuzzy numbers under study are all with the same kind of asymmetry of membership functions (see [6]). The property of continuity, given in Theorem 4, means that if two fuzzy numbers are close, with respect to metric D, then their approximations are also close, with respect to metric D. Continuity is of extreme importance in applications, especially in fuzzy control systems, where it is sometimes indicated as robustness and discontinuous approximation operators seem unnatural.

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