Fuzzy Subgroups and the Teichmüller Space

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Abstract— *There exists a generalization of the Teichmüller space of a covering group. In this paper we combine this generalized Teichmüller space $T(G)$ and any fuzzy subgroup $\mathcal{A}: G \longrightarrow \mathcal{F}$ where G is a subgroup of the group consisting of such orientation preserving and orientation reversing Möbius transformations which act in the upper half-plane of the extended complex plane. A partially ordered set $\mathcal{F} = (\mathcal{F}, \leq)$ consists of stabilizers of G and all of their intersections. After preliminaries we present two new results concerning this special case of fuzzy subgroups. These conclusions are then applied to the known theory of the parametrization of the generalized Teichmüller space . As consequence, the equivalence classes of fuzzy subgroups (with an equivalence relation) become the elements of $T(G)$ where G is generated by a finite set of hyperbolic Möbius transformations. Let the number of the generators be n. Then there is an embedding $\psi : T(G) \longrightarrow \mathbb{R}^{3n-3}$ and therefore a homeomorphism $T(G) \longrightarrow \psi(T(G))$. Through parametrization of $T(G)$ $\psi(\mathcal{A}(G))$ has $3n-3$ real coordinates which are also the coordinates of $\mathcal{A}(G)$ up to identification.*

Keywords— Fuzzy subgroups, Möbius groups, Möbius transformations, compact Riemann surfaces, Teichmüller spaces.

1 Introduction

Orientation preserving and orientation reversing Möbius transformations form a group denoted by $\widehat{M}$. Following [2], we say that every subgroup of $\widehat{M}$ is a Möbius group. In Section 2 we first take some basic knowledge of Möbius transformations and then list all the types of Möbius transformations acting in the upper half-plane H which means that they are self-mappings of H. Let G and G' be Möbius groups fixing H. Then we define geometric isomorphisms $j : G \longrightarrow G'$ which play an important role in our representation.

In Section 3 we present two new Propositons 1 and 2 which the author of this paper has proved in [1]. It is not relevant to give the proofs in this connection because they are too long to show as well as many preliminaries are required. The propositions constitute the relationship between subgroups of the group of Möbius transformations and certain kind of fuzzy subgroups.

Let G, G', G_1 and G_2 be Möbius groups with a fixed G. Defining a delation $\delta(j)$ of an isomorphism $j : G \longrightarrow G'$ by means of multipliers of Möbius transformations $g \in G$ we say that any two isomorphisms $j_1 : G \longrightarrow G_1$ and $j_2 : G \longrightarrow G_2$ with some properties are equivalent precisely when $\delta(j_2 j_1^{-1}) = 1$. The set $E(G)$ of equivalence classes $[j]$ forms a metric space $(E(G), d)$ with a metric d. In the theory of compact Riemann surfaces this space can be regarded as a generalization of the Teichmüller space of a covering group G. So, $(E(G), d)$ is denoted by $T(G)$. All the presented premiliminaries and the parametrization of the generalized Teichmüller space can be found from [2], [3] and [4]. We will consider them in Section 4 so much as needed for Section 5.

Suppose that there exists a set F in G such that the parametrization condition in Proposition 3 is satisfied. Then we obtain an embedding $\psi : T(G) \longrightarrow \mathbb{R}^{3n-3}$ where G is finitely generated by n hyperbolic elements. More precisely, $T(G) \longrightarrow \psi(T(G))$ is a homeomorphism. Then, up to homeomorphism, $[j]$ is an element of $\mathbb{R}^{3n-3}$ with coordinates $k(j(g))$ which are the multipliers of Möbius transformations $j(g)$ for every $g \in F$.

In Section 5 we apply fuzzy subgroups to the theory of the parametrization of the generalized Teichmüller space. According to Corollary 1 the type-preserving isomorphisms $j_1 : G \longrightarrow G_1$ and $j_2 : G \longrightarrow G_2$ as well as the fuzzy subgroups $\mathcal{A}(j_1(g))$ and $\mathcal{A}(j_2(g))$ are simultaneously equivalent. The definition of the fuzzy subgroups is given by (3) in Proposition 1 and the equivalence between two fuzzy subgroups is defined by conjugation with a Möbius transformation. Roughly speaking, it is now possible to replace any element $[j]$ by the corresponding $[\mathcal{A}(j(g))]$ in Proposition 4. This leads to the representation of $[\mathcal{A}(j(g))]$ with real coordinates (up to identification) in the Euclidean space $\mathbb{R}^{3n-3}$.

In the following special case, we refer to [3]: If G is a covering group of the upper half-plane H over some compact Riemann surface S, then $[j]$ has coordinates in $\mathbb{R}^{6s-6+3n}$ where S is of genus s and n is the number of conformal disks removed from S. The coordinates of $[\mathcal{A}(j(g))]$ are given in subsection 5.2.

2 Preliminaries for Möbius transformations

In this section we refer to [2].

2.1 Möbius transformations

Directly conformal automorphisms of the extended complex plane $\widehat{\mathbb{C}}$ are orientation preserving Möbius transformations

$$g(z) = \frac{az+b}{cz+d} \quad a,\, b,\, c,\, d \in \mathbb{C}, \quad ad-bc = 1, \tag{1}$$

and $g(-\frac{d}{c}) = \infty$, $g(\infty) = \frac{a}{c}$.

Indirectly conformal automorphisms of $\widehat{\mathbb{C}}$ are of the form

$$g(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \quad a,\, b,\, c,\, d \in \mathbb{C}, \quad ad-bc = -1 \tag{2}$$

which are orientation reversing Möbius transformations. The mappings (1) and (2) form the group $\widehat{M}$. Two transformations g_1 and g_2 in $\widehat{M}$ are conjugate if $g_1 = h g_2 h^{-1}$ for some Möbius transformation h.

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A point z is said to be a fixed point of g if $g(z) = z$. Every non-identity orientation preserving Möbius transformation g has one fixed point or two fixed points. Then

- g with one fixed point is parabolic and it is conjugate to $z \mapsto z+1$.
- If g has two fixed points and it is conjugate to $z \mapsto kz$ for some $k \in \mathbb{C} \setminus \{0\}$, then g is loxodromic if $|k| \neq 1$ and elliptic if $|k| = 1$, $k \neq 1$.
- Loxodromic transformations fixing the upper half-plane H are called hyperbolic. Otherwise they are strictly loxodromic.

In the loxodromic case we set

$$P(g) = \lim_{n\to\infty} g^n \text{ and } N(g) = \lim_{n\to\infty} g^{-n},$$

which are the attracting and repelling fixed points of g. Because the fixed points of any loxodromic g are real iff $g(H) = H$, hyperbolic transformations have fixed points on the real axis $\mathbb{R}$. Therefore $P(g)$ and $N(g)$ are real if g is hyperbolic. A circle or line in H perpendicular to to the real axis is a non-Euclidean line. Now the non-Euclidean line through $P(g)$ and $N(g)$ is called the axis $ax(g)$ of g. In the elliptic case $g(H) = H$ iff the fixed points of g are complex conjugates.

The multiplier of a Möbius transformation g is defined by means of the cross-ratio

$$k(g) = (g(z),z,x,y) = \frac{g(z)-x}{g(z)-y}\,\frac{z-y}{z-x},$$

where x and y are two different fixed points of g. In the parabolic case we set $k(g) = 1$. For the above complex number k we have $k = k(g)$. The multiplier is invariant in conjugation.

We are only interested in such orientation reversing Möbius transformations which act in H (fix H). This leads to the next consideration: the axis $ax(\sigma) = \{z \mid \sigma(z) = z\}$ of a reflection σ fixing H is also a circle or line orthogonal to $\mathbb{R}$. Denote by x and y the real fixed points of σ. Then

- the reflection $\sigma(z) = \eta(\bar{z})$ where η is the elliptic transformation defined by $k(\eta) = -1$, $\eta(x) = x$ and $\eta(y) = y$,
- a glide-reflection s fixing H is of the form $s = \tau\sigma$ where τ is a hyperbolic transformation fixing H, σ is a reflection fixing H and $ax(\tau) = ax(\sigma)$.

The reflection σ has an infinite number of fixed points but two real fixed points (if $ax(\sigma)$ is not a line). For the multiplier we set $k(\sigma) = -1$. The glide-reflection s and the hyperbolic τ have the same two fixed points. Moreover, the multiplier $k(s) = -k(\tau)$.

It is known that transformations (1) and (2) fix the upper-half plane H iff the coefficients a,b,c,d are real. In fact, there are the following types of Möbius transformations fixing H: hyperbolic, parabolic and elliptic transformations, the identity transformation, reflections and glide-reflections.

Suppose that G and G' are groups of Möbius transformations (or Möbius groups) acting in H. An isomorphism $j : G \longrightarrow G'$ is induced by a Möbius transformation h if $j(g) = hgh^{-1}$ for all $g \in G$. We say that $j : G \longrightarrow G'$ is type-preserving if g and $j(g)$ are of the same type for all $g \in G$. A type-preserving isomorphism $j : G \longrightarrow G'$ is geometric on $\mathbb{R}$ if there exists a homeomorphism $\varphi : \widehat{\mathbb{R}} \longrightarrow \widehat{\mathbb{R}}$ inducing j on $\widehat{\mathbb{R}}$, which is the boundary of the upper half-plane. Especially $\varphi(P(g)) = P(j(g))$ and $\varphi(N(g)) = N(j(g))$ for any hyperbolic g. Since every Möbius transformation is a homeomorphism $\widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, then for every φ which induces a geometric isomorphism j we have $h \mid \widehat{\mathbb{R}} = \varphi$ ($h : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a Möbius transformation).

3 Connection between fuzzy subgroups and Möbius transformations

Let $(P,\leq)$ be a partially ordered set and A a nonempty set.

(i) A mapping $\mathcal{A} : A \longrightarrow P$ is a P-(fuzzy)set on A.

(ii) For every $p \in P$, $A_p = \{x \in A \mid \mathcal{A}(x) \geq p\}$ is a p-level set subset (in short a level subset) of $\mathcal{A}$.

(iii) Let $\mathcal{G} = (G,\circ)$ be a group such that $(A_p,\circ)$ is a subgroup of $\mathcal{G}$ for every $p \in P$. Then a P-set on G, $\mathcal{A} : G \to P$, is said to be a P-(fuzzy) subgroup of $\mathcal{G}$.

Proposition 1. *[1] Let G be a subgroup of the group of Möbius transformations and let $G_z = \{g \in G \mid g(z) = z\}$ be a stabilizer of G at $z \in \widehat{\mathbb{C}}$. If $\mathcal{F}$ is a set of stabilizers of G and all of their intersections, and $\mathcal{F} = (\mathcal{F},\leq)$ is a partially ordered set under $p \leq q$ iff $p \supseteq q$ $(p,q \in \mathcal{F})$, then $\mathcal{A} : G \to \mathcal{F}$*

$$\mathcal{A}(g) = \bigcap(p \in \mathcal{F} \mid g \in p) = \bigcap(G_z \mid g \in G_z) \qquad (3)$$

is a $\mathcal{F}$-subgroup of G.

We set an equivalence relation between fuzzy subgroups (3) by conjugation: $\mathcal{A}(g_1) \sim \mathcal{A}(g_2)$ iff there exists a Möbius transformation h such that $\mathcal{A}(g_1) = h\mathcal{A}(g_2)h^{-1}$. Observe that stabilizers form a group and the conjugation of the intersection of the groups G_z occurs by elements in formula (3). Recall that a Möbius group is a subgroup of the group $\widehat{M}$.

Proposition 2. *[1] Let G and G' be Möbius groups acting in the upper half-plane, $\mathcal{F}$ a set of stabilizers of G and all of their intersections. Let $\mathcal{A} : G \longrightarrow \mathcal{F}$ be defined by (3) and suppose that there is a type-preserving isomorphism $j : G \longrightarrow G'$. Then the following conditions are equivalent:*

(i) $\mathcal{A}(g) \sim \mathcal{A}(j(g))$ for every $g \in G$,

(ii) $k(g) = k(j(g))$ for every $g \in G$,

(iii) j is induced by a Möbius transformation,

(iv) $j : G \longrightarrow G'$ is a geometric isomorphism.

Further, let $j_1 : G \longrightarrow G_1$ and $j_2 : G \longrightarrow G_2$ be type-preserving isomorphisms. Then

$$\mathcal{A}(j_1(g)) \sim \mathcal{A}(j_2(g)) \quad \text{for every } g \in G$$

iff

$$j_2 j_1^{-1} : G_1 \longrightarrow G_2 \quad \text{is induced by a Möbius transformation.}$$

4 Coordinates of the generalized Teichmüller space

In this section we refer to [3].

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4.1 The set E

Let G and G' be Möbius groups acting in the upper half-plane H and $j : G \longrightarrow G'$ an isomorphism. We suppose that G has a set $E = \{g_1, g_2, \cdots\}$ of hyperbolic generators satisfying the following conditions:

(i) $ax(g_1) \cap ax(g_2)$ is a set of two points,

(ii) $(ax(g_1) \cup ax(g_2)) \cap ax(g_i) = \emptyset$, $i = 3, 4, \cdots$,

(iii) $(N(g_i), P(g_i), N(g_1), P(g_1)) < 1$, $i = 3, 4, \cdots$.

4.2 The dilation

Let $j : G \longrightarrow G'$ be an isomorphism between Möbius groups. The dilation $\delta(j)$ of j is the smallest number of the numbers $a \geq 1$ satisfying

$$|k(g)|^{\frac{1}{a}} \leq |k(j(g))| \leq |k(g)|^a \tag{4}$$

for all $g \in G$. If there exists no number $a \geq 1$ we define $\delta(j) = \infty$.

For a fixed G, let $J(G)$ be a set of all isomorphisms $j : G \longrightarrow G'$ with the following properties:

(i) G and G' are Möbius groups acting in H,

(ii) the dilation $\delta(j)$ is finite,

(iii) there exists a homeomorphism $\varphi : \widehat{\mathbb{R}} \longrightarrow \widehat{\mathbb{R}}$ such that $\varphi(P(g_i)) = P(j(g_i))$ and $\varphi(N(g_i)) = N(j(g_i))$ for all $g_i \in E$.

Define in $J(G)$ an equivalence relation $\sim$ by setting $j_1 \sim j_2$ iff $\delta(j_2 j_1^{-1}) = 1$. Then the set $E(G)$ of equivalence classes $[j]$ becomes a metric space $(E(G), d)$ with the dilation metric d defined by $d([j_1], [j_2]) = log\, \delta(j_2 j_1^{-1})$ [4].

Definition 1. *[3] The space $T(G) = (E(G), d)$ is called the generalized Teichmüller space of a Möbius group G.*

4.3 Parametrization of the Teichmüller space $T(G)$

Let us construct a set F containing the next elements:

(1) g_i, $i = 1, 2, \cdots$,

(2) $g_i g_1$, $i = 2, 3, \cdots$,

(3) $g_2 g_i g_2^{-1} g_i^{-1}$, $i = 3, 4, \cdots$,

where every $g_i \in E$. The following proposition shows the significance of the set F if we want to find the condition for the parametrization of $T(G)$. We say that F parametrizes the space $T(G)$ or the set $J(G)$.

Proposition 3. *[3] Suppose that there exists a homeomorphism $\varphi : \widehat{\mathbb{R}} \longrightarrow \widehat{\mathbb{R}}$ for which $\varphi(P(g_i)) = P(j(g_i))$ and $\varphi(N(g_i)) = N(j(g_i))$ for all $g_i \in E$. If*

$$k(j(g)) = k(g)$$

for all $g \in F$, then j is induced by a Möbius transformation $h : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$.

Since multipliers are invariant in conjugation the converse holds trivially.

Proposition 4. *[3] Suppose that $E = \{g_1, \cdots, g_n\}$ is finite. Then there is an embedding*

$$\psi : T(G) \longrightarrow \mathbb{R}^{3n-3}, \;\; \psi([j]) = (x_1, \cdots, x_{3n-3}),$$

where

$$\begin{aligned} x_i &= k(j(g_i)), \;\; i = 1, \cdots, n, \\ x_{n-1+i} &= k(j(g_i) j(g_1)), \;\; i = 2, \cdots n, \\ x_{2n-3+i} &= k(j(g_2) j(g_i) j(g_2)^{-1} j(g_i)^{-1}), \;\; i = 3, \cdots, n. \end{aligned}$$

Proof. The mapping $\psi : T(G) \longrightarrow \mathbb{R}^{3n-3}$ is continuous with the dilation metric d [3], [4]. By Proposition 3, ψ is injective [3]. Furthermore, the inverse $\psi^{-1} : \psi(T(G)) \longrightarrow T(G)$ is continuous. Hence $\psi : T(G) \longrightarrow \psi(T(G))$ is a homeomorphism. □

5 Fuzzy subgroups in the Teichmüller space

5.1 The main results

Let G, G_1 and G_2 be Möbius groups. According to Proposition 2 and inequalities (4) we conclude

Corollary 1. *Let $j_1 : G \longrightarrow G_1$ and $j_2 : G \longrightarrow G_2$ be type-preserving isomorphisms. Then the following statements are equivalent:*

(i) $j_1 \sim j_2$,

(ii) $\delta(j_2 j_1^{-1}) = 1$,

(iii) $k(j_2 j_1^{-1}(g)) = k(g)$ *for every* $g \in G$,

(iv) $j_2 j_1^{-1} : G_1 \longrightarrow G_2$ *is induced by a Möbius transformation,*

(v) $\mathcal{A}(j_1(g)) \sim \mathcal{A}(j_2(g))$ *for every* $g \in G$.

Let $j \in J(G)$ and suppose that $j : G \longrightarrow G'$ is geometric. Then j is type-preserving and satisfies the condition (iii) in subsection 4.2. Following the definition of $E(G)$, let us denote the set of equivalence classes $[\mathcal{A}(j(g))]$ by $E'(G)$. By Corollary 1,(i),(v), $(E'(G), d)$ becomes also a metric space having the same metric d. This leads to the new form of the generalized Teichmüller space, $T'(G) = (E'(G), d)$. Reformulation of Proposition 4 for the fuzzy subgroups $\mathcal{A}(j(g))$ yields

Proposition 5. *Suppose that $E = \{g_1, \cdots, g_n\}$ is finite. Then there is an embedding*

$$\psi' : T'(G) \longrightarrow \mathbb{R}^{3n-3}, \;\; \psi'([\mathcal{A}(j(g)]) = (x_1, \cdots, x_{3n-3}),$$

for every $g \in G$, and where

$$\begin{aligned} x_i &= k(j(x_i)), \;\; i = 1, \cdots, n, \\ x_{n-1+i} &= k(j(g_i) j(g_1)), \;\; i = 2, \cdots n, \\ x_{2n-3+i} &= k(j(g_2) j(g_i) j(g_2)^{-1} j(g_i)^{-1}), \;\; i = 3, \cdots, n. \end{aligned}$$

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5.2 *The covering group of a compact Riemann surface*

Let G be a Möbius group acting in the upper half-plane H and suppose that G is a covering group of H over the compact Riemann surface S. Since any non-identity cover transformation of G has no fixed points in H, we know that all elements of G are hyperbolic or parabolic Möbius transformations, or it is the identity mapping. We say that G is of signature (s,n) if $S = H/G$ is of genus s from which n conformal disks are removed.

Assume that G is a fixed covering group of signature (s,n) with $s > 0$ and $n > 0$. Then G has $2s+n$ hyperbolic generators satisfying the defining relation:

$$c_n \cdots c_1 \, b_s^{-1} \, a_s^{-1} \, b_s \, a_s \cdots b_1^{-1} \, a_1^{-1} \, b_1 \, a_1 \; = \; id$$

from which c_n can be solved. As conclusion the set

$$E \; = \; \{a_1, b_1, a_2^{-1}, b_2, \cdots, a_s^{-1}, b_s, c_1^{-1}, \cdots, c_{n-1}^{-1}\}$$

generates G freely. Moreover, E satisfies the conditions $(i)-(iii)$ in subsection 4.1. Then the mapping $\psi' : T'(G) \longrightarrow \mathbb{R}^{3n-3}$ in Proposition 5 takes the form

$$\psi' : T'(G) \longrightarrow \mathbb{R}^{6s-6+3n}, \;\; \psi'([\mathcal{A}(j(g)]) = (x_1, \cdots, x_{6s-6+3n})$$

with coordinates

$$\begin{aligned}
x_t \;\; &= \;\; k(j(a_i)), \;\; i = 1, \cdots, s, \\
&\;\;\;\;\;\; k(j(b_i)), \;\; i = 1, \cdots, s \\
&\;\;\;\;\;\; k(j(c_i)), \;\; i = 1, \cdots, n-1, \\
&\;\;\;\;\;\; \text{for } \; t = 1, \cdots, 2s+n-1, \\
x_{2s+n-2+t} \;\; &= \;\; k(j(b_i) j(a_1)), \;\; i = 1, \cdots, s, \\
&\;\;\;\;\;\; k(j(a_i)^{-1}) j(a_1)), \;\; i = 2, \cdots, s, \\
&\;\;\;\;\;\; k(j(c_i)^{-1}) j(a_1)) \;\; i = 1, \cdots, n-1, \\
&\;\;\;\;\;\; \text{for } \; t = 2, \cdots 2s+n-1, \\
x_{4s+2n-5+t} \;\; &= \;\; k(j(b_1) j((a_i)^{-1}) j((b_1)^{-1}) j(a_i)), \\
&\;\;\;\;\;\; i = 2, \cdots, s \\
&\;\;\;\;\;\; k(j(b_1)(j(b_i) j((b_1)^{-1}) j((b_i)^{-1})), \\
&\;\;\;\;\;\; i = 2, \cdots, s \\
&\;\;\;\;\;\; k(j(b_1) j((c_i)^{-1}) j((b_1)^{-1}) j(c_i)), \\
&\;\;\;\;\;\; i = 1, \cdots, n-1, \\
&\;\;\;\;\;\; \text{for } \; t = 3, \cdots, 2s+n-1.
\end{aligned}$$

Example 1. Let us construct a compact Riemann surface $S = H/G$ with a covering group G of signature $(1,1)$. Then S consists of one handle from which one conformal disk is removed. Therefore G has a set $E = \{a, b, c^{-1}\}$ of hyperbolic generators with the defining relation

$$c^{-1} \, a \, b \; = \; id,$$

where c is a boundary mapping of the disk satisfying $c = ab$. Then a and b generates G freely, $G =< a, b >$. In fact, the hyperbolic a and b have no comman fixed points and G is a purely hyperbolic group. It does not contain parabolic or elliptic elements. The first one is concluded from [3] and the second one holds because G is a covering group.

An embedding

$$\begin{aligned}
&\psi' : E'(G) \longrightarrow \mathbb{R}^3, \\
&\psi'([\mathcal{A}(j(g)]) = (k(j(a_1)), k(j(b_1)), k(j(b_1) j(a_1)))
\end{aligned}$$

gives the coordinates $(k(j(a_1)), k(j(b_1)), k(j(b_1) j(a_1)))$ of $[\mathcal{A}(j(g)]$ up to homeomorphism for any $g \in G$ and some geometric isomorphism j of G and $j \in J(G)$.

References

[1] P. Kukkurainen, Fuzzy subgroups, algebraic and topological points of view and complex analysis. Thesis for the degree of Doctor of Philosophy, Acta Universitatis Lappeenrantaensis, 2009. In printed.

[2] M. Seppälä and T. Sorvali, Geometry of Riemann Surfaces and Teichmüller Spaces, Elsevier Science Publishers B.V., North-Holland Amsterdam, 1992.

[3] T. Sorvali, Parametrization of free Möbius groups, Annales Academiae Scientiarum Fennicae. Series A. I.Mathematica 579, 1974, 1-12.

[4] T. Sorvali, The boundary mapping induced by an isomorphism of covering groups, Annales Academiae Scientiarum Fennicae. Series A. I.Mathematica 526, 1972, 4-31.

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