A Semi-infinite Programming Approach to Possibilistic Optimization under Necessity Measure Constraints

Masahiro Inuiguchi[1]

1.Graduate School of Engineering Science, Osaka University
Toyonaka, Osaka, Japan
Email: inuiguti@sys.es.osaka-u.ac.jp

Abstract— *In this paper, possibilistic linear programming problems are investigated. After reviewing relations among conjunction and implication functions, necessity fractile optimization models with various implication functions are applied to the possibilistic linear problems. We show that the necessity fractile optimization models are reduced to semi-infinite linear programming problems. A simple numerical example is given to demonstrate the correctness of the result. The paper is concluded with some remarks for further developments.*

Keywords— Possibilistic linear programming, semi-infinite linear programming, necessity measure, implication function, conjunction function

1 Introduction

Fuzzy and possibilistic programming approaches are proposed to mathematical programming problems with ambiguity and vagueness [1, 2, 3]. By those approaches, we obtain reasonable solutions under conflicting soft constraints and goals, robust solutions under hard and soft constraints, hopeful solutions of attaining high-level goals, and so on. In possibilistic programming approaches, possibility and necessity measures are used to reduce the problems to the conventional programming problems. Many results demonstrate that possibilistic linear programming problems preserve the linearity in the reduced problems when possibility and necessity measures are defined by minimum operation and Dienes implication function. However, cases with the other conjunction and implication functions have not yet considerably investigated while several alternative approaches [4, 5] have been proposed in calculation of linear functions with fuzzy coefficients.

In this paper, we treat necessity fractile optimization model of possibilistic linear programming problems with soft constraints. We note that soft constraints include hard constraints as a special case. Then the problems are more general than problems with hard constraints. To construct necessity measures, we allow arbitrary implication satisfying weak requirements. Moreover, implication functions can be different among constraints. By the properties of implication functions, we show that necessity fractile optimization problems are reduced to a semi-infinite linear programming problems. Therefore the problems can be approximated by linear programming problems or the solutions can be obtained by a relaxation procedure together with the simplex method.

In next section, we briefly review the closure of generation procedures among conjunction and implication functions. Moreover, we introduce possibility and necessity measures. Possibilistic linear programming problems with soft constraints are given and reduced to the conventional programming problems through necessity fractile optimization models in Section 3. The main results are shown together with a conceivable solution procedure. in Section 4. In Section 5, a simple numerical example is given to demonstrate the correctness of the main result.

2 Conjunction and Implication Functions

2.1 Definitions

In this paper, a conjunction function is defined as a two-place function $T : [0,1] \times [0,1] \to [0,1]$ satisfying

- (T0) T is lower semi-continuous, (semi-continuity)
- (T1) $T(0,0) = T(0,1) = T(1,0) = 0$ and $T(1,1) = 1$, (boundary condition)
- (T2) $T(a,b) \le T(c,d)$ if $0 \le a \le c \le 1$ and $0 \le b \le d \le 1$. (monotonicity)

A conjunction function T satisfies the following properties (t1) $T(a,1) = T(1,a) = a$ for any $a \in [0,1]$, (T3) $T(a,b) = T(b,a)$ for any $a,b \in [0,1]$ (commutativity) and (T4) $T(a,T(b,c)) = T(T(a,b),c)$ for any $a,b,c \in [0,1]$ (associativity) is known as a triangular norm.

In this paper, an implication function is a two-place function $I : [0,1] \times [0,1] \to [0,1]$ satisfying

- (I0) I is upper semi-continuous, (semi-continuity)
- (I1) $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$, (boundary condition)
- (I2) $I(a,b) \le I(c,d)$ if $0 \le c \le a \le 1$ and $0 \le b \le d \le 1$. (monotonicity)

Given a conjunction function and/or a strong negation n, an implication function can be generated through a transformation. The following three transformations are frequently adopted in the literature for implication generators:

$$I^{\mathrm{R}}[T](a,b) = \sup\{s \in [0,1] \mid T(a,s) \le b\}, \quad (1)$$
$$I^{\mathrm{S}}[T](a,b) = n(T(a,n(b))), \quad (2)$$
$$I^{\mathrm{r-R}}[T](a,b) = \sup\{s \in [0,1] \mid T(n(b),s) \le n(a)\}. \quad (3)$$

The first one, I^{R}, is encountered in the maximum solution of a fuzzy relation equation [6] and adopted in view of modus ponens. The second one, I^{S}, is introduced in analogy to Boolean logic. The last one, $I^{\mathrm{r-R}}$ is reciprocal to the first one which is obtained by taking a contraposition of the first one. When T is a t-norm, $I^{\mathrm{R}}[T]$, $I^{\mathrm{S}}[T]$ and $I^{\mathrm{r-R}}[T]$ are called R-implication (residual implication), S-implication and reciprocal R-implication, respectively. Whereas I^{S} produces an implication function from an arbitrary conjunction function T,

ISBN: 978-989-95079-6-8

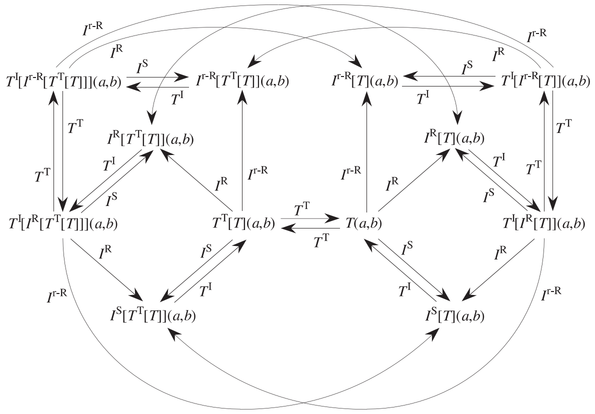

Figure 1: Conjunction and implication generation

I^{R} and $I^{\mathrm{r-R}}$ produce an implication function from a conjunction function which satisfies

$$T(1, a) > 0 \text{ for any } a > 0. \tag{4}$$

On the other hand, a conjunction function can be generated from an implication function through a transformation,

$$T^{\mathrm{I}}[I](a, b) = n(I(a, n(b))). \tag{5}$$

This transformation is symmetrical to I^{S}. From an implication function, a conjunction function is produced through T^{I}.

A conjunction function in this paper is not commutative. Thus, a new conjunction function may be generated from a conjunction function through

$$T^{\mathrm{T}}[T](a, b) = T(b, a). \tag{6}$$

2.2 Closure of generation processes

For the transformations (1)–(3), (5) and (6), the following equalities can be easily shown:

$$T^{\mathrm{I}} \circ I^{\mathrm{S}} = \text{id.},\ \ I^{\mathrm{S}} \circ T^{\mathrm{I}} = \text{id.},\ \ T^{\mathrm{T}} \circ T^{\mathrm{T}} = \text{id.}, \tag{7}$$
$$I^{\mathrm{S}} \circ T^{\mathrm{T}} \circ T^{\mathrm{I}} \circ I^{\mathrm{R}} = I^{\mathrm{r-R}}, \tag{8}$$

where '$\circ$' denotes a composition, for example, $T^{\mathrm{I}} \circ I^{\mathrm{R}}$ is a composite transformation of I^{R} and T^{I}, i.e.,

$$T^{\mathrm{I}} \circ I^{\mathrm{R}}[T](a, b) = T^{\mathrm{I}}[I^{\mathrm{R}}[T]](a, b).$$

The notation 'id.' stands for the identical transformation.

From (T0), we have the following equalities [7]:

$$I^{\mathrm{R}} \circ T^{\mathrm{I}} \circ I^{\mathrm{R}}[T] = I^{\mathrm{S}}[T], \tag{9}$$
$$I^{\mathrm{r-R}} \circ T^{\mathrm{I}} \circ I^{\mathrm{R}} \circ T^{\mathrm{T}}[T] = I^{\mathrm{S}}[T], \tag{10}$$
$$I^{\mathrm{R}} \circ T^{\mathrm{I}} \circ I^{\mathrm{r-R}} \circ T^{\mathrm{T}}[T] = I^{\mathrm{r-R}}[T], \tag{11}$$
$$I^{\mathrm{r-R}} \circ T^{\mathrm{I}} \circ I^{\mathrm{r-R}} \circ T^{\mathrm{T}}[T] = I^{R}[T], \tag{12}$$

Equations (7)–(12) are summed up by Figure 1. As shown in Figure 1, the generation processes from a lower semi-continuous conjunction function as well as from an upper semi-continuous implication function are closed. This result is given by Inuiguchi and Sakawa [7] as the generalization of the result by Dubois and Prade [8]. Note that the semi-continuity is preserved through the generation processes [7].

As shown in Figure 1, we have six conjunction functions and six implication functions. The mappings among those twelve functions are given in Table 1. The inverse mappings are also indicated in Table 1. Some of those functions often appear in literatures on fuzzy relation equations, fuzzy logic, approximate reasoning and so on.

2.3 Possibility and necessity measures

Given a possible range V, the possibility and necessity measures $\Pi_V(S)$ and $N_V(S)$ of a fuzzy event S are defined by

$$\Pi_V(S) = \sup_{u \in U} T(\mu_V(u), \mu_S(u)), \tag{13}$$
$$N_V(S) = \inf_{u \in U} I(\mu_V(u), \mu_S(u)), \tag{14}$$

where μ_V and μ_S are membership functions of fuzzy sets V and S, respectively. U is the universal set. $\Pi_V(S)$ indicates the degree to what extent S is possible under possible range V, while $N_V(S)$ indicates the degree to what extent S is certain under possible range V.

Neither possibility measure nor necessity measure is unique. Therefore, we need to select possibility and necessity measures suitable for the given problem. By their definitions, selections of possibility and necessity measures are equivalent to selections of conjunction and implication functions. In the literature, minimum operation, i.e., $T(a, b) = \min(a, b)$ and Dienes implication, i.e., $I(a, b) = \max(1 - a, b)$ are often used because they are simple and used in the original definitions of possibility and necessity measures [9, 10]. However, there is no guarantee that those possibility and necessity measures represent the decision maker's preferences well. Considering the variety of decision maker's preferences, the original possibility and necessity measures are not sufficiently flexible. Then the selections of possibility and necessity measures are important issues.

A conceivable approach to their selections are proposed by Inuiguchi and Tanino [11] and by Inuiguchi et al. [12]. The approach is based on the images of relations between two fuzzy sets represented by two modifier functions. Because of space limitations, we do not introduce this approach but we treat various kinds of necessity measures in the framework of possibilistic linear programming problems.

3 Linear Program with Necessity Measures

We consider a possibilistic linear programming problem,

$$\begin{array}{rl} \text{maximize} & \mathbf{c}^{\mathrm{T}}\mathbf{x}, \\ \text{subject to} & \mathbf{a}_i^{\mathrm{T}}\mathbf{x} \lesssim_i b_i,\ i = 1, 2, \ldots, m, \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \tag{15}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is a decision vector. b_i, $i = 1, 2, \ldots, m$ are constants. Components c_j and a_{ij} of $\mathbf{c}$ and $\mathbf{a}_i$ are not known exactly but the possible ranges of those values are known as fuzzy numbers C_j and A_{ij}, respectively. A fuzzy number is a normal, convex and bounded fuzzy set on the real line whose membership function is upper semi-continuous. The notation $\lesssim_i$ is a fuzzified inequality so that $\lesssim_i b_i$ corresponds to a fuzzy set B_i with verbal expression 'a set of real numbers which are roughly smaller than b_i'. We assume that the membership function μ_{B_i} of B_i is non-increasing and upper semi-continuous and satisfies $\mu_{B_i}(b_i) = 1$.

ISBN: 978-989-95079-6-8

Table 1: Mappings among twelve conjunction and implication functions

conjunction → implication		
Φ	$\Phi(a,b)$	inverse
$I^{\mathrm{R}}[T]$	$\sup\{s\in[0,1]\mid T(a,s)\le b\}$	$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{I}}[I]]]$
$I^{\mathrm{S}}[T]$	$n(T(a,n(b)))$	$T^{\mathrm{I}}[I]$
$I^{\mathrm{r-R}}[T]$	$\sup\{s\in[0,1]\mid T(n(b),s)\le n(a)\}$	$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]]$
$I^{\mathrm{R}}[T^{\mathrm{T}}[T]]$	$\sup\{s\in[0,1]\mid T(s,a)\le b\}$	$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{I}}[I]]]$
$I^{\mathrm{S}}[T^{\mathrm{T}}[T]]$	$n(T(n(b),a))$	$T^{\mathrm{T}}[T^{\mathrm{I}}[I]]$
$I^{\mathrm{r-R}}[T^{\mathrm{T}}[T]]$	$\sup\{s\in[0,1]\mid T(s,n(b))\le n(a)\}$	$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]]$
implication → conjunction		
Φ	$\Phi(a,b)$	inverse
$T^{\mathrm{I}}[I]$	$n(I(a,n(b)))$	$I^{\mathrm{S}}[T]$
$T^{\mathrm{T}}[T^{\mathrm{I}}[I]]$	$n(I(b,n(a)))$	$I^{\mathrm{S}}[T^{\mathrm{T}}[T]]$
$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{I}}[I]]]$	$\inf\{s\in[0,1]\mid I(a,s)\ge b\}$	$I^{\mathrm{R}}[T]$
$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{I}}[I]]]$	$\inf\{s\in[0,1]\mid I(b,s)\ge a\}$	$I^{\mathrm{R}}[T^{\mathrm{T}}[T]]$
$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]]$	$\inf\{s\in[0,1]\mid I(n(s),n(a))\ge b\}$	$I^{\mathrm{r-R}}[T]$
$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]]$	$\inf\{s\in[0,1]\mid I(n(s),n(b))\ge a\}$	$I^{\mathrm{r-R}}[T^{\mathrm{T}}[T]]$
conjunction → conjunction		
Φ	$\Phi(a,b)$	inverse
$T^{\mathrm{T}}[T]$	$T(b,a)$	$T^{\mathrm{T}}[T]$
$T^{\mathrm{I}}[I^{\mathrm{R}}[T]]$	$n(\sup\{s\in[0,1]\mid T(a,s)\le n(b)\})$	$T^{\mathrm{I}}[I^{\mathrm{R}}[T]]$
$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T]]$	$n(\sup\{s\in[0,1]\mid T(b,s)\le n(a)\})$	$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{T}}[T]]]$
$T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{T}}[T]]]$	$n(\sup\{s\in[0,1]\mid T(s,a)\le n(b)\})$	$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T]]$
$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{T}}[T]]]$	$n(\sup\{s\in[0,1]\mid T(s,b)\le n(a)\})$	$T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{T}}[T]]]$
implication → implication		
Φ	$\Phi(a,b)$	inverse
$I^{\mathrm{R}}[T^{\mathrm{I}}[I]]$	$\sup\{s\in[0,1]\mid I(a,n(s))\ge n(b)\}$	$I^{\mathrm{R}}[T^{\mathrm{I}}[I]]$
$I^{\mathrm{r-R}}[T^{\mathrm{I}}[I]]$	$\sup\{s\in[0,1]\mid I(n(b),n(s))\ge a\}$	$I^{\mathrm{R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$
$I^{\mathrm{R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$	$\sup\{s\in[0,1]\mid I(s,n(a))\ge n(b)\}$	$I^{\mathrm{r-R}}[T^{\mathrm{I}}[I]]$
$I^{\mathrm{S}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$	$I(n(b),n(a))$	$I^{\mathrm{S}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$
$I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$	$\sup\{s\in[0,1]\mid I(s,b)\ge a\}$	$I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I]]]$

By the extension principle, the possible ranges of $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ and $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$ are obtained as fuzzy sets $\mathbf{C}^{\mathrm{T}}\mathbf{x}$ and $\mathbf{A}_i^{\mathrm{T}}\mathbf{x}$, respectively, defined by the following membership functions:

$$\mu_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}(y)=\sup_{\substack{r_1,\dots,r_n\\ \boldsymbol{r}^{\mathrm{T}}\boldsymbol{x}=y}}\min\left(\mu_{C_1}(r_1),\dots,\mu_{C_n}(r_n)\right),\quad(16)$$

$$\mu_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(y)=\sup_{\substack{r_1,\dots,r_n\\ \boldsymbol{r}^{\mathrm{T}}\boldsymbol{x}=y}}\min\left(\mu_{A_{i1}}(r_1),\dots,\mu_{A_{in}}(r_n)\right),\quad(17)$$

where $\mathbf{r}=(r_1,\dots,r_n)^{\mathrm{T}}$.

Using a necessity measure N^i defined by an upper semi-continuous implication function I^i, in this paper, we formulate Problem (15) as a necessity fractile optimization model (see Inuiguchi and Ramík [3]):

$$\begin{aligned}&\text{maximize} && q,\\ &\text{subject to} && N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q,+\infty))\ge h^0,\\ &&& N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i)\ge h^i,\ i=1,2,\dots,m,\\ &&& \mathbf{x}\ge\mathbf{0},\end{aligned}\quad(18)$$

where q is an auxiliary variable. $h^0\in(0,1]$ and $h^i\in(0,1]$, $i=1,2,\dots,m$ are certainty levels of goal achievement and constraint satisfactions specified by the decision maker. Note that we obtain

$$N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q,+\infty))=\inf_{r<q}I^0(\mu_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}(r),0),\quad(19)$$

$$N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i)=\inf_{r}I^i(\mu_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(r),\mu_{B_i}(r)).\quad(20)$$

The selections of necessity measures and certainty levels depend on the required robustness of goal achievement/constraint satisfactions, the meanings of total goal achievement/constraint satisfactions, the estimation of fuzzy coefficients and so on. The method proposed by Inuiguchi and Tanino [11] and by Inuiguchi et al. [12] would be useful for selecting suitable necessity measures.

Let $[S]_h$ be a h-level set of a fuzzy set S, i.e., $[S]_h=\{u\in U\mid \mu_S(u)\ge h\}$. Then, because fuzzy numbers C_i and A_{ij} are bounded and have upper semi-continuous mem-

ISBN: 978-989-95079-6-8

bership functions, we have (see Dubois and Prade [13])

$$[\mathbf{C}^{\mathrm{T}}\mathbf{x}]_h = \sum_{j=1}^{n} [C_j]_h x_j, \tag{21}$$

$$[\mathbf{A}_i^{\mathrm{T}}\mathbf{x}]_h = \sum_{j=1}^{n} [A_{ij}]_h x_j. \tag{22}$$

Let $c_j^{\mathrm{L}}(h) = \inf[C_j]_h$, $c_j^{\mathrm{R}}(h) = \sup[C_j]_h$, $a_{ij}^{\mathrm{L}}(h) = \inf[A_{ij}]_h$ and $a_{ij}^{\mathrm{R}}(h) = \sup[A_{ij}]_h$. Then, considering the non-negativity of $\mathbf{x}$, we have

$$[\mathbf{C}^{\mathrm{T}}\mathbf{x}]_h = \left[\sum_{j=1}^{n} c_j^{\mathrm{L}}(h)x_j, \sum_{j=1}^{n} c_j^{\mathrm{R}}(h)x_j\right], \tag{23}$$

$$[\mathbf{A}_i^{\mathrm{T}}\mathbf{x}]_h = \left[\sum_{j=1}^{n} a_{ij}^{\mathrm{L}}(h)x_j, \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(h)x_j\right]. \tag{24}$$

To reduce Problem (18), the following theorem is useful.

Theorem 1. *Let N^i be a necessity measure defined by an implication function I^i. Then for any fuzzy sets V and S of a universal set U, we have*

$$\begin{aligned}
&N_V^i(S) \geq h \\
&\Leftrightarrow \forall u \in U,\ \forall k \in [0,1]; \\
&\quad \mu_V(u) \geq k \text{ implies } \mu_S(u) \geq T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{I}}[I^i]]](k,h), \\
&\Leftrightarrow \forall k \in [0,1];\ [V]_k \subseteq [S]_{f^i(k,h)},
\end{aligned} \tag{25}$$

where $f^i(k,h) = T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{I}}[I^i]]](k,h)$.

(**Proof**) Because of the upper semi-continuity of I^i, Figure 1 and Table 1, we have

$$\begin{aligned}
&N_V^i(S) = \inf_{u \in U} I^i(\mu_V(u), \mu_S(u)) \geq h \\
&\Leftrightarrow \forall u \in U;\ I^i(\mu_V(u), \mu_S(u)) \geq h \\
&\Leftrightarrow \forall u \in U;\ \mu_V(u) \leq \sup\{s \in [0,1] \mid I^i(s, \mu_S(u)) \geq h\} \\
&\Leftrightarrow \forall u \in U;\ \mu_V(u) \leq I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I^i]]](h, \mu_S(u)) \\
&\Leftrightarrow \forall u \in U,\ \forall k \in [0,1]; \\
&\quad \mu_V(u) \geq k \text{ implies } I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I^i]]](h, \mu_S(u)) \geq k \\
&\Leftrightarrow \forall u \in U,\ \forall k \in [0,1];\ \mu_V(u) \geq k \text{ implies} \\
&\quad \mu_S(u) \geq \inf\{s \in [0,1] \mid I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I^i]]](h,s) \geq k\} \\
&\Leftrightarrow \forall u \in U,\ \forall k \in [0,1];\ \mu_V(u) \geq k \text{ implies} \\
&\quad \mu_S(u) \geq T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{I}}[I^{\mathrm{r-R}}[T^{\mathrm{T}}[T^{\mathrm{I}}[I^i]]]]]](k,h) \\
&\Leftrightarrow \forall u \in U,\ \forall k \in [0,1];\ \mu_V(u) \geq k \text{ implies} \\
&\quad \mu_S(u) \geq T^{\mathrm{I}}[I^{\mathrm{R}}[T^{\mathrm{I}}[I^i]]](k,h).
\end{aligned}$$

(Q.E.D.)

From the assumptions of B_i, we have $[B_i]_k = (-\infty, \bar{b}_i(k)]$, where $\bar{b}_i : [0,1] \to [0,+\infty)$ is defined by $\bar{b}_i(k) = \sup\{r \mid \mu_{B_i}(r) \geq k\}$. From (23), (24) and Theorem 1, Problem (18) is reduced to the following linear semi-infinite programming problem:

$$\begin{aligned}
\text{maximize}\quad & q, \\
\text{subject to}\quad & \sum_{j=1}^{n} c_j^{\mathrm{L}}(k)x_j \geq \bar{q}(f^0(k,h^0)),\ \forall k \in [0,1], \\
& \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k)x_j \leq b_i(f^i(k,h^i)),\ \forall k \in [0,1], \\
& \qquad\qquad i = 1,2,\ldots,m, \\
& \mathbf{x} \geq \mathbf{0},
\end{aligned} \tag{26}$$

where $\bar{q} : [0,1] \to \{-\infty, q\}$ is defined by

$$\bar{q}(k) = \begin{cases} q, & \text{if } k > 0, \\ -\infty, & \text{if } k = 0. \end{cases} \tag{27}$$

We showed that fractile optimization models based on necessity measures defined by any implication functions of possibilistic linear programming problems are reduced to linear semi-infinite programming problems. Thus, we can solve the problems by using linear semi-infinite programming techniques [14]. For example, let $\varepsilon > 0$ be a sufficiently small number and let $k_v = (v-1)(1-\varepsilon)/(l-1)+\varepsilon$, $v = 1,2,\ldots,l$. Then Problem (26) can be approximated by the following linear programming problem:

$$\begin{aligned}
\text{maximize}\quad & q, \\
\text{subject to}\quad & \sum_{j=1}^{n} c_j^{\mathrm{L}}(k_v)x_j \geq \bar{q}(f^0(k_v,h^0)),\ v = 1,\ldots,l, \\
& \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_v)x_j \leq b_i(f^i(k_v,h^i)), \\
& \qquad v = 1,\ldots,l,\ i = 1,\ldots,m, \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned} \tag{28}$$

Let $M = \{1,2,\ldots,m\}$. As another approach to computation of an approximate solution to Problem (26), the following algorithm based on a relaxation procedure is conceivable:

Algorithm

S1. Let $z = 0$ and $\varepsilon > 0$ be a sufficiently small positive number. Solve the following linear programming problem:

$$\begin{aligned}
&\text{maximize } q, \\
&\text{subject to} \\
&\sum_{j=1}^{n} c_j^{\mathrm{L}}(k_0)x_j \geq \bar{q}(f^0(k_0,h^0)), k_0 = \varepsilon, 1, \\
&\sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i)x_j \leq b_i(f^i(k_i,h^i)),\ k_i = \varepsilon, 1,\ i \in M, \\
&\mathbf{x} \geq \mathbf{0}.
\end{aligned}$$

Let $\mathbf{x}^z = (x_1^z, \ldots, x_n^z)^{\mathrm{T}}$ be the obtained optimal solution and q^z be the optimal value.

S2. Calculate $d_0 = \inf_{r\in[0,1]} \sum_{j=1}^{n} c_j^{\mathrm{L}}(r)x_j^z - \bar{q}(f^0(r,h^0))$. If $d_0 < 0$, we define $k_0^z = \arg\inf_{r\in[0,1]} \sum_{j=1}^{n} c_j^{\mathrm{L}}(r)x_j^z - \bar{q}(f^0(r,h^0))$ and otherwise, $k_0^z = 0$.

S3. For $i = 1,2,\ldots,m$, the following processes (a) and (b) are applied. (a) calculate $d_i = \sup_{r\in[0,1]} \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(r)x_j^z - b_i(f^i(r,h^i))$. (b) If $d_0 > 0$, we define $k_0^z = \arg\sup_{r\in[0,1]} \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(r)x_j^z - b_i(f^i(r,h^i))$ and otherwise, $k_i^z = 0$.

ISBN: 978-989-95079-6-8

Table 2: Fuzzy numbers and constraints

$C_1 = \langle 5, 1\rangle$	$C_2 = \langle 7, 0.7\rangle$	
$A_{11} = \langle 2, 0.7\rangle$	$A_{12} = \langle 3, 0.5\rangle$	$B_1 = (230, 60)$
$A_{21} = \langle 4, 1.5\rangle$	$A_{22} = \langle 2, 0.3\rangle$	$B_2 = (370, 100)$
$A_{31} = \langle 1, 0.5\rangle$	$A_{32} = \langle 3, 0.3\rangle$	$B_3 = (140, 90)$

S4. If $\sum_{i=0}^{m} k_i^z = 0$, terminate the algorithm. An approximate solution is obtained as $\mathbf{x}^z$ and q^z.

S5. Let $z = z + 1$. Solve the following linear programming problem:

$$\begin{array}{l} \text{maximize } q, \\ \text{subject to} \\ \displaystyle\sum_{j=1}^{n} c_j^{\mathrm{L}}(k_0)x_j \geq \bar{q}(f^0(k_0, h^0)), k_0 = \varepsilon, 1, \\ \displaystyle\sum_{j=1}^{n} c_j^{\mathrm{L}}(k_0^w)x_j \geq \bar{q}(f^0(k_0^w, h^0)), w = 1, \ldots, z \\ \displaystyle\sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i)x_j \leq b_i(f^i(k_i, h^i)),\ k_i = \varepsilon, 1,\ i \in M, \\ \displaystyle\sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^w)x_j \leq b_i(f^i(k_i^w, h^i)),\ w = 1, \ldots, z,\ i \in M, \\ \mathbf{x} \geq \mathbf{0}. \end{array}$$

Let $\mathbf{x}^z = (x_1^z, \ldots, x_n^z)^{\mathrm{T}}$ be the obtained optimal solution and q^z be the optimal value. Go to S2.

4 A Numerical Example

In order to demonstrate that Problem (26) is equivalent to Problem (18), in other words, an optimal solution to Problem (28) is an approximate solution to Problem (18), we consider the possibilistic linear programming problem (15) with $m = 3$. Fuzzy numbers C_j and A_{ij} are given by symmetric triangular fuzzy numbers whose center values and spreads are shown in Table 2, where a symmetric triangular fuzzy number Q=$\langle q^{\mathrm{C}}, q^{\mathrm{S}}\rangle$ ($q^{\mathrm{S}} > 0$) is characterized by a membership function,

$$\mu_Q(r) = \begin{cases} 1 - \dfrac{|r - q^{\mathrm{C}}|}{q^{\mathrm{S}}} & \text{if } r \in [q^{\mathrm{C}} - q^{\mathrm{S}}, q^{\mathrm{C}} + q^{\mathrm{S}}], \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

As shown in Table 2, fuzzy constraints B_i are defined by fuzzy sets $B_i = (b_i^{\mathrm{C}}, b_i^{\mathrm{S}}\rangle$ with a linear membership function,

$$\mu_{B_i}(r) = \begin{cases} 1 & \text{if } r < b_i^{\mathrm{C}}, \\ 1 - \dfrac{r - b_i^{\mathrm{C}}|}{b_i^{\mathrm{S}}} & \text{if } r \in [b_i^{\mathrm{C}}, b_i^{\mathrm{C}} + b_i^{\mathrm{S}}], \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

We apply Problem (18) with reciprocal Goguen implication $I^0(a, b) = \min(1, (1-a)/(1-b))$ with definition $(1-a)/0 = +\infty$ for any $a \leq 1$, Lukasiewicz implication $I^1(a, b) = \min(1, 1-a+b)$, Reichenbach implication $I^2(a, b) = 1-a+ab$ and Goguen implication $I^3(a, b) = \min(1, b/a)$ with definition $b/0 = +\infty$ for any $b \geq 0$. Then, for $h > 0$ and $k > 0$, we obtain $f_0(k, h) = \max(0, 1 - (1 - k)/h)$, $f_1(k, h) = \max(0, k + h - 1)$, $f_2(k, h) = \max(0, (k + h - 1)/k)$ and $f_3(k, h) = kh$. Moreover, we set $h^0 = 0.5$, $h^1 = 0.5$, $h^2 = 0.62$ and $h^3 = 0.62$.

Let $\varepsilon = 0.0000001$ and $l = 20$, we solve Problem (28), i.e.,

$$\begin{array}{ll} \multicolumn{2}{l}{\text{maximize } q,} \\ \multicolumn{2}{l}{\text{subject to}} \\ 4.500x_1 + 6.650x_2 \geq q, & 4.550x_1 + 6.685x_2 \geq q, \\ 4.600x_1 + 6.720x_2 \geq q, & 4.650x_1 + 6.755x_2 \geq q, \\ 4.700x_1 + 6.790x_2 \geq q, & 4.750x_1 + 6.825x_2 \geq q, \\ 4.800x_1 + 6.860x_2 \geq q, & 4.850x_1 + 6.895x_2 \geq q, \\ 4.900x_1 + 6.930x_2 \geq q, & 4.950x_1 + 6.965x_2 \geq q, \\ 5x_1 + 7x_2 \geq q, & 2.350x_1 + 3.250x_2 \leq 290.00, \\ 2.315x_1 + 3.225x_2 \leq 287.00, & 2.280x_1 + 3.200x_2 \leq 284.00, \\ 2.245x_1 + 3.175x_2 \leq 281.00, & 2.210x_1 + 3.150x_2 \leq 278.00, \\ 2.175x_1 + 3.125x_2 \leq 275.00, & 2.140x_1 + 3.100x_2 \leq 272.00, \\ 2.105x_1 + 3.075x_2 \leq 269.00, & 2.070x_1 + 3.050x_2 \leq 266.00, \\ 2.035x_1 + 3.025x_2 \leq 263.00, & 2x_1 + 3x_2 \leq 260, \\ 4.900x_1 + 2.180x_2 \leq 465.00, & 4.825x_1 + 2.165x_2 \leq 454.44, \\ 4.750x_1 + 2.150x_2 \leq 446.00, & 4.675x_1 + 2.135x_2 \leq 439.09, \\ 4.600x_1 + 2.120x_2 \leq 433.33, & 4.525x_1 + 2.105x_2 \leq 428.46, \\ 4.450x_1 + 2.090x_2 \leq 424.29, & 4.375x_1 + 2.075x_2 \leq 420.67, \\ 4.300x_1 + 2.060x_2 \leq 417.50, & 4.225x_1 + 2.045x_2 \leq 414.71, \\ 4.150x_1 + 2.030x_2 \leq 412.22, & 4.075x_1 + 2.015x_2 \leq 410.00, \\ 4x_1 + 2x_2 \leq 408, & 1.500x_1 + 3.300x_2 \leq 230.00, \\ 1.475x_1 + 3.285x_2 \leq 227.21, & 1.450x_1 + 3.270x_2 \leq 224.42, \\ 1.425x_1 + 3.255x_2 \leq 221.63, & 1.400x_1 + 3.240x_2 \leq 218.84, \\ 1.375x_1 + 3.225x_2 \leq 216.05, & 1.350x_1 + 3.210x_2 \leq 213.26, \\ 1.325x_1 + 3.195x_2 \leq 210.47, & 1.300x_1 + 3.180x_2 \leq 207.68, \\ 1.275x_1 + 3.165x_2 \leq 204.89, & 1.250x_1 + 3.150x_2 \leq 202.10, \\ 1.225x_1 + 3.135x_2 \leq 199.31, & 1.200x_1 + 3.120x_2 \leq 196.52, \\ 1.175x_1 + 3.105x_2 \leq 193.73, & 1.150x_1 + 3.090x_2 \leq 190.94, \\ 1.125x_1 + 3.075x_2 \leq 188.15, & 1.100x_1 + 3.060x_2 \leq 185.36, \\ 1.075x_1 + 3.045x_2 \leq 182.57, & 1.050x_1 + 3.030x_2 \leq 179.78, \\ 1.025x_1 + 3.015x_2 \leq 176.99, & 1x_1 + 3x_2 \leq 174.2, \\ x_1 \geq 0,\ x_2 \geq 0. & \end{array} \tag{31}$$

Solving this problem, we obtain an optimal solution $\mathbf{x}^* = (x_1^*, x_2^*) = (79.508, 31.564)$ with the optimal value $q^* = 567.69$. As shown in Figures 2$\sim$5, we can observe that the obtained solution approximately satisfies constraints, $N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q^*, +\infty)) \geq 0, 5$ and $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\mathbf{x}}(B_i) \geq h^i$, $i = 1, 2, 3$.

5 Concluding Remarks

Utilizing the closure of generation processes of conjunction and implication functions, we have shown that necessity fractile optimization models of possibilistic linear programming problems are reduced to semi-infinite linear programming problems. Therefore, we can solve the problems approximately by linear programming techniques. As a solution procedure for the reduced semi-infinite linear programming problem, we have described a relaxation procedure. In order to demonstrate the main result, we have given a simple numerical example. In the example, the reduced semi-infinite linear programming problem is approximated by a linear programming problem having coarsely-sampled constraints. It has been confirmed that the obtained solution approximately satisfies the required constraints on necessity measures by figures.

The investigation on possibilistic linear programming problems with necessity measures is important to have flexible

ISBN: 978-989-95079-6-8

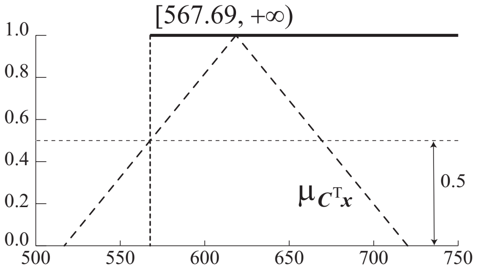

Figure 2: The relation between $\mathbf{C}^{\mathrm{T}}\mathbf{x}^*$ and the optimal value

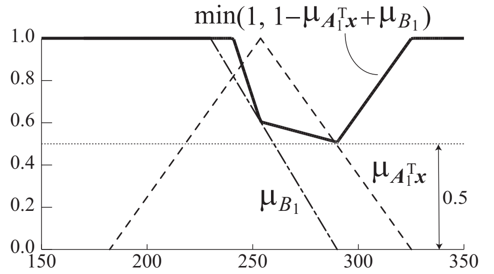

Figure 3: The satisfaction of $N^1_{\boldsymbol{A}_1^{\mathrm{T}}\mathbf{x}}(B_1) \geq 0.5$

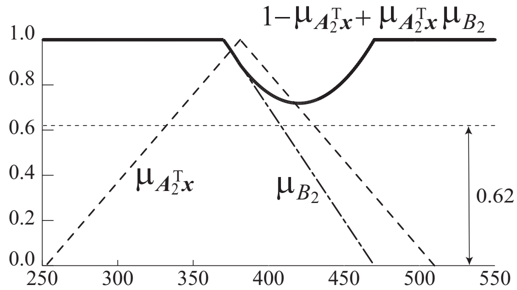

Figure 4: The satisfaction of $N^2_{\boldsymbol{A}_2^{\mathrm{T}}\mathbf{x}}(B_2) \geq 0.62$

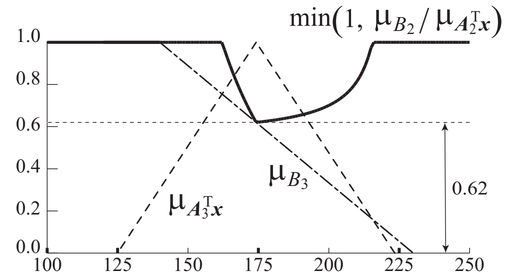

Figure 5: The satisfaction of $N^3_{\boldsymbol{A}_3^{\mathrm{T}}\mathbf{x}}(B_3) \geq 0.62$

models of robust optimization reflecting decision makers' various attitudes toward the uncertainty. Fortunately, models with necessity measures are often easier than models with possibility measures. Analyzing some special implication functions defining necessity measures and special shapes of fuzzy coefficients, the reduced semi-infinite constraints can further reduced to finite constraints. Moreover, we may solve necessity measure optimization models of possibilistic linear programming problems by the simplex method together with the bisection method. These would be future topics in our studies.

Acknowledgment

The author acknowledges that this work has been partially supported by the Grant-in-Aid for Scientific Research (B) No. 17310098, and thanks Yuji Matsumoto (Osaka University) for his help in solving a linear programming problem.

References

[1] M. Inuiguchi, H. Ichihashi and H.T̃anaka. Fuzzy programming: A survey of recent developments. in: R. Słowiński and J. Teghem (eds.). *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*. Kluwer Academic Publishers, Dordrecht, 45–68, 1990.

[2] H. Rommelfanger and R. Słowiński. Fuzzy linear programming with single or multiple objective functions. in: R. Słowiński (ed.). *Fuzzy Sets in Decision Analysis, Operations Research and Statistics*. Kluwer Academic Publishers, Boston, 179–213, 1998.

[3] M. Inuiguchi and J. Ramík. Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems*, 111: 29–45, 2000.

[4] H. Rommelfanger and T. Keresztfalvi. Multicriteria fuzzy optimization based on Yager's parameterized t-norm. *Foundations of Computing and Decision Sciences*, 16: 99–110, 1991.

[5] M. Inuiguchi. Linear programming with interactive fuzzy numbers. *Proceedings of 40th Annual Convention of ORSI*, 27–37, 2007.

[6] E. Sanchez. Resolution of composite fuzzy relation equations. *Information and Control*, 30: 38–48, 1976.

[7] M. Inuiguchi and M. Sakawa. On the closure of generation processes of implication functions from a conjunction function. in: T. Yamakawa and G. Matsumoto (eds.). *Methodologies for the Conception, Design, and Application of Intelligent Systems*. World Scientific, Singapore, 1: 327–330, 1996.

[8] D. Dubois and H. Prade. A theorem on implication functions defined from triangular norms. *Stochastica*, 8: 267–279, 1984.

[9] L.A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1: 3–28, 1978.

[10] D. Dubois and H. Prade. *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York, 1980.

[11] M. Inuiguchi and T. Tanino. Necessity measures and parametric inclusion relations of fuzzy sets. *Fundamenta Informaticae*. 42: 279–302, 2000.

[12] M. Inuiguchi, S. Greco, R. Słowiński and T. Tanino. Possibility and necessity measure specification using modifiers for decision making under fuzziness. *Fuzzy Sets and Systems*. 137: 151–175, 2003.

[13] D. Dubois and H. Prade. Fuzzy numbers: An overview. in: J.C. Bezdek (ed.). *Analysis of Fuzzy Information, Vol.I: Mathematics and Logic*. CRC Press, Boca Raton, FL, pp.3–39, 1987.

[14] M.A. Goberna and M.A. López. *Linear Semi-infinite Optimization*. John Wiley & Sons, Chichester, 1998.

ISBN: 978-989-95079-6-8