The Collapsing Method: Does the Direction of Collapse Affect Accuracy?

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Abstract— *The Greenfield-Chiclana Collapsing Defuzzifier comes in many variants, depending on the order of slice collapse. The accuracy of the fundamental variants of forward, backward, inward and outward, and the composite variants of forward-backward and outward right-left is compared experimentally for the discretised interval type-2 fuzzy set.*

Keywords— Centroid, Collapsing, Defuzzification, Interval Type-2 Fuzzy Set, Representative Embedded Set.

1 Introduction

A fuzzy inferencing system (FIS) is a computerised system that uses fuzzy sets and rules to support decision making. *Type-2* FISs are being developed for an increasing number of applications such as [1], [2], [3]. There are five main stages to any FIS: fuzzification, antecedent computation, implication, aggregation and defuzzification. In the case of a type-2 FIS (where at least one fuzzy set is type-2), defuzzification consists of two parts — type-reduction and defuzzification proper, as shown in figure 1. Type-reduction is the procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set, known as the type-reduced set (TRS). The TRS is then easily defuzzified to give a crisp number.

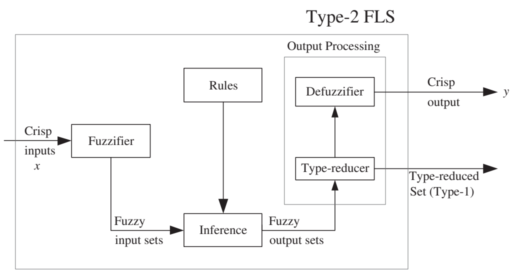

Figure 1: Type-2 FIS (from Mendel [4]).

The type-reduction stage of type-2 defuzzification is problematic owing to its computational complexity. This is the type-reduction algorithm originally described by Mendel ([4], pages 248-252):

1. All possible type-2 embedded sets ([4], definition 3-10, page 98) are enumerated.
2. For each embedded set the minimum secondary membership grade is found.
3. For each embedded set the domain value of the type-1 centroid of the type-2 embedded set is calculated.
4. For each embedded set the secondary grade is paired with the domain value to produce a set of ordered pairs (x,z). It is possible that for some values of x there will be more than one corresponding value of z.
5. For each domain value, the maximum secondary grade is selected. This creates a set of ordered pairs $(x, zMax)$ such that there is a one-to-one correspondence between x and $zMax$. This completes the type-reduction of the type-2 set to the type-1 TRS.

The resultant TRS, as with any type-1 fuzzy set, is readily defuzzified by finding its centroid.

Thus type-reduction involves the processing of *all* the embedded sets within the type-2 set. This is why we term the procedure 'exhaustive defuzzification'. Embedded sets are very numerous. For instance, when a prototype type-2 FIS performed an inference using sets which had been discretised into 51 slices across both the x and y-axes, the number of embedded sets in the aggregated set was calculated to be in the order of 2.9×10^{63}. Though individually easily processed, embedded sets in their totality give rise to a processing bottleneck simply by virtue of their high cardinality. Consequently, exhaustive defuzzification is an impractical technique.

A computationally simpler alternative to the exhaustive method is the *Greenfield-Chiclana Collapsing Defuzzifier*, introduced in [5]. This technique converts an interval type-2 fuzzy set into a type-1 fuzzy set which approximates to the *representative embedded set (RES)*, whose defuzzified value is equal to that of the original type-2 set. As a type-1 set, the RES may then be defuzzified straightforwardly.

In this paper we build on the work reported in [5]. The next section covers assumptions and definitions; section 3 presents an overview of the collapsing defuzzifier, after which section 4 introduces the theme of this article, the notion that *there are variants of the collapsing method*. Sections 5 and 6 are concerned with finding experimentally the most accurate variant.

2 Preliminaries

2.1 Assumptions

Discretisation The work presented here is concerned only with defuzzification of *discretised* type-2 fuzzy sets.

Interval Type-2 Fuzzy Set This paper is concerned with the *interval* type-2 fuzzy set.

Centroid Method of Defuzzification It is assumed that the centroid method of defuzzification ([6], page 336) is used.

2.2 Definitions

Definition 1 (Degree of Discretisation) *The* degree of discretisation *of a discretised fuzzy set is the separation of the slices.*

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Scalar Cardinality For type-1 fuzzy sets, Klir and Folger ([7], p17) define scalar cardinality as follows:

Definition 2 (Scalar Cardinality) *The* scalar cardinality *of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A. Thus,*

$$| A |= \sum_{x \in X} \mu_A(x). \qquad ([7], p17)$$

To distinguish scalar cardinality from cardinality in the classical sense, we adopt the '$\| \ \|$' symbol for scalar cardinality.

3 Overview of the Collapsing Method

An interval type-2 set may be regarded as a blurred type-1 set. The collapsing method is a technique for deriving a type-1 fuzzy set from a type-2 fuzzy set, and may be thought of as a reversal of blurring. The type-1 set's membership function is calculated so that its defuzzified value approximates that of the type-2 fuzzy set. It is a simple matter to defuzzify the type-1 set, and to do so would be to find the defuzzified value of the original type-2 fuzzy set. The collapsing process approximates the output of the type-reducer followed by the type-1 defuzzifier, and in so doing reduces the computational complexity of type-2 defuzzification. We term this special type-1 set the *'representative embedded set approximation (RESA)'*. (Full details of the collapsing algorithm, including proof of the associated theorem, may be found at [5].) We formally state the Simple[1] Representative Embedded Set Approximation:

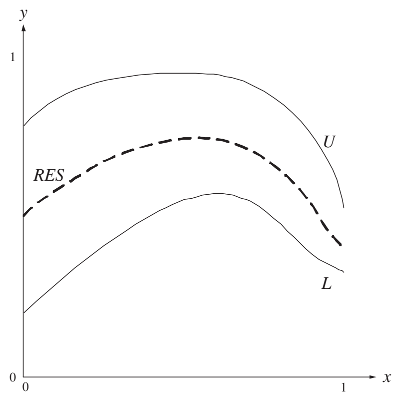

Figure 2: A Representative Embedded Set.

Theorem 1 (Simple Representative Embedded Set Approx.) *The membership function of the* embedded set *R derived by dynamically collapsing slices of a discretised type-2 interval fuzzy set $\tilde{F}$, having lower membership function L, and upper membership function U, is:*

$$\mu_R(x_i) = \mu_L(x_i) + r_i$$

with

$$r_i = \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j \right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j \right) + b_i}, \qquad (1)$$

and $b_i = \mu_U(x_i) - \mu_L(x_i),\ r_0 = 0.$

This is an iterative formula. Collapsing proceeds vertical slice by vertical slice. The first slice is collapsed, the first y-value of the RESA calculated, the next slice is collapsed and the second y-value of the RESA calculated, and so on until all the slices have been collapsed. In this formula each b_i is the blur for vertical slice i, i.e. the difference between the upper membership function and the lower membership function for slice i. Each r_i is the amount by which the y-value of L must be increased to give the y-value of the RESA R.

4 Variants of the Collapsing Method

As we have seen, equation (1) is the formula for collapsing. This is in fact *a version* of collapsing — the most intuitive variant, whereby the slices are collapsed in the order of increasing domain value ($x = 0$ to $x = 1$). We term this *collapsing forward*. However slice collapse may be performed in any slice order giving slightly different RESAs. If the domain of the interval type-2 fuzzy set is discretised into s vertical slices, the number of permutations of these slices is $s!$ ([8], page 139). Therefore there must be $s!$ RESAs obtainable by varying the order of slice collapse. The question this piece of research addresses is, "Does the order in which the slices are collapsed affect the accuracy of the method?"

There are four fundamental variants, which we term *forward*, *backward*, *outward* and *inward*. Inward and outward may each be approached in two different ways. For the inward variant, slice collapse might start from the left or from the right. For the outward variant, the first slice is in the middle[2], but the second slice may be to the right or to the left. Added to these, there are composite variants, such as *forward-backward*, which is the mean of the defuzzified values found by collapsing forward and collapsing backward.

5 Experimental Comparison

5.1 Test Sets

We chose to test defuzzification in isolation from the rest of an FIS, on specially created test sets. Our methodology was to run different collapsing variants against each other to see which gave the most accurate results.

Symmetric Horizontal Test Set The lower membership function is the line $y = 0.2$; the upper membership function the line $y = 0.8$. The shape of this test set may be described as a horizontal stripe. The symmetry of this set tells us that its defuzzified value is 0.5.

Symmetric Triangular Test Set This is a normal test set. The lower and upper membership functions are both triangular in shape, both with vertices at $(0.4, 1)$. The symmetry of this set reveals its defuzzified value to be 0.4.

Asymmetric Gaussian Test Set This test set was deliberately designed to be asymmetrical, and hence a more realistic simulation of an FIS aggregated set. Both the lower and upper membership functions are Gaussian. As this set has no symmetry, exhaustive defuzzification (section 1) had to be employed to determine the actual defuzzified value, which, as

[1] In [5], we used the term 'simple' to describe an interval set in which each vertical slice consists of only two points, corresponding to L and U. The term is redundant in the context of this paper.

[2] We always employ an odd number of slices, giving a determinate middle slice.

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would be expected, varies slightly with the degree of discretisation.

5.2 Test Strategy

A preliminary set of tests was performed on the fundamental variants: forward, backward, inward, outward, and the composite variant forward-backward. As will be reported in the next section, the outward variant outperformed the others. This led on to further tests to discover the most accurate version of collapsing outward.

For the Gaussian test set, the only way of knowing its defuzzified value was to employ exhaustive defuzzification (section 1). This procedure only works properly for 21 slices or under. In contrast, the horizontal and triangular test sets reveal their defuzzified values by symmetry, so the number of slices used can be much higher, allowing finer discretisation.

6 Results and Conclusion

6.1 Preliminary Tests

Table 1 gives the results for the horizontal test set; table 2 gives the associated errors. Table 3 shows the triangular test set results, and table 4 the errors. The defuzzification results for the Gaussian test set are shown in table 5, with the errors in table 6. For all three test sets, the best performing variant was outward, followed by inward, then forward and backward. For the symmetrical sets (horizontal and triangular), the errors of collapsing forward were equal and opposite to those of collapsing backward. Therefore in these cases we would expect collapsing forward-backward to give exact results. This has been confirmed by experiments. For the Gaussian test set, backward performed more poorly than forward. In this case the composite of forward-backward performed worse than forward, though better than backward.

6.2 Further Tests

The outward variant may be performed in two ways, outward right and outward left (section 4). Collapsing right-left is the mean of collapsing right and collapsing left. The results and associated errors for the three versions of the outward variant as applied to the three test sets are shown in tables 7 to 9.

For the symmetrical horizontal test set, outward right and outward left gave rise to equal but opposite errors. For the composite outward right-left, these errors cancelled to zero.

The triangular test set, though symmetrical, is not placed symmetrically about $x = 0.5$. The errors of collapsing right and collapsing left were of equal sign and either equal or very close in quantity. When the errors were not equal, those of outward left were marginally smaller than those of outward right. Unsurprisingly, the performance of the composite of outward right-left fell between that of outward right and outward left.

For the Gaussian test set, the errors were all of negative sign. At all degrees of discretisation, outward left gave the best results, outward right the poorest, and outward right-left came in between.

For two of the three test sets outward left outperformed outward right. Our conjecture is that the position of the centroid is an important factor affecting which performs better out of outward right and outward left. This topic requires further research using a wider range of test sets, but for now we conclude that the optimum strategy is the composite outward right-left.

6.3 Why is Outward the Most Accurate Variant?

This explanation is based on the symmetrical horizontal test set. As each slice is collapsed, $\|L\| + \sum_{j=1}^{j=i-1} r_j$ in both the numerator and denominator of the collapsing formula (equation (1)) increases, which means that as the collapse progresses, the r_i for each collapsed slice i is a closer approximation to $\frac{1}{2}b_i$, i.e. half the 'blur' term. Thus with every successive collapsed slice, the RESA tends towards the midline of L and U, as shown in figure 3 for the forward and backward variants.

For the symmetrical horizontal test set, we take the RES to be the midline of L and U for two reasons. Firstly, by symmetry we would expect the RES to be a horizontal line. Secondly, as the number of slices is increased (either as the collapse progresses, or as the degree of discretisation is made finer), the RESA gets closer to the midline of L and U.

Therefore, as the slices are collapsed, the RESA approaches the RES. This means that the earlier slices in the RESA deviate more from the RES than the later ones. To get the best results, the collapse need to proceed symmetrically. Both the inward and outward variants meet this criterion; the inaccuracies are distributed symmetrically. However the greatest inaccuracy is associated with the first collapsed slice. To achieve maximum accuracy, the ideal place for this first slice to be positioned is centrally, as the effect on the defuzzified value obtained is then minimal. For this reason outward (figure 4) gives a more accurate defuzzified values than inward. We would expect the same reasoning to apply to all type-2 fuzzy test sets. However further investigation, using radically contrasting test sets, is planned.

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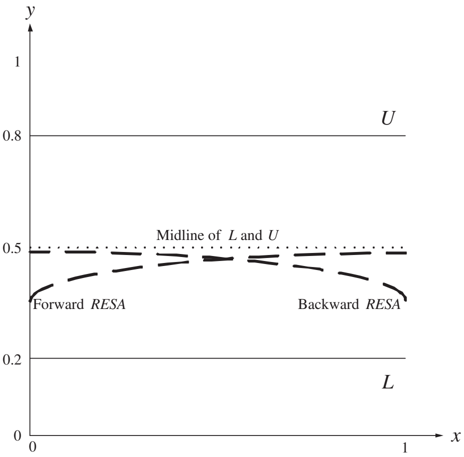

Figure 3: Forward RESA and Backward RESA.

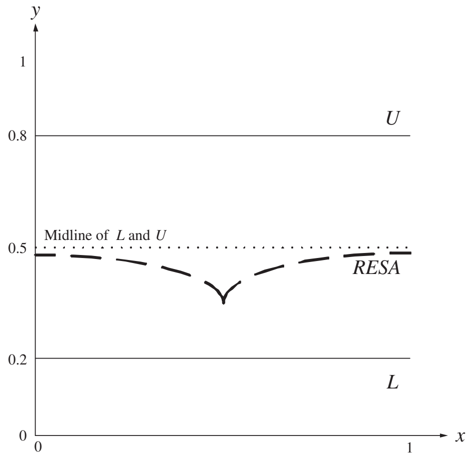

Figure 4: Outward RESA.

Table 1: Defuzzified Values Obtained by Collapsing the Symmetrical Horizontal Test Set.

Degree of Discretisation	Defuzzified Value	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward
0.1	0.5	0.5038320922	0.4961679078	0.4993086838	0.4995891494
0.05	0.5	0.5019998917	0.4980001083	0.4998049953	0.4998891777
0.02	0.5	0.5008177226	0.4991822774	0.4999665227	0.4999815068
0.01	0.5	0.5004115350	0.4995884650	0.4999914377	0.4999953154
0.005	0.5	0.5002064040	0.4997935960	0.4999978353	0.4999988213
0.002	0.5	0.5000827100	0.4999172900	0.4999996513	0.4999998107
0.001	0.5	0.5000413793	0.4999586207	0.4999999126	0.4999999526
0.0001	0.5	0.5000041401	0.4999958599	0.4999999991	0.4999999995
0.00001	0.5	0.5000004140	0.4999995860	0.5000000000	0.5000000000

Table 2: Errors Incurred in Collapsing the Symmetrical Horizontal Test Set.

Degree of Discretisation	Defuzzified Value	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward
0.1	0.5	0.0038320922	-0.0038320922	-0.0006913162	-0.0004108506
0.05	0.5	0.0019998917	-0.0019998917	-0.0001950047	-0.0001108223
0.02	0.5	0.0008177226	-0.0008177226	-0.0000334773	-0.0000184932
0.01	0.5	0.0004115350	-0.0004115350	-0.0000085623	-0.0000046846
0.005	0.5	0.0002064040	-0.0002064040	-0.0000021647	-0.0000011787
0.002	0.5	0.0000827100	-0.0000827100	-0.0000003487	-0.0000001893
0.001	0.5	0.0000413793	-0.0000413793	-0.0000000874	-0.0000000474
0.0001	0.5	0.0000041401	-0.0000041401	-0.0000000009	-0.0000000005
0.00001	0.5	0.0000004140	-0.0000004140	0.0000000000	0.0000000000

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Table 3: Defuzzified Values Obtained by Collapsing the Symmetrical Triangular Test Set.

Degree of Discretisation	Defuzzified Value	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward
0.1	0.4	0.4001359091	0.3998640909	0.4001131909	0.3998916916
0.05	0.4	0.4000597189	0.3999402811	0.4000498280	0.3999505451
0.02	0.4	0.4000230806	0.3999769194	0.4000195457	0.3999808751
0.01	0.4	0.4000115326	0.3999884674	0.4000098170	0.3999904744
0.005	0.4	0.4000057773	0.3999942227	0.4000049299	0.3999952381
0.002	0.4	0.4000023153	0.3999976847	0.4000019784	0.3999980943
0.001	0.4	0.4000011585	0.3999988415	0.4000009904	0.3999990469
0.0001	0.4	0.4000001159	0.3999998841	0.4000000992	0.3999999047
0.00001	0.4	0.4000000116	0.3999999884	0.4000000099	0.3999999905

Table 4: Errors Incurred in Collapsing the Symmetrical Triangular Test Set.

Degree of Discretisation	Defuzzified Value	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward
0.1	0.4	0.0001359091	-0.0001359091	0.0001131909	-0.0001083084
0.05	0.4	0.0000597189	-0.0000597189	0.0000498280	-0.0000494549
0.02	0.4	0.0000230806	-0.0000230806	0.0000195457	-0.0000191249
0.01	0.4	0.0000115326	-0.0000115326	0.0000098170	-0.0000095256
0.005	0.4	0.0000057773	-0.0000057773	0.0000049299	-0.0000047619
0.002	0.4	0.0000023153	-0.0000023153	0.0000019784	-0.0000019057
0.001	0.4	0.0000011585	-0.0000011585	0.0000009904	-0.0000009531
0.0001	0.4	0.0000001159	-0.0000001159	0.0000000992	-0.0000000953
0.00001	0.4	0.0000000116	-0.0000000116	0.0000000099	-0.0000000095

Table 5: Defuzzified Values Obtained by Collapsing the Gaussian Test Set.

Degree of Discretisation	Defuzzified Value (EM)	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward	Collapsing Forward-Backward
0.5	0.2899142309	0.2947300898	0.4090097593	0.2940174555	0.2884666838	0.3518699246
0.25	0.2906756945	0.2925398791	0.3712146394	0.2923170555	0.2900969651	0.3318772592
0.125	0.3043413255	0.3052741624	0.3526142975	0.3051864643	0.3041285835	0.3289442299
0.1	0.3074987724	0.3082433183	0.3477346996	0.3081777251	0.3073450280	0.3279890090
0.0625	0.3125118626	0.3129728510	0.3393585073	0.3129362993	0.3124323840	0.3261656791
0.05	0.3142610070	0.3146278507	0.3362363800	0.3145998182	0.3142020426	0.3254321154

Table 6: Errors Incurred in Collapsing the Gaussian Test Set.

Degree of Discretisation	Defuzzified Value (EM)	Collapsing Forward	Collapsing Backward	Collapsing Inward	Collapsing Outward	Collapsing Forward-Backward
0.5	0.2899142309	0.0048158589	0.1190955284	0.0041032246	-0.0014475471	0.0619556937
0.25	0.2906756945	0.0018641846	0.0805389449	0.0016413610	-0.0005787294	0.0412015647
0.125	0.3043413255	0.0009328369	0.0482729720	0.0008451388	-0.0002127420	0.0246029044
0.1	0.3074987724	0.0007445459	0.0402359272	0.0006789527	-0.0001537444	0.0204902366
0.0625	0.3125118626	0.0004609884	0.0268466447	0.0004244367	-0.0000794786	0.0136538165
0.05	0.3142610070	0.0003668437	0.0219753730	0.0003388112	-0.0000589644	0.0111711084

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Table 7: Defuzzified Values and Errors Obtained for the Symmetrical Horizontal Test Set, Collapsed Outward.

		Collapsing Defuzzified Values			**Errors**		
Degree of Disc.	Defuzzified Value	Outward Right	Outward Left	Outward Right-Left	Outward Right	Outward Left	Outward Right-Left
0.1	0.5	0.4995891494	0.5004108506	0.5000000000	-0.0004108506	0.0004108506	0.0000000000
0.05	0.5	0.4998891777	0.5001108223	0.5000000000	-0.0001108223	0.0001108223	0.0000000000
0.02	0.5	0.4999815068	0.5000184932	0.5000000000	-0.0000184932	0.0000184932	0.0000000000
0.01	0.5	0.4999953154	0.5000046846	0.5000000000	-0.0000046846	0.0000046846	0.0000000000
0.005	0.5	0.4999988213	0.5000011787	0.5000000000	-0.0000011787	0.0000011787	0.0000000000
0.002	0.5	0.4999998107	0.5000001893	0.5000000000	-0.0000001893	0.0000001893	0.0000000000
0.001	0.5	0.4999999526	0.5000000474	0.5000000000	-0.0000000474	0.0000000474	0.0000000000
0.0001	0.5	0.4999999995	0.5000000005	0.5000000000	-0.0000000005	0.0000000005	0.0000000000
0.00001	0.5	0.5000000000	0.5000000000	0.5000000000	0.0000000000	0.0000000000	0.0000000000

Table 8: Defuzzified Values and Errors Obtained for the Symmetrical Triangular Test Set, Collapsed Outward.

		Collapsing Defuzzified Values			**Errors**		
Degree of Disc.	Defuzzified Value	Outward Right	Outward Left	Outward Right-Left	Outward Right	Outward Left	Outward Right-Left
0.1	0.4	0.3998916916	0.3998916916	0.3998916916	-0.0001083084	-0.0001083084	-0.0001083084
0.05	0.4	0.3999505451	0.3999522908	0.3999514180	-0.0000494549	-0.0000477092	-0.0000485820
0.02	0.4	0.3999808751	0.3999812363	0.3999810557	-0.0000191249	-0.0000187637	-0.0000189443
0.01	0.4	0.3999904744	0.3999905679	0.3999905212	-0.0000095256	-0.0000094321	-0.0000094788
0.005	0.4	0.3999952381	0.3999952617	0.3999966681	-0.0000047619	-0.0000047383	-0.0000047501
0.002	0.4	0.3999980943	0.3999980981	0.3999980962	-0.0000019057	-0.0000019019	-0.0000019038
0.001	0.4	0.3999990469	0.3999990479	0.3999990474	-0.0000009531	-0.0000009521	-0.0000009526
0.0001	0.4	0.3999999047	0.3999999047	0.3999999047	-0.0000000953	-0.0000000953	-0.0000000953
0.00001	0.4	0.3999999905	0.3999999905	0.3999999905	-0.0000000095	-0.0000000095	-0.0000000095

Table 9: Defuzzified Values and Errors Obtained for the Gaussian Test Set, Collapsed Outward.

		Collapsing Defuzzified Values			**Errors**		
Degree of Disc.	Defuzzified Value	Outward Right	Outward Left	Outward Right-Left	Outward Right	Outward Left	Outward Right-Left
0.5	0.2899142309	0.2884666838	0.2890645675	0.2887656257	-0.0014475471	-0.0008496634	-0.0011486052
0.25	0.2906756945	0.2900969651	0.2902918203	0.2901943927	-0.0005787294	-0.0003838742	-0.0004813018
0.125	0.3043413255	0.3041285835	0.3041906758	0.3041285835	-0.0002127420	-0.0001506497	-0.0001816959
0.1	0.3074987724	0.3073450280	0.3073862653	0.3073656467	-0.0001537444	-0.0001125071	-0.0001331257
0.0625	0.3125118626	0.3124323840	0.3124493111	0.3124408476	-0.0000794786	-0.0000625515	-0.0000710150
0.05	0.3142610070	0.3142020426	0.3142130418	0.3142075422	-0.0000589644	-0.0000479652	-0.0000534648

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