On Some Fuzzy Categories of Many-valued Topological Spaces

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Abstract— *The aim of this paper is to construct some new fuzzy categories related to many-valued sets and many valued topologies and to consider their properties.*

Keywords— Fuzzy categories, GL-monoids, L-valued equalities, L-topologies and L-fuzzy topologies.

1 Introduction

Since the inception of the notion of a fuzzy set by Zadeh [19] and its generalization as an (L)-fuzzy set where L is a complete lattice by Goguen [3], the efforts of many researchers were forced to introduce the appropriate fuzzy counterparts of the classical mathematical concepts and to develop substantial corresponding theories. However from the categorical point of view in most cases the work was done in the context of ordinary, that is crisp categories. To explain our idea we give some examples

1. Following the works of Chang [2] and Goguen [4], an L-fuzzy topological space is a pair (X, τ) where X is a set and τ is a subfamily of L^X satisfying axioms which are natural analogues of classical topological axioms. The continuity of mappings between L-fuzzy topological spaces is defined just by "translating" the definition of continuity in fuzzy enviroment.

2. In [18] Mingsheng Ying introduced the concept of a fuzzifying topological space by semantical analysis of usual topological axioms, cf also [6]. Namely a fuzzifying topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} : L^X \to L$ is a mapping satisfying the axioms which are obtained by logical interpretation of the standard topological axioms. Continuity of mappings between two fuzzifying topological spaces is defined by logical interpretation of the axiom "the preimage of an open set is open".

3. In [11] A. Rosenfeld defined a fuzzy subgroup of a group $(G, \cdot, e)$ as a mapping $H : G \to [0, 1]$ such that $H(x \cdot y) \geq H(x) \wedge H(y)$ and $H(x) = H(x^{-1})$ for all $x, y \in G$. A homomorphism between fuzzy subgroups H_1 and H_2 is defined as a homomorhism between the underlying groups $f : G_1 \to G_2$ such that $H_1(x) \leq H_2(f(x))$ for every $x \in G_1$.

Thus in all these cases as well as in many other situations, the authors work in the following context: they consider a (usual) category whose objects are certain mathematical structures involving fuzzy sets as objects and mappings satisfying certain properties between these objects as morphisms.

On the other hand in [13] the concept of a fuzzy category was introduced. A fuzzy category is some category-type conglomerate whose objects and morphisms may be such only to a certain degree, see the next section for the precise definition. The foundations of the theory of fuzzy categories were developed in a series of subsequent papers, see e.g. [14], [10], [15], etc. These works contain also description of some concrete fuzzy categories related to algebra and topology, which are obtained by a certain fuzzification of classical categories.

The aim of this paper is to construct some new fuzzy categories related to many-valued sets and many valued topologies and to consider their properties. They can be regarded as fuzzy counterparts of the category $\mathbf{SET}(L)$ [7] and categories $\mathbf{TOP}(L)$ and $\mathbf{FTOP}(L)$ introduced in our work [16]. The fuzzy categories considered in this paper were first defined in our talk at the Czech-Latvian Seminar *"Advanced methods in Soft Computing",* in Trojanovice, Czech Republic, November 19-22, 2008.

2 Fuzzy categories

First we recall the concept of an (L)-fuzzy category (where L is a GL-monoid) in a special form appropriate for our merits.

Recall (see [7]) that a $GL-$monoid is an infinitely distributive lattice $(L, \leq, \wedge, \vee)$ [1] enriched with a monotone, commutative and associative binary operation $*$ such that

1. $a * 1 = a$ and $a * 0 = 0$ for all $a \in L$ where 0 and 1 are the bottom and the top elements of L respectively;
2. $a * \left(\bigvee_{i\in\mathcal{I}} b_i\right) = \bigvee_{i\in\mathcal{I}} (a * b_i)\ \forall a \in L,\ \forall\{b_i : i \in \mathcal{I}\} \subseteq L$;
3. If $a \leq b$, then there exists $c \in L$ such that $a = b * c$. (The last property of a GL-monoid is called divisability.)

Such operation $*$ will be refered to as the conjunction in the GL-monoid. Important examples of GL-monoids are Heyting algebras (in this case $* = \wedge$) and Łukasiewicz MV-algebra (in this case L = [0,1] and $a * b = \max\{a + b - 1, 0\}$) [5].

It is known that every $GL-$monoid is residuated, i.e. there exists a further binary operation " $\mapsto$ " (residuation) on L linked to $*$ with the Galois condition:

$$a * b \leq c \Longleftrightarrow a \leq (b \mapsto c) \qquad \forall a, b, c \in L.$$

Explicitely implication is given by

$$a \mapsto b = \bigvee\{\lambda \in L \mid a * \lambda \leq b\}.$$

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Let $L = (L, \leq, \wedge, \vee, *)$ be a GL-monoid. *An (L)-fuzzy category* [13] is a pair $(\mathcal{C}, \mu)$ where $\mathcal{C}$ is an ordinary category with the class of objects $\mathcal{O}b(\mathcal{C})$, the class of morphisms $\mathcal{M}or(\mathcal{C})$, and where $\mu : \mathcal{M}or(\mathcal{C}) \to L$ is an L-subclass of the class of morphism such that:

(i) $\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever composition $g \circ f$ is defined in the category $\mathcal{C}$;

(ii) for each $X \in \mathcal{O}b(\mathcal{C})$ $\mu(id_X) = 1$ where id_X is the identity morphism.

3 Category SET(L) and fuzzy category $\mathcal{F}$-SET(L)

Category **SET**(L) was first defined in [7], see also [8]. The objects of the category **SET**(L) are L-valued sets, that is pairs (X, E) where X is a set and E is an L-valued equality, that is a mapping $E : X \times X \to L$ such that

1) $E(x, x) = 1$;

2) $E(x, y) = E(y, x)$;

3) $E(x, y) * E(y, z) \leq E(x, z)\ \forall x, y, z \in X$.

An L-subset A of an L-valued set (X, E) (that is a mapping $A : X \to L$) is called extensional if $A(x) * E(x, x') \leq A(x')$ for all $x, x' \in X$. The family of all extensional L-subsets of (X, E) will be denoted L^X_E.
The morphisms $f : (X, E_X) \to (Y, E_Y)$ in the category **SET**(L) are extensional mappings, that is mappings $f : X \to Y$ such that

$$E_X(x, x') \leq E_Y(f(x), f(x')) \quad \forall x, x' \in X.$$

Consider the category

$$\mathcal{C}(L) = (\mathcal{O}b(\mathbf{SET}(L)), \mathcal{M}or(\mathbf{SET})),$$

that is $\mathcal{C}(L)$ has objects from **SET**(L) and morphisms from **SET**, that is all mappings between the corresponding sets.
We define the measure μ of extensionality for a mapping $f : X \to Y$ by

$$\mu(f) = \bigwedge_{x,x' \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))).$$

Theorem 3.1 *The triple $(\mathcal{O}b(\mathbf{SET}(L)), \mathcal{M}or(\mathbf{SET}), \mu)$ is an L-fuzzy category. It will be denoted by $\mathcal{F}$-**SET**(L) and called the L-fuzzification of the category **SET**(L).*

Proof: Let $id_X : (X, E) \to (X, E)$ be the identity mapping. Then, obviously, $\mu(id_X) = 1$. Hence, to prove the theorem, we have to show that if $f : (X, E_X) \to (Y, E_Y)$ and $g : (Y, E_Y) \to (Z, E_Z)$ are potential morphisms (that is mappings between the corresponding sets), then $\mu(g \circ f) \geq \mu(g) * \mu(f)$. Indeed,

$$\begin{gathered}\mu(g \circ f) = \\ = \bigwedge_{x,x' \in X} (E_X(x, x') \mapsto E_Z((g \circ f)(x), (g \circ f)(x'))) \geq \\ \bigwedge_{x,x' \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x'))) * \\ \bigwedge_{x,x' \in X} (E_Y(f(x), f(x')) \mapsto E_Z(g(f(x)), g(f(x')))) \geq \\ \mu(f) * \bigwedge_{y,y' \in Y} (E_Y(y, y') \mapsto E_Z(g(y), g(y'))) = \\ \mu(f) * \mu(g).\end{gathered}$$

□

4 Category TOP(L) and fuzzy category $\mathcal{F}$-TOP(L).

Definition 4.1 *Let $L = (L, \wedge, \vee, *)$ be a GL-monoid and (X, E) be an L-valued set. A family $\tau \subseteq L^X$ is called an L-topology on an L-valued set (X, E) if*

1. $0_X \in \tau; 1_X \in \tau$ *(where $0_X, 1_X$ are constant L-sets with values 0 and 1 respectively;*

2. *if $U, V \in \tau$, then $U \wedge V \in \tau$;*

3. *if $U_i \in \tau \quad \forall i \in \mathcal{I}$, then $\bigvee_{i \in \mathcal{I}} U_i \in \tau$;*

4. *if $U \in \tau$, then $U(x) * E(x, x') \leq U(x')\ \forall x, x' \in X$.*

The last condition means that all L-sets in τ are extensional and hence $\tau \subseteq L^X_E$.
The triple (X, E, τ) where τ is an L-topology on an L-valued set is called an L-valued L-topological space. Respectively, the elements $U \in \tau$ are called open L-sets in this L-valued L-topological space.

Definition 4.2 *A mapping $f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$ is called continuous if*

1. $E_X(x, x') \leq E_Y(f(x), f(x'))$ *for all $x, x' \in X$,* that is f is an extensional mapping between the corresponding L-valued sets (X, E_X) and (Y, E_Y), and

2. $f^{-1}(V) \in \tau_X$ *whenever* $V \in \tau_Y$.

Obviously, if $f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$ and

$$g : (Y, E_Y, \tau_Y) \to (Z, E_Z, \tau_Z)$$

are continuous, then the composition

$$g \circ f : (X, E_X, \tau_X) \to (Z, E_Z, \tau_Z)$$

is continuous and the identity mapping

$$id_X : (X, E_X, \tau_X) \to (X, E_X, \tau_X)$$

is continuous. Hence L-valued L-topological spaces and continuous mappings between them form a category **TOP**(L).
One can easily see that in case E is crisp, that is $E(x, x') = 0$ if $x \neq x'$ and $E(x, x) = 1$, the L-valued toplogical space is just an L-topologcal space in the sense of [2], [4] and the category of L-topological spaces is actually a complete subcategory of **TOP**(L).

Notice that, by lower-semicontinuity of conjunction, for extensional L-sets U_i, we have

$$\begin{gathered}\left(\bigvee_{i \in \mathcal{I}} U_i(x)\right) * E(x, x') \leq \\ \bigvee_{i \in \mathcal{I}} (U_i(x) * E(x, x')) \leq \bigvee_{i \in \mathcal{I}} U_i(x') \quad \forall x, x' \in X.\end{gathered}$$

Thus the supremum of extensional L-subsets of an L-valued set is extensional itself. Therefore, in an analogy with classical topology we can define the interior $int(A)$ of an L-subset A of an L-valued L-topological space (X, E, τ) as the largest $(\geq)$ one of all open L-subsets of (X, E, τ) contained $(\leq)$ in A. Equivalently, it can be defined by the formula

$$int(A) = \bigvee \{U \in \tau \mid U \leq A\}.$$

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One can easily verify that the resulting operator

$$int : L^X \to L^X$$

satisfies all properties, analogous to the properties of the interior operator in classical topology.

The interior operator allows us to measure "the degree of openess" of an L-set A in an L-valued L-topological space by evaluating, to what extent an L-set A is contained in its interior. To realize this we need to extend an L-valued equality E from a set X to its extensional L-powerset L_E^X. This can be done as follows (see [17]):
Given $A, B \in L_E^X$ let

$$\mathcal{R}(A, B) = \bigwedge_{x,z \in X} ((E(x, z) * A(x)) \mapsto B(z)) .$$

It is shown in [17] that $\mathcal{R} : L_E^X \times L_E^X \to L$ is an L-fuzzy order relation on L_E^X, that is it is refelexive ($\mathcal{R}(A, A) = 1$ for all $A \in L_E^X$) and transitive ($\mathcal{R}(A, B) * \mathcal{R}(B, C) \leq \mathcal{R}(A, C)$ for all $A, B, C \in L_E^X$). By setting $\mathcal{E}(A, B) = \mathcal{R}(A, B) \wedge \mathcal{R}(B, A)$ an L-valued equality on L_E^X is obtained [17].
Now to fuzzify the category **TOP**(L) we consider the category

$$(\mathcal{O}b(\mathbf{TOP}(L)), \mathcal{M}or(\mathbf{SET}), \circ),$$

where objects are L-valued L-topological space (X, E, τ), but morphisms are from the category $\mathbf{SET}$, that is mapings between corresponding sets. For a given mapping

$$f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$$

we define the degree of extensionality by

$$\mu_1(f) = \bigwedge_{x,x' \in X} (E_X(x, x') \mapsto E_Y(f(x), f(x')) ,$$

the degree of continuity by

$$\mu_2(f) = \bigwedge_{V \in \tau_Y} \mathcal{R}(f^{-1}(V), int(f^{-1}(V)))$$

and finally the degree of being a morphism by

$$\mu(f) = \mu_1(f) \wedge \mu_2(f).$$

Theorem 4.3 $(\mathcal{O}b(\mathbf{TOP}(L)), \mathcal{M}or(\mathbf{SET}), \circ, \mu)$ *is a fuzzy category.*

Proof Let $f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$ and

$$g : (Y, E_Y, \tau_Y) \to (Z, E_Z, \tau_Z).$$

The inequality

$$\mu_1(g \circ f) \geq \mu_1(g) * \mu_1(f)$$

was established in the previous section. To establish the inequality

$$\mu_2(g \circ f) \geq \mu_2(g) * \mu_2(f)$$

we are reasoning as follows:

Let $f : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$,
$g : (Y, E_Y, \tau_Y) \to (Z, E_Z, \tau_Z)$.
We fix $W \in \tau_Z$. Then by transitivity of $\mathcal{R}$

$$\mathcal{R}(f^{-1}(g^{-1}(W)), int_X(f^{-1}(g^{-1}(W)))) \geq$$
$$\mathcal{R}(f^{-1}(g^{-1}(W)), f^{-1}(int_Y(g^{-1}(W))))*$$
$$\mathcal{R}(f^{-1}(int_Y(g^{-1}(W))), int_X(f^{-1}(g^{-1}(W)))).$$

We estimate each member separately:

$$\mathcal{R}(f^{-1}(int_Y(g^{-1}(W))), int_X(f^{-1}(g^{-1}(W)))) \geq$$
$$\mathcal{R}(f^{-1}(int_Y(g^{-1}(W))), int_X(f^{-1}(int_Y g^{-1}(W)))) \geq$$
$$\mu_2(f).$$
$$\mathcal{R}(f^{-1}(g^{-1}(W)), f^{-1}(int_Y(g^{-1}(W)))) =$$
$$\bigwedge_{x,x'} ((E(x, x') * g^{-1}(W)(f(x))) \mapsto$$
$$int_Y(g^{-1}(W)(f(x')))) \geq$$

(by extentionality of f)

$$\geq \bigwedge_{x,x'} ((E(f(x), f(x')) * g^{-1}(W)(f(x))) \mapsto$$
$$int_Y(g^{-1}(W)(f(x')))) \geq$$
$$\bigwedge_{y,y'} ((E(y, y') * g^{-1}(W)(y)) \mapsto int_Y(g^{-1}(W)(y'))) \geq$$
$$\mu_2(g).$$

Thus

$$\mathcal{R}(f^{-1}(g^{-1}(W)), int_X(f^{-1}(g^{-1}(W)))) \geq \mu_2(g) * \mu_2(f).$$

Since this is valid for any $W \in \tau_Z$ we obtain the requested

$$\mu_2(g \circ f) \geq \mu_2(g) * \mu_2(f).$$

Finally referring to the properties of a GL-monoid, we get

$$\mu(g \circ f) = \mu_1(g \circ f) \wedge \mu_2(g \circ f) \geq$$
$$(\mu_1(f) * \mu_1(g)) \wedge (\mu_2(f) * \mu_2(g)) \geq$$
$$(\mu_1(f) \wedge \mu_2(f)) * (\mu_1(g) \wedge \mu_2(g)) = \mu(f) * \mu(g).$$

We conclude the proof by noticing that obviously $\mu(id_X) = 1$ for the identity morphism

$$id_X : (X, E_X, \tau_X) \to (X, E_X, \tau_X).$$

□

5 Category FTOP(L) and fuzzy category $\mathcal{F}$-FTOP(L)

The concept of an L-valued L-fuzzy topological space as a generalization of the concept of an L-fuzzy topological space [12], [9] and the corresponding category **FTOP**(L) were introduced in [17]. The objects of this category are triples $(X, E, \mathcal{T})$ where (X, E) is an L-valued set and $\mathcal{T} : L^X \to L$ is an L-fuzzy topology, that is an extensional subset of L_E^X such that

1) $\mathcal{T}(1) = \mathcal{T}(0) = 1$;
2) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V) \quad \forall U, V \in L_E^X$;
3) $\mathcal{T}\left(\bigvee_{i \in I} U_i\right) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{T}(U_i) \quad \forall \{U_i \mid i \in \mathcal{I}\} \subseteq L_E^X$.

and morphisms are extensional mappings

$$f : (X, E_X) \to (Y, E_Y)$$

such that $\mathcal{T}(f^{-1}(V)) \geq \mathcal{T}(V)$.
One can easily see that in case E is crisp, that is $E(x, x') = 0$

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if $x \neq x'$ and $E(x,x) = 1$, the L-valued L-fuzzy toplogical space is just an L-fuzzy topological space as it was defined in [12], [4] and the category of L-fuzzy topological spaces is actually a complete subcategory of $\mathbf{FTOP}(L)$.

Let $(X, E_X, \mathcal{T}_X)$, $(Y, E_Y, \mathcal{T}_Y)$ be L-valued L-fuzzy topological spaces and $f : X \to Y$ be a mapping of the corresponding underlying sets. The measure of the extensionality μ_1 of the mapping $f : (X, E_X) \to (Y, E_Y)$ is defined as above. The measure of continuity μ_2 of a mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is defined by

$$\mu_2(f) = \bigwedge_{V \in L_E^Y} \left(\mathcal{T}_Y(V) \mapsto \mathcal{T}_X(f^{-1}(V))\right).$$

Further, for a mapping $f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$ we set $\mu(f) = \mu_1(f) \wedge \mu_2(f)$.

Theorem 5.1

$$\mathcal{F}-\mathbf{FTOP}(L) = (\mathcal{O}b(\mathbf{FTOP}(L)), \mathcal{M}or(\mathbf{SET}), \mu)$$

is an L-fuzzy category.

Proof: Since for the identity morphism

$$id_X : (X, E_X, \mathcal{T}_X) \to (X, E_X, \mathcal{T}_X)$$

obviously $\mu_1(id_X) = 1$ and $\mu_2(id_X) = 1$, we have $\mu(id_X) = 1$. Therefore to prove the theorem we have to establish that given two functions $f : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$ and $g : (Y, E_Y, \mathcal{T}_Y) \to (Z, E_Z, \mathcal{T}_Z)$, viewed as the mappings of the corresponding underlying sets it holds

$$\mu(g \circ f) \geq \mu(g) * \mu(f).$$

The inequality $\mu_1(g \circ f) \geq \mu_1(g) * \mu_1(f)$ was established in subsection 3. The inequality

$$\mu_2(g \circ f) \geq \mu_2(g) * \mu_2(f)$$

was established in [14], see also [15]. Now, referring to the properties of a GL-monoid, we get

$$\begin{gathered}\mu(g \circ f) = \mu_1(g \circ f) \wedge \mu_2(g \circ f) \geq \\ (\mu_1(g) * \mu_1(f)) \wedge (\mu_2(g) * \mu_2(f)) \geq \\ (\mu_1(g) \wedge \mu_2(g)) * (\mu_1(f) \wedge \mu_2(f)) = \mu(g) * \mu(f).\end{gathered}$$

□

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