On Aristotle's NC and EM Principles in Three-valued Logics*

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Abstract— *By interpreting* 'p *is* impossible' *by* 'p *is* self-contradictory', *and* 'p *is* always' *by* 'not p *is* self-contradictory', *this paper studies which of the three-valued systems of Łucasiewicz, Gödel, Kleene, Bochvar, and Post, do verify the Aristotle's principles of Non-Contradiction (NC), and Excluded-Middle (EM).*

Keywords— Systems of three-valued logic, Non-Contradiction, Excluded-Middle.

1 Introduction

As it is well known, the systems of multiple-valued logic do not verify the *principles of non-contradiction* and *excluded-middle*, once presented in the forms (see [5])

$$a \cdot a' = 0, \; a + a' = 1.$$

Nevertheless, in [7] a different way of looking at these two principles, and based on the concept of self-contradiction, respectively $a \cdot a' \leq (a \cdot a')'$, and $(a + a')' \leq ((a + a')')'$, for all a, b, was introduced. These expressions, that are even in more agreement with the original formulation given by Aristotle (see [1]), allow the verification of the two principles for a wide class of structures that include all algebras of fuzzy sets, as well as De Morgan algebras (see [5]). All that meant some progress for what concerns the verification of the two principles.

This paper tries to study the verification of the two Aristotle's principles once interpreted as in [7] , by some of the most well known systems of three-valued logic, namely, by those of the Łukasiewicz, Gödel, Kleene, Bochvar and Post. It is first shown that in Łukasiewicz, Gödel and Kleene cases the two principles are verified, that in the case of Bochvar only one of them is verified, and that in the case of Post no one is verified. Everything is done in agreement with the implication's semantics, as given at each case by its truth-table. At the end, although far from the semantical interpretation at each case, it is shown how the five systems could verify the two principles in the sense introduced in [7].

2 Systems of MV logic

2.1

Let Ω be a set consisting of propositions $p, q, r, \ldots$ and such that,

- If $p \in \Omega$, then '*not p*'$\in \Omega$
- If $p, q \in \Omega$, then '*p and q*', '*p or q*', and '*If p, then q*', are in Ω

2.2

Let L be a set with at least three elements, and endowed with a unary operation $' : L \to L$, and three binary operations $\cdot, +, \to : L \times L \to L$. Suppose there exists 0, 1 in L such that $0 \neq 1$, and $0' = 1$.

2.3

(Ω, L, t) is a *system of multiple-valued logic*, if $t : \Omega \to L$ (truth-function) verifies:

- $t(\text{not } p) = t(p)'$
- $t(p \text{ and } q) = t(p) \cdot t(q)$
- $t(p \text{ or } q) = t(p) + t(q)$
- $t(\text{if } p, \text{ then } q) = t(p) \to t(q)$

for all p, q in Ω. If L is with n elements, the system is an *n-valued* one.

2.4

Two systems of multiple-valued logic, (Ω_1, L_1, t_1) and (Ω_2, L_2, t_2), are *isomorphic* if there exist a pair of bijections (Φ, φ) such that

- $\Phi : \Omega_1 \to \Omega_2, \quad \varphi : L_1 \to L_2$
- $\varphi(a') = \varphi(a)', \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b), \varphi(a+b) = \varphi(a) + \varphi(b), \varphi(a \to b) = \varphi(a) \to \varphi(b)$ with the corresponding operations in L_1, L_2
- $t_1 = \varphi^{-1} \circ t_2 \circ \Phi$, or $t_2(\Phi(p)) = \varphi(t_1(p))$ for all $p \in \Omega_1$.

Example.

The three-valued system of Łukasiewicz, $Ł_3$, (see [5]) is obtained with $L = \{T, F, I\}$ and the four operations given by the following tables:

	$'$
T	F
I	I
F	T

$\bullet$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

$+$	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

$\to$	T	I	F
T	T	I	F
I	T	T	I
F	T	T	T

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where $T, I, F \in L$, $T \neq F$, and $F' = T$.

With $L_0 = \{1, \frac{1}{2}, 0\}$, and $\varphi : L \to L_0$ given by $\varphi(T) = 1, \varphi(I) = \frac{1}{2}, \varphi(F) = 0$, the tables are translated into,

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\bullet$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

Then, *any system* $Ł_3(\Omega, L, t)$ *is isomorphic to* (Ω, L_0, t^*), with $t^*(p) = \varphi(t(p))$, for all $p \in \Omega$.

Notice that it is,

$a' = 1 - a, a \cdot b = \min(a, b), a + b = \max(a, b)$,

$a \to b = \min(1, 1 - a + b)$, for all $a, b \in L_0$.

Hence,

$t(\text{not } p) = 1 - t(p), t(p \text{ and } q) = \min(t(p), t(q))$,

$t(p \text{ or } q) = \max(t(p), t(q))$,

$t(\text{If } p, \text{ then } q) = \min(1, 1 - t(p) + t(q))$,

for all $p, q \in \Omega$.

Notice that $(a')' = a'' = a$.

3 The other L_0 three-valued systems

Analogously, we can obtain the corresponding systems of Gödel, Bochvar, Kleene and Post (see [5], [2]), by their isomorphic images with $\varphi : L \to L_0$ given by $\varphi(T) = 1, \varphi(I) = \frac{1}{2}, \varphi(F) = 0$.

3.1

Gödel (G_3), is translated into L_0 by

	$'$
1	0
$\frac{1}{2}$	0
0	1

$\bullet$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	0
0	1	1	1

or, for all $a \in L_0$,

$$a' = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{if } a \neq 0 \end{cases}$$

$a \cdot b = \min(a, b)$

$a + b = \max(a, b)$

$$a \to b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$$

Notice that, the negation $'$ is not strong, since $(\frac{1}{2}')' = 0' = 1$.

3.2

Kleene (K_3), is translated into L_0 by

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\bullet$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

or, for all $a \in L_0$,

$a' = 1 - a$,

$a \cdot b = \min(a, b)$,

$a + b = \max(a, b)$,

$a \to b = \max(1 - a, b)$.

Notice that $(a')' = a'' = a$, for all $a \in L_0$ (the negation is strong).

3.3

Bochvar (B_3) is translated into L_0 by

	$'$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\bullet$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$+$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	1

or,

$a' = 1 - a$,

$$a \cdot b = \begin{cases} \frac{1}{2}, & \text{if } (a, b) \in \{(\frac{1}{2}, 0), (0, \frac{1}{2})\} \\ \min(a, b), & \text{otherwise} \end{cases},$$

$$a + b = \begin{cases} \frac{1}{2}, & \text{if } (a, b) \in \{(1, \frac{1}{2}), (\frac{1}{2}, 1)\} \\ \max(a, b), & \text{otherwise} \end{cases},$$

$a \to b = a' + b$.

Notice that $(a')' = a'' = a$, for all $a \in L_0$ (the negation is strong).

3.4

Post (P_3), is translated into L_0 by

	$'$
1	$\frac{1}{2}$
$\frac{1}{2}$	0
0	1

$\bullet$	1	$\frac{1}{2}$	0
1	0	0	$\frac{1}{2}$
$\frac{1}{2}$	0	1	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$+$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

or,

$$a' = \begin{cases} 1, & \text{if } a = 0 \\ |\frac{1}{2} - a|, & \text{if } a \neq 0 \end{cases},$$

$a \cdot b = (a' + b')'$,

$a + b = \max(a, b)$,

$a \to b = a' + b$.

Notice that, for all $a \in L_0$, is $(a')' = a'' \neq a$ (the negation is not strong).

Remark. 3.1. In the systems $Ł_3, K_3, G_3, B_3$ but not in P_3, holds the following:

- $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0, 1 \cdot 1 = 1$
- $0 + 1 = 1 + 0 = 1 + 1 = 1, 0 + 0 = 0$
- $0' = 1, 1' = 0$

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- $1 \to 1 = 0 \to 1 = 0 \to 0 = 1, 1 \to 0 = 0$

Then, these four three-valued systems do contain the classical system of two-valued logic based on the values $\{0, 1\}$ that, notwithstanding, is not contained in P_3, a system in which only the operation + preserves such classical case.

4 The two principles

4.1

Triplets $(L, \models, ')$ consisting of a non-empty set L, a transitive relation $\models \subset L \times L$, and a mapping $' : L \to L$ reversing $\models$: if $a \models b$, then $b' \models a'$, are called *transitive reversing systems* (TRS for short) (see [7])

4.2

An operation $\cdot : L \times L \to L$, is a *type-1 operation* in $(L, \models, ')$, provided $a \cdot a' \models a$ and $a \cdot a' \models a'$, for all $a \in L$.

An operation $+ : L \times L \to L$, is a *type-2 operation* in $(L, \models, ')$, provided $a \models a + a'$ and $a' \models a + a'$, for all $a \in L$.

Theorem 4.1. *(ANC, for Aristotle's Non-contradiction).*
If $(L, \models, ')$ is a TRS, and $\cdot$ is a type-1 operation, it holds

$$a \cdot a' \models (a \cdot a')' \quad (1)$$

for all $a \in L$ (see [7])

Theorem 4.2. *(AEM, for Aristotle's Excluded-middle).*
If $(L, \models, ')$ is a TRS, and $+$ is a type-2 operation, it holds

$$(a + a')' \models ((a + a')')' \quad (2)$$

for all $a \in L$ (see [7])

Remark. 4.3. By defining 'a is selfcontradictory' as $a \models a'$, and by taking $'$ representing 'not', $\cdot$ representing 'and', it is clear that (1) can be read

'a and not a, is self-contradictory'.

Hence, interpreting 'impossible' by 'self-contradictory' (1) represents 'a and not a, is impossible'. That is, (1) can be read as an algebraic translation of the Aristotle's Non-contradiction principle.

Remark. 4.4. Like in the previous remark, and taking + representing 'or', (2) can be read as 'not (a or not a) is self - contradictory', or 'not (a or not a) is impossible', an algebraic translation of the Aristotle's Excluded-middle principle usually stated as 'a or not a is always'.

4.3

Theorem 4.5. *Given a triplet $(L, \cdot, ')$, with $L \neq \emptyset$, $\cdot : L \times L \to L$, and $' : L \to L$, the relation $\models_{NC} \subset L \times L$, given by the set of pairs*

$$\models_{NC} = \{(a \cdot a', (a \cdot a')'); a \in L\},$$

assures the verification of the principle ANC.

Proof. Obviously, $a \cdot a' \models_{NC} (a \cdot a')'$, for all $a \in L$. □

Theorem 4.6. *Given a triplet $(L, +, ')$, with $L \neq \emptyset$, $+ : L \times L \to L$, and $' : L \to L$, the relation $\models_{EM} \subset L \times L$, given by the set of pairs*

$$\models_{EM} = \{((a + a')', ((a + a')')'); a \in L\},$$

assures the verification of the principle AEM

Proof. Obviously, $(a + a')' \models_{NC} (a + a')')'$, for all $a \in L$. □

Theorem 4.7. *Given a quadruplet $(L, \cdot, +, ')$, with $L \neq \emptyset, \cdot : L \times L \to L$, $+ : L \times L \to L$, and $' : L \to L$, the relation*

$$\models = \models_{NC} \cup \models_{EM} \subset L \times L$$

assures the verification of the two principles ANC and AEM.

Proof. Obviously, for all $a \in L$, is

- $a \cdot a' \models_{NC} (a \cdot a')'$ and $a \cdot a' \models (a \cdot a')'$
- $(a + a')' \models_{EM} ((a + a')')'$, and $(a + a')' \models ((a + a')')'$. □

It is to be noticed that *when $'$ verifies* $(a')' = a'' = a$, ($'$ is *strong*), *the triplets* $(L, \models_{NC}, ')$ and $(L, \models_{EM}, ')$ are TRS, since:

- $a \cdot a' \models_{NC} (a \cdot a')'$, and $(a \cdot a')'' = a \cdot a'$
- $(a+a')' \models_{EM} ((a+a')')'$ is equivalent to $(a+a')' \models_{EM} a + a'$

Of course, *this does not mean that those triplets are TRS only when $'$ is strong*.

Remark. 4.8. It is obvious that the set of relations $\models$ for which $(L, \cdot, +, ')$ verifies, respectively, the principle ANC or the principle AEM, is not empty. If $\models$ allows to verify ANC, it is $\models_{NC} \subset \models$, if $\models$ allows to verify AEM is $\models_{EM} \subset \models$, and if $\models$ allows both principles is $\models_{NC} \cup \models_{EM} \subset \models$

Remark. 4.9. It should be pointed out that the last theorems ignore what follows. *The unary operation $'$ should represent 'not', and the relation $a \models b$ should represent* '*If a, then b*'. Otherwise, the interpretation of 'If a, then not a' by means of $a \models a'$ (a is self-contradictory) has no sense at all. Hence, *and provided $a \cdot b$ does represent* 'a and b', *those representations are crucial for asserting that $a \cdot a' \models (a \cdot a')'$ is an interpretation of* "'a and not a' is a self-contradictory statement". And analogously with $(a + a')' \models ((a + a')')'$.

4.4

Let us call *Modern Non-contradiction Principle* (MNC, for short), the statement '$a \cdot a' = 0$ for all $a \in L$', and *Modern Excluded-middle Principle* (MEM, for short), '$a + a' = 1$ for all $a \in L$'

Theorem 4.10. *Given a triplet $(L, \models, ')$, an operation $\cdot : L \times L \to L$, and $0 \in L$, such that*

- $0 \models 0'$
- $a \cdot a' = 0$ *(MNC)*

then, it holds ANC.

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Proof. $a \cdot a' = 0 \models 0' = (a \cdot a')'$. $\square$

Theorem 4.11. *Given a triplet* $(L, \models, ')$*, an operation* $+ : L \times L \to L$*, and* $1 \in L$*, such that*

- $1' \models (1')'$
- $a + a' = 1$ *(MEM)*

then, it holds AEM.

Proof. $(a + a')' = 1' \models (1')' = ((a + a')')'$. $\square$

Remark. 4.12. If $1' = 0$, then $1' \models (1')'$ is equivalent to $0 \models 0'$

Remark. 4.13. In the sense of theorems 4.10 and 4.11, MNC and MEM are particular cases of ANC and AEM, respectively.

Remark. 4.14. Apart of the three operations $(\cdot, +, ')$, the principles ANC and AEM only involve the relation $\models$, but the principles MNC and MEM do involve the relation =, and the 'singular' elements 0,1.

4.5

In what follows this paper is devoted to study if the three-valued systems $Ł_3, G_3, K_3, B_3$, and P_3, do verify the principles ANC or AEM once expressed in the above algebraic terms.

5 Posing the problem for the five three-valued systems

Given a MVL system (Ω, L_0, t), consider its 'values' part $(L_0, ', \cdot, +, \to)$, and the binary relations $\leq$ and $\models$ given, respectively, by

$a \models b \Leftrightarrow a \to b = 1$, and $\leq$ is the linear order in $L_0 = \{1, \frac{1}{2}, 0\}$, inherited from $\mathbb{R}$.

- *Case* $Ł_3$.

  It is $a \to b = 1 \Leftrightarrow \min(1, 1 - a + b) = 1 \Leftrightarrow a \leq b$. That is, $\models_Ł = \leq$.

  Since $a' = 1 - a$ implies "$a \leq b \Leftrightarrow b' \leq a'$", $(L_0, \leq, ')$ is a TRS in which, obviously, the operation $\cdot = \min$ is one of type-1, and $+ = \max$ is one of type-2. Hence, it holds

  $$a \cdot a' \leq (a \cdot a')',\ (a + a')' \leq ((a + a')')',$$

  for all $a \in L_0$.

- *Case* G_3.

  It is $a \to b = 1 \Leftrightarrow a \leq b$. That is, $\models_G = \leq$, like in the case $Ł_3$. Since it is also $\cdot = \min$, $+ = \max$, it also holds

  $$a \cdot a' \leq (a \cdot a')',\ (a + a')' \leq ((a + a')')',$$

  for all $a \in L_0$.

- *Case* K_3.

  Since, $\models_K = \{(1,1), (\frac{1}{2}, 1), (0,1), (0, \frac{1}{2}), (0,0)\} \subset \leq$, and $\models'_K = \{(0,0), (0, \frac{1}{2}), (0,1), (\frac{1}{2}, 1), (1,1)\} = \models_K$, it means that $(L_0, \models_K, ')$ is a TRS. Since, $a' = 1 - a$, it is also $(L_0, \leq, ')$ a TRS.

  From $\frac{1}{2} \cdot \frac{1}{2}' = \frac{1}{2}$, $(\frac{1}{2} \cdot \frac{1}{2}')' = \frac{1}{2}$, and $(\frac{1}{2}, \frac{1}{2}) \notin \models_K$, follows that it is not $a \cdot a' \models (a \cdot a')'$, for all $a \in L_0$.

  Analogously, $(\frac{1}{2} + \frac{1}{2}')' = \frac{1}{2}$, $(\frac{1}{2} + \frac{1}{2})'' = \frac{1}{2}$, shows that it is not $(a + a')' \models ((a \cdot a')')$, for all $a \in L_0$.

  Nevertheless, since $\cdot$ denoting $\min$ is a type-1 operation in $(L_0, \leq, ')$, and $+$ denoting $\max$ is a type-2 operation in $(L_0, \leq, ')$, it follows

  $$a \cdot a' \leq (a \cdot a')',\ (a + a')' \leq ((a + a')')',$$

  for all $a \in L_0$.

- *Case* B_3.

  It is $\models_B = \{(1,1), (0,1), (0,0)\} \subset \leq$, and $\models'_B = \{(b', a'); (a,b) \in \models_B\} = \{(0,0), (0,1), (1,1)\} = \models_B$. That is, $(L_0, \models_B, ')$ and $(L_0, \leq, ')$ are TRSs.

  Since, $\frac{1}{2} \cdot \frac{1}{2}' = \frac{1}{2}$ and $(\frac{1}{2} \cdot \frac{1}{2}')' = \frac{1}{2}$, but $(\frac{1}{2}, \frac{1}{2}) \notin \models_B$, it is not $a \cdot a' \models_B (a \cdot a')'$, for all $a \in L_0$.

  Since, $(\frac{1}{2} + \frac{1}{2}')' = \frac{1}{2}' = \frac{1}{2}$ and $((\frac{1}{2} + \frac{1}{2}')')' = \frac{1}{2}$, it is not $(a + a')' \models_B ((a + a')')'$, for all $a \in L_0$.

  Notwithstanding,

  $$\left.\begin{array}{lll} 1 \cdot 1' = 0, (1 \cdot 1')' = 0' = 1 & \Rightarrow & 0 \leq 1 \\ \frac{1}{2} \cdot \frac{1}{2}' = \frac{1}{2}, (\frac{1}{2} \cdot \frac{1}{2}')' = \frac{1}{2} & \Rightarrow & \frac{1}{2} \leq \frac{1}{2} \\ 0 \cdot 0' = 0, (0 \cdot 0')' = 0' = 1 & \Rightarrow & 0 \leq 1 \end{array}\right\} \Rightarrow$$

  $a \cdot a' \leq (a \cdot a')'$, for all $a \in L_0$.

  Analogously,

  $$\left.\begin{array}{lll} (1 + 1')' = 1, ((1 + 1')')' = 1' + 1 = 1 & \Rightarrow & 1 \leq 1 \\ (\frac{1}{2} + \frac{1}{2}')' = \frac{1}{2}, ((\frac{1}{2} + \frac{1}{2}')')' = \frac{1}{2} & \Rightarrow & \frac{1}{2} \leq \frac{1}{2} \\ (0 + 0')' = 0, ((0 + 0')')' = 0 + 0' = 1 & \Rightarrow & 0 \leq 1 \end{array}\right\} \Rightarrow$$

  $(a + a')' \leq ((a + a')')'$, for all $a \in L_0$.

- *Case* P_3.

  It is $\models_P = \{(1,1), (\frac{1}{2}, 1), (0,1), (0, \frac{1}{2}), (0,0)\} \subset \leq$, but since $(\frac{1}{2}, \frac{1}{2}) \in \models'_P$ it is not $\models'_P \subset \models_P$, and $(L_0, \models_P, ')$ is not a TRS. In addition, also $(L_0, \leq_P, ')$ it is not a TRS, since $\frac{1}{2} \leq 1$, but $1' = \frac{1}{2} \not\leq 0 = \frac{1}{2}'$.

  From, $0 \cdot 0' = 0 \cdot 1 = \frac{1}{2}$, $(0 \cdot 0')' = \frac{1}{2}' = 0$, and neither $(\frac{1}{2}, 0) \models 0$, nor $\frac{1}{2} \leq 0$, it follows that it is not $a \cdot a' \models_P (a \cdot a')'$, nor $a \cdot a' \leq (a \cdot a')'$, for all $a \in L_0$.

  From, $(0 + 0')' = \frac{1}{2}$, $((0 + 0')')' = 0$, it follows that it is neither $(a + a')' \models_P ((a + a')')'$, nor $(a + a')' \leq ((a + a')')'$, for all $a \in L_0$.

6 The solutions to the problem

6.1

Given a multiple-valued system (Ω, L_0, t), consider the two relations:

$$p \models q \Leftrightarrow t(p) \models t(q),\ \text{and}\ \ p \leq q \Leftrightarrow t(p) \leq t(q),$$

and define:

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- The MV-system (Ω, L_0, t) verifies the principle ANC respect to $\vDash, \leq$, respectively, whenever

  p and not p $\vDash$ *not* (*p and not p*),
  or, *p and not p* $\leq$ *not* (*p and not p*)

- The MV-system (Ω, L_0, t) verifies the principle AEM respect to $\vDash, \leq$, respectively, whenever

  not (*p or not p*) $\vDash$ *not*(*not* (*p or not p*)),
  or, *not*(*p or not p*) $\leq$ *not*(*not* (*p or not p*)),

  for all $p \in \Omega$.

6.2

Hence, from the results in section 5, follows:

1. Three-valued *Łukasiewicz* systems *do verify* ANC *and* AEM, *with respect to* $\vDash_{Ł} = \leq$
2. Three-valued *Gödel* systems *do verify* ANC *and* AEM, *with respect to* $\vDash_G = \leq$
3. Three-valued *Kleene* systems *verify neither* ANC ,*nor* AEM, *with respect to* $\vDash_K \subset \leq$
4. Three-valued *Kleene* systems *do verify* ANC *and* AEM, *with respect to* $\leq$
5. Three-valued *Bochvar* systems *verify neither* ANC, *nor* AEM, *with respect to* $\vDash_B \subset \vDash_K \subset \leq$
6. Three-valued *Bochvar* systems *do verify* ANC *and* AEM, *with respect to* $\leq$
7. Three-valued *Post* systems *verify neither* ANC, *nor* AEM, *with respect to* $\vDash_P = \vDash_K \subset \leq$
8. Three-valued *Post* systems *verify neither* ANC, *nor* AEM, *with respect to* $\leq$.

Anyway, and in the line of theorems in section 4.2, it is possible to go further for what concerns points 3, 5, 7 and 8.

6.3

Obviously, *the most surprising case is* P_3. By just adding $(\frac{1}{2}, 0)$ to $\vDash_P$, that is, with the relation

$$\vDash_P^* = \vDash \cup \{(\tfrac{1}{2}, 0)\},$$

it is easy to prove that *both* ANC *and* AEM *hold with respect to* $\vDash_P^*$. But the triplet $(L_=, \vDash_P^*, ')$ is not a TRS, since $(1,1) \in \vDash_P^*$ but $(1', 1') = (\frac{1}{2}, \frac{1}{2}) \notin \vDash_P^*$.

Taking $\vDash_P^{**} = \vDash_P^* \cup \{(\frac{1}{2}, \frac{1}{2})\} = \{(\frac{1}{2}, 0), (0, \frac{1}{2}), (0,0), (1,1), (\frac{1}{2}, 1), (0,1), (\frac{1}{2}, \frac{1}{2})\}$, it is obtained the TRS $(L_0, \vDash_P^{**}, ')$, and *it also holds* ANC *and* AEN *with respect to* $\vDash_P^{**}$. In addition, and as it is easy to check, *the operation* $\cdot$ *in* P_3 *is a type-1 operation in* $(L_0, \vDash_P^{**}, ')$, and *the operation* + in P_3 *is a type-2 operation in* $(L_0, \vDash_P^{**}, ')$.

It should be pointed out that neither $\cdot$ is a type-1 operation in $(L_0, \vDash_P^*, ')$, nor + is a type-2 operation in $(L_0, \vDash_P^*, ')$. The reasons are, respectively, that $\frac{1}{2} \cdot \frac{1}{2}' = \frac{1}{2}$, and $\frac{1}{2} + \frac{1}{2}' = \frac{1}{2}$, but $(\frac{1}{2} \cdot \frac{1}{2}) \notin \vDash_P^*$.

6.4

In the case K_3, taking $\vDash_K^* = \vDash_K \cup \{(\frac{1}{2}, \frac{1}{2})\} = \leq$, the triplet $(L_0, \vDash_K^*, ')$ is a TRS in which the operation $\cdot$ denoting $\min$ is a type-1 operation, and the operation + is a type-2 operation. Hence, *it holds ANC with respect to* $\vDash_K^*$ *, and also AEM with respect to the same* $\vDash_K^*$.

6.5

Concerning the case B_3, with $\vDash_B^* = \vDash_B \cup \{(\frac{1}{2}, \frac{1}{2})\}$, the TRS $(L_0, \vDash_B^*, ')$ is reached in which *it holds neither ANC, nor AEM, since*

$$(1+1')' = 1, ((1+1')')' = 1' = 0, \text{ but } (1,0) \notin \vDash_B^* .$$

Anyway, with $\vDash_B^{**} = \vDash_B^* \cup \{(1,0)\} = \{(1,1), (0,1), (0,0), (\frac{1}{2}, \frac{1}{2}), (1,0)\}$, it is obtained the TRS $(L_0, \vDash_B^{**}, ')$, *in which the two principles ANC and AEM obviously do hold.*

6.6

At the end, in the five cases there exist a TRS $(L_0, \vDash, ')$ for which the two principles ANC and AEM do hold. In the cases $Ł_3$ and G_3, it is $\vDash\, = \,\leq$ with $\leq$ the lineal order of the real line. In the case K_3, there is a solution with $\leq$, but also with $\vDash_K^*$ that, nevertheless, is again a part of the order $\leq$.

What is different are the cases of Bochvar and Post, where $\vDash_B^{**}$ is not a part of $\leq$, and $\vDash_P^{**}$ is not a part of $\leq$, but on the contrary, it is $\leq \subset \vDash_P^{**}$.

This fact, of not being a part of $\leq$, seems to be the red line between the three system $Ł_3, G_3, K_3$ and the two systems B_3, P_3. The trick to reach the principles ANC and AEM seems to be based on the identification of some pairs $(a, b) \in L_0 \times L_0$ that, added to $\vDash$ in the line of section 4.2, give a new relation for which (without altering the original negation) the two principles do hold.

6.7

Form Remark 4.8, it follows that the relation considered from 6.2 to 6.5 are larger than $\vDash_{NC}$, and $\vDash_{EM}$. As it is easy to check,

- In $Ł_3$, $\vDash_{NC} = \vDash_{EM} = \{(0,1), (\frac{1}{2}, \frac{1}{2})\} \subset \vDash_{Ł} = \leq$
- In G_3, $\vDash_{NC} = \vDash_{EM} = \{(0,1)\} \subset \vDash_G = \leq$
- In K_3, $\vDash_{NC} = \vDash_{EM} = \{(0,1), (\frac{1}{2}, \frac{1}{2})\} \subset \leq$, but $\vDash_{NC} \not\subseteq \vDash_K$, and $\vDash_{NC} \not\subseteq \vDash_K^*$
- In B_3, $\vDash_{NC} = \vDash_{EM} = \{(0,1), (\frac{1}{2}, \frac{1}{2})\} \subset \leq$, but $\vDash_{NC} \not\subseteq \vDash_B$, and $\vDash_{NC} \not\subseteq \vDash_B^*$
- In P_3, $\vDash_{NC} = \vDash_{EM} = \{(0,1), (\frac{1}{2}, 0)\} \subset \vDash_P^*$, but $\vDash_{NC} \not\subseteq \leq$, and $\vDash_{NC} \not\subseteq \vDash_P$.

Remark. 6.1. An element $a \in L$ is $\vDash$-self-contradictory, if $a \vDash a'$.

In $Ł_3$, since $\vDash_{Ł_3} = \leq$ and $a' = 1 - a$, it is $a \leq a'$ whenever $a \leq \frac{1}{2}$, that is, the self-contradictory elements are 0 and $\frac{1}{2}$. In G_3, with $\vDash_G = \leq$, the only element that is self-contradictory is 0.

In K_3, with $\vDash_K \subset \leq$, the only self-contradictory is 0, but with $\vDash_K^*$ also $\frac{1}{2}$ is self-contradictory.

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In B_3, with $\models_B \subset \leq$, the only element that is self-contradictory is 0, but with $\models_B^*$ and $\models_B^{**}$ also $\frac{1}{2}$ is self-contradictory.

In P_3, with $\models_P$, the only element that is self-contradictory is 0, but with $\models_P^*$ and $\models_P^{**}$ also $\frac{1}{2}$ is self-contradictory.

7 Conclusion

7.1

When the principles NC and EM are interpreted in the usual 'modern' forms,

$$a \cdot a' = 0,\ a + a' = 1$$

for all $a \in L_0$, neither Ł$_3$ nor G_3, K_3, B_3, P_3 do verify one of them. For example

- in Ł$_3, G_3, K_3, B_3$: $\frac{1}{2} \cdot \frac{1}{2}' = \frac{1}{2} \neq 0, \frac{1}{2} + \frac{1}{2}' = \frac{1}{2} \neq 1$
- in $P_3 : 0 \cdot 0' = \frac{1}{2} \neq 0, \frac{1}{2} + \frac{1}{2}' = \frac{1}{2} \neq 1$.

Hence, the results presented in this paper mean some progress in what concerns, jointly with what is in [7] and [6], the general verification of the non-contradiction and excluded-middle principles. If they can fail at the just mentioned 'modern' level, it seems to exist at least another level (that of self-contradiction) at which they do not fail.

7.2

Nevertheless, right now the authors are not yet able to completely answer the question of *why the systems Ł$_3, G_3, K_3$ do have solution in $\leq$, but the systems B_3, P_3 do have to exit from $\leq$ to reach a solution, and the system K_3 also admits a second solution in $\leq$*. In all these cases, the pairs that are to be added to the initial relation given by $a \to b = 1$, verify $a \to b \neq 1$. For example,

- In K_3, the added pair $(\frac{1}{2}, \frac{1}{2})$, verifies $\frac{1}{2} \to \frac{1}{2} = \frac{1}{2} \neq 1$
- In B_3, the added pairs $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$, verify $\frac{1}{2} \to \frac{1}{2} = \frac{1}{2} \neq 1, 1 \to 0 = 0 \neq 1$
- In P_3, the added pairs $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, verify $\frac{1}{2} \to 0 = 0 \neq 1, \frac{1}{2} \to \frac{1}{2} = \frac{1}{2} \neq 1$

Notice that in the cases Ł$_3$ and G_3 it is $a \to b = 1 \Leftrightarrow a \leq b$. Instead, in K_3 it is only $a \to b = \max(1-a, b) = 1 \Rightarrow a = 0$, or $b = 1 \Rightarrow a \leq b$, but not reciprocally. It seems that the relation 'If a, then b' given by $a \to b = 1$, needs to be enlarged to reach both principles but without an agreement with the truth table of $\to$. The case K_3, with the change of $\frac{1}{2} \to \frac{1}{2} = \frac{1}{2}$ by $\frac{1}{2} \to \frac{1}{2} = 1$, results to be coincidental with Ł$_3$. This is a subject that, related to the semantics of the system, deserves more thinking.

7.3

Another consideration about the operators $\to$ drives us to see into its conditional behaviour related to the $\leq$ or $\models$ relations. That is [4], whether or not the *Modus Ponens* rule of inference, $a \cdot (a \to b) \leq b$, or $a \cdot (a \to b) \models b$, do hold.

Only the Gödel system, G_3, verifies them, as is easy to prove. For the rest of the cases the following counter-examples answer the question.

1. Ł$_3$, with respect to $\models_{Ł} = \leq$. It is $\frac{1}{2} \cdot (\frac{1}{2} \to 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \nleq 0$, that is $(\frac{1}{2}, 0) \notin \leq$.
2. K_3 , with respect to $\models_K$, and $\models_K^* = \leq$. It is $\frac{1}{2} \cdot (\frac{1}{2} \to 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$, and $(\frac{1}{2}, 0) \notin \leq = \models_K^*$ and $(\frac{1}{2}, 0) \notin \models_K$ too.
3. B_3 , with respect to $\models_B, \models_B^*, \models_B^{**}$, and $\leq$. It is $\frac{1}{2} \cdot (\frac{1}{2} \to 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$, and $(\frac{1}{2}, 0) \notin \leq$, $(\frac{1}{2}, 0) \notin \models_B$, $(\frac{1}{2}, 0) \notin \models_B^*$ and $(\frac{1}{2}, 0) \notin \models_B^{**}$ too.
4. P_3 , with respect to both $\models_P$ and $\leq$. It is $\frac{1}{2} \cdot (\frac{1}{2} \to 0) = \frac{1}{2} \cdot 0 = \frac{1}{2}$, and $(\frac{1}{2}, 0) \notin \leq$, $(\frac{1}{2}, 0) \notin \models_P$. With respect to both $\models_P^*$ and $\models_P^{**}$. It is $\frac{1}{2} \cdot (\frac{1}{2} \to \frac{1}{2}) = \frac{1}{2} \cdot \frac{1}{2} = 1$, and $(1, \frac{1}{2}) \notin \models_P^*$, $(1, \frac{1}{2}) \notin \models_P^{**}$.

Hence, it is difficult to see the interest of such systems from the point of view of inference.

7.4

Last, but not least significant, is to notice how the different operators $\to$ verify the five properties characterizing implication functions [3], namely, i) $0 \to y = 1$, ii) $1 \to y = y$, iii) $x \to (y \to z) = y \to (x \to z)$, iv) $\to$ is decreasing in the first variable, and v) $\to$ is increasing in the second variable.

A simple calculation shows that in Ł$_3$, G_3, and K_3 the operation $\to$ verifies the five properties, therefore in these systems $\to$ is an implication function. On the contrary, B_3 does verify only properties ii and iii, and P_3 only i, iii, and iv. Hence, neither in B_3, nor in P_3, $\to$ is an implication function.

In the same vein, since in P_3 it is $1' = \frac{1}{2}$, but not $1' = 0$, operation $'$ is not, properly speaking, what is usually designed as a negation.

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