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Abstract— Real options are a typical framework in economics that involves uncertainty. Very often, in fact, managers have vague ideas about the future expected cash flows, the cost of the project and many other variables that are fundamental in the process decision among many investments. The calculation of the value function of real options can take advantage of a model of uncertainty that include stochastic processes and fuzzy numbers.

A special version of the multiple population differential evolution algorithm is designed to compute the level-cuts of the fuzzy extension of the multidimensional real valued function of fuzzy numbers in the resulting optimization problems.

We perform some computational experiments about the option to defer investment, that is an American call option on the present value of the completed expected cash flows with the exercise price equal to the required outlay. We show that fuzziness may help for a more profitable decision.

Keywords— Fuzzy Numbers, Parametric Representation, Real Options, Sensitivity Analysis.

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Nowadays the most suitable valuation methodology for corporate investment decisions is the real options theory (ROT) because it takes into account management's flexibility to adapt ongoing projects in response to uncertain conditions. Since Myers' ([7]) innovative idea of viewing firm's future investment opportunities as real options – that is, the right but not the obligation to undertake some business decision at a cost during a certain period of time -, a vast literature has developed, which elaborate both theoretical and empirical methods for quantifying the values of various real (call or put) options embedded in investment opportunities. Dixit and Pindyck ([3]) develop a systematic treatment of ROT, providing the fundamentals of this method and also emphasizing the market implications of such valuation of investment decisions under uncertainty. Trigeorgis ([12]) provides a classification of real options that maps different categories of investments into the space of different types of financial options.

The value of real options depends on some basic variables:(i) the underlying asset, which is the current value of (gross) expected future operating cash flows, (ii) the exercise price, which is the cost of the project; (iii) the time to expiration of the option, that is the time up to which the project can be undertaken (either finite or infinite); (iv) the standard deviation of the value of the underlying risky asset; (v) the risk-free rate of interest over the life of the option. All the above-mentioned variables are uncertain and therefore various stochastic models have been introduced in ROT to deal with the uncertainty surrounding most corporate decisions. In general expected future cash flows are assumed to evolve according to a geometric Brownian motion, but it often happens that reality is more complex than a normal distribution.

The imprecision associated with the subjective judgement and estimation of future cash flows, which is typical of management's project decisions, needs to be incorporated in the treatment of uncertainty. By introducing fuzzy numbers, we are able to capture the somewhat vague and imprecise ideas the manager possesses about the future expected cash flows, the profitability of the project, the costs of the project, etc. To the best of our knowledge, such an approach has never been discussed in the literature, with the exception of Carlsson and Fuller [1], that interpret the possibility of making an investment decision in terms of a European option, while we use an American option. In addition, we elaborate a computing methodology which is more general and we can represent the shape of the value functions.

The paper is organized as follows. In section 2 we present some basic elements of fuzzy theory which will be used in the numerical implementation of real option models. In section 3 we describe the introduction of fuzziness in real options and section 4 collects some of the computational experiments that have been performed in order to capture how and how much fuzziness affects the decision. Finally, section 5 concludes.

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Fuzzy numbers (more details in [4]) are a very powerful and flexible way to describe uncertainty or possibilistic values for given variables for which a precise quantification is not possible or one is interested in evaluating the effects of variations around a specified value. In fact a fuzzy number models the different specifications of intervals around a given precise value; it is defined, informally, as a "cascade" of intervals, which start with a given number $a \in \mathbb{R}$ and grow increasing to a final interval which gives the most uncertain set of possible values. The levels of the cascade are usually parametrized by a parameter $\alpha \in [0, 1]$ which represents the so called membership value (or possibilistic degree) of a given interval, with the convention that $\alpha = 1$ corresponds to the possibly exact value (the core of the fuzzy number) while $\alpha = 0$ corresponds to the highest uncertainty (the support of the fuzzy number). With the same convention on α we can say that $1 - \alpha$ is the level of uncertainty of the corresponding interval.

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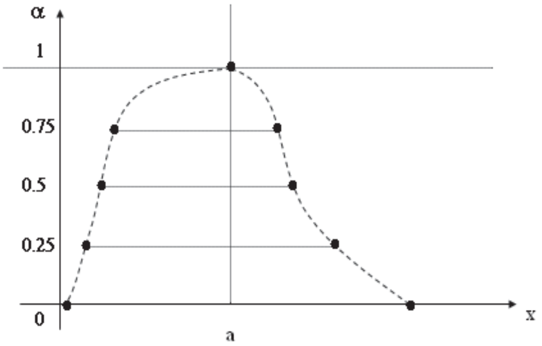
A wide class of fuzzy numbers with the core at $a \in \mathbb{R}$ is obtained by considering its membership function $\mu : \mathbb{R} \longrightarrow [0,1]$ such that, denoting $[a^-, a^+]$ the interval representing the support,

$$\mu(x) = \begin{cases} L(x) & if \quad a^- \leq x \leq a \\ R(x) & if \quad a \leq x \leq a^+ \\ 0 & otherwise \end{cases} \quad \text{for } x \in \mathbb{R} \tag{1}$$

where $L(x)$ is the left branch, an increasing function with $L(a^-) = 0$, $L(a) = 1$ and $R(x)$ is the right branch, a decreasing function with $R(a) = 1$, $R(a^+) = 0$. A fuzzy number obtained by the form (1) is called LR-fuzzy number (with usual notation $u = \langle a^-, a, a^+ \rangle_{L,R}$).



Dbll hR(: A fuzzy number as a "cascade" of intervals representing increasing uncertainty around the given value a.

For values of $\alpha \in]0,1]$, the $\alpha - cut$ is defined to be the compact interval $[u]_\alpha = \{x | \mu(x) \geq \alpha\}$ and the support is $[u]_0 = cl\{x|\mu(x) > 0\}$ ($cl(A)$ is the closure of set A). The *level-cuts* of a fuzzy number are "nested" closed intervals and this property is the basis for the LU representation (L for lower, U for upper).

: Rfie lklf e (*An LU-fuzzy quantity (number or interval) u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \longrightarrow \mathbb{R}$, defining the end-points of the $\alpha - cuts$, satisfying the three conditions:(i) $u^- : \alpha \longrightarrow u^-_\alpha \in \mathbb{R}$ is a bounded monotonic nondecreasing left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;(ii) $u^+ : \alpha \longrightarrow u^+_\alpha \in \mathbb{R}$ is a bounded monotonic nonincreasing left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;(iii) $u^-_\alpha \leq u^+_\alpha$ $\forall \alpha \in [0,1]$. In particular, the $\alpha - cuts$ of a fuzzy number or interval are nonempty, compact intervals of the form $[u]_\alpha = [u^-_\alpha, u^+_\alpha] \subset \mathbb{R}$.*

The support of u is the interval $[u^-_0, u^+_0]$ and the core is $[u^-_1, u^+_1]$. We refer to the functions $u^-_{(.)}$ and $u^+_{(.)}$ as the lower and upper branches on u, respectively. The membership function can be written as $\mu_u(x) = \sup\{\alpha | x \in [u^-_\alpha, u^+_\alpha]\}$, where the left branch is the increasing inverse of $u^-_{(.)}$ on $[u^-_0, u^-_1]$ and the right is the decreasing inverse of $u^+_{(.)}$ on $[u^+_1, u^+_0]$.

To model the monotonic branches u^-_α and u^-_α we start with an increasing shape function p such that $p(0) = 0$ and $p(1) = 1$ and a decreasing function q such that $q(0) = 1$ and $q(1) = 0$, with the four numbers $u^-_0 \leq u^-_1 \leq u^+_1 \leq u^+_0$ defining the support $\left[u^-_0, u^+_0\right]$ and the core $\left[u^-_1, u^+_1\right]$ and we define

$$\begin{aligned} u^-_\alpha &= u^-_1 - (u^-_1 - u^-_0)p(\alpha) \text{ and} \\ u^+_\alpha &= u^+_1 - (u^+_1 - u^+_0)q(\alpha) \text{ for all } \alpha \in [0,1]\,. \end{aligned} \tag{2}$$

The two shape functions p and q, as suggested in [11], are selected in a family of parametrized monotonic functions where the parameters are related to the first derivatives of p and q in 0 and 1; there are many ways to define p and q as illustrated in [9]. For each decomposition we require (in the differentiable case) $4(N+1)$ parameters $u = (\alpha_i; u^-_i, \delta u^-_i, u^+_i, \delta u^+_i)_{i=0,1,...,N}$ satisfying the following conditions:

$$\begin{aligned} u^-_0 &\leq u^-_1 \leq ... \leq u^-_N \leq u^+_N \leq u^+_{N-1} \leq ... \leq u^+_0 \\ \delta u^-_i &\geq 0, \delta u^+_i \leq 0. \end{aligned} \tag{3}$$

and on each sub-interval $[\alpha_{i-1}, \alpha_i]$ we use the data $u^-_{i-1} \leq u^-_i \leq u^+_i \leq u^+_{i-1}$ and the slopes $\delta u^-_{i-1}, \delta u^-_i \geq 0$ and $\delta u^+_{i-1}, \delta u^+_i \leq 0$.

The simplest representation is obtained on the trivial decomposition of the interval $[0,1]$, with $N = 1$ (without internal points) and $\alpha_0 = 0, \alpha_1 = 1$. In this simple case, u can be represented by a vector of 8 components

$$u = (u^-_0, \delta u^-_0, u^+_0, \delta u^+_0; u^-_1, \delta u^-_1, u^+_1, \delta u^+_1). \tag{4}$$

In the search for the value of a real option, the fundamental step is the computation of fuzzy-valued functions. Given a function $y = f(x_1, x_2, ..., x_n)$ of n real (crisp) variables $x_1, x_2, ..., x_n$, its fuzzy extension is obtained to evaluate the effect of uncertainty on the x_j modelled by the corresponding fuzzy number u_j. If $v = f(u_1, u_2, ..., u_n)$ denotes the fuzzy extension of a continuous function f in n variables, then for each level α the resulting interval $[v^-_\alpha, v^+_\alpha]$ represents the propagation of uncertainty from x_j to y. In particular, if the uncertainty on the original variables is modelled by fuzzy numbers, the obtained v is yet a fuzzy number starting from a single value (at level $\alpha = 1$) to the most uncertain interval (at level $\alpha = 0$).

It is well known that the fuzzy extension of f to normal upper semicontinuous fuzzy intervals (with compact support) has the level-cutting property, i.e. the $\alpha - cuts$ $[v^-_\alpha, v^+_\alpha]$ of v are the images of the $\alpha - cuts$ of $(u_1, u_2, ..., u_n)$ and are obtained by solving the box-constrained optimization problems $(EP)_\alpha$:

$$\begin{cases} v^-_\alpha = \min\left\{f(x_1,...,x_n) | x_k \in [u^-_{k,\alpha}, u^+_{k,\alpha}],\ k = 1,...,n\right\} \\ v^+_\alpha = \max\left\{f(x_1,...,x_n) | x_k \in [u^-_{k,\alpha}, u^+_{k,\alpha}],\ k = 1,...,n\right\} \end{cases}. \tag{5}$$

With the exception of simple elementary cases for which the optimization problems above can be solved analytically, the direct application of *(EP)* may be difficult and computationally expensive. Usually, the $\alpha - cuts$ $[v^-_\alpha, v^+_\alpha]$ of v are computed at a prefixed given number of values α_j of interest (say from 10 to 100 points) and the membership function is approximated pointwise. As we will see, an advantage of the LU-parametrization is to obtain the extended fuzzy numbers $v = f(u_1, u_2, ..., u_n)$ in the same parametric form as for $u_1, u_2, ..., u_n$ with a possible important reduction in the computational effort and with a good approximation. As for the

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fuzzy numbers, we will consider the fuzzy extension of multivariate differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$.

For a vector of n fuzzy numbers $u = (u_1, u_2, ..., u_n)$ let

$$u_k = (u^-_{k,i}, \delta u^-_{k,i}, u^+_{k,i}, \delta u^+_{k,i})_{i=0,1,...,N} \text{ for } k = 1, 2, ..., n$$

be the LU representation of the $k - th$ component. Let $v = f(u_1, u_2, ..., u_n)$ and $v = (v^-_i, \delta v^-_i, v^+_i, \delta v^+_i)_{i=0,1,...,N}$ be its LU representation; the $\alpha - cuts$ of v are obtained by solving the box-constrained optimization problems (5). For each $\alpha = \alpha_i$, $i = 0, 1, ..., N$ the $\min$ and the $\max$ (5) can occur either at a point whose components $x_{k,i}$ are internal to the corresponding intervals $[u^-_{k,i}, u^+_{k,i}]$ or are coincident with one of the extreme values; denote by $\widehat{x}^-_i = (\widehat{x}^-_{1,i}, ..., \widehat{x}^-_{n,i})$ and $\widehat{x}^+_i = (\widehat{x}^+_{1,i}, ..., \widehat{x}^+_{n,i})$ the points where the $\min$ and the $\max$ take place; then

$$v^-_i = f(\widehat{x}^-_{1,i}, \widehat{x}^-_{2,i}, ..., \widehat{x}^-_{n,i}) \text{ and } v^+_i = f(\widehat{x}^+_{1,i}, \widehat{x}^+_{2,i}, ..., \widehat{x}^+_{n,i}) \tag{6}$$

and the slopes δv^-_i, δv^+_i are computed (as f is differentiable) by

$$\begin{aligned} \delta v^-_i &= \sum_{\substack{k=1 \\ \widehat{x}^-_{k,i} = u^-_{k,i}}}^{n} f'^-_k \delta u^-_{k,i} + \sum_{\substack{k=1 \\ \widehat{x}^-_{k,i} = u^0_{k,i}}}^{n} f'^-_k \delta u^+_{k,i} \\ \delta v^+_i &= \sum_{\substack{k=1 \\ \widehat{x}^0_{k,i} = u^-_{k,i}}}^{n} f'^+_k \delta u^-_{k,i} + \sum_{\substack{k=1 \\ \widehat{x}^0_{k,i} = u^0_{k,i}}}^{n} f'^+_k \delta u^+_{k,i} \end{aligned} \tag{7}$$

where $f'^-_k = \frac{\partial f(\widehat{x}^-_{1,i},..,\widehat{x}^-_{n,i})}{\partial x_k}$ and $f'^+_k = \frac{\partial f(\widehat{x}^0_{1,i},..,\widehat{x}^0_{n,i})}{\partial x_k}$.To solve the optimization problems (5), we use an implementation of a multiple population differential evolution algorithm extensively described and analyzed in [10].

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The *option to defer investment* is an American call option on the present value of the completed expected cash flows with the exercise price being equal to the required outlay. A project that can be postponed allows learning more about potential project outcomes before making a commitment. A seminal contribution on the option to defer is McDonald and Siegel [6] where the optimal time to invest and an explicit formula for the value of the option to invest are derived for an irreversible project whose net profits follow a geometric Brownian motion.

A firm is supposed to consider the following investment opportunity: at any time t the firm can pay some estimated cost K to install an investment project whose expected future net cash flows conditional on undertaking the project have an estimated present value Π. The installation of such project is irreversible. Let Π follow a geometric Brownian motion of the form:

$$d\Pi = \Pi(\mu dt + \sigma dW_t) \tag{8}$$

where $\mu < r$ is the appreciation rate, r is the risk-free interest rate and σ is the volatility ($\mu \in R, \sigma > 0$) and W is a standard Wiener process. For simplicity, let us assume that the time to expiration of this investment opportunity is infinite, which facilitates the derivation of a closed-form solution. If $V = V(\Pi)$ is the option value then it holds:

$$\frac{1}{2}\sigma^2\Pi^2 V''(\Pi) + \mu\Pi V'(\Pi) - rV = 0$$

for $\Pi < \Pi^*$ with the initial condition $V(0) = 0$ and smooth-pasting $V(\Pi^*) = \Pi^* - K, V'(\Pi^*) = 1$. The solution is

$$\begin{cases} \Pi^* = K\frac{\phi}{\phi-1} \\ V(\Pi) = (\Pi^* - K)\left(\frac{\Pi}{\Pi^*}\right)^{\phi} \end{cases} \tag{9}$$

with $\phi = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left(\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right)^{\frac{1}{2}} > 1$.

In our methodology fuzziness is present in three steps. The estimated present value Π of future net cash flows of the project follows the stochastic differential equation (8) and we model the uncertainty of its parameters across intervals of values. The intervals are built with differentiated levels of uncertainty; given a crisp value, the levels produce a shape that can be characterized by asymmetries or nonlinearities depending on subjective beliefs and available information of the decision maker. It follows that fuzzy parameters play the lead role in a sensitivity analysis that starts gradually from a null variation to the greatest variation of the uncertainty consistent with data. In particular W_t remains a standard Brownian motion (the theory and applications of fuzzy set-valued stochastic differential equations has received recently several significant contributions from the seminal paper of Feng [5]).

Fuzziness comes out also in the valuation function of the option (obtained with the extension principle) that depends not only on Π, σ and μ but also on r and K, which we assume to be fuzzy too.

Finally, fuzziness affects the crucial value Π^*: as soon as Π reaches the threshold value Π^*, the firm finds it optimal to invest (case of the option to defer investment) or disinvest and liquidate (case of the option to abandon). Thus, the decision is based on the threshold value, which depends on all the parameters of the model. In the valuation method based on fuzzy variables, $\{\Pi_t, t \geq 0\}$ is assumed to be a fuzzy stochastic process, which is specified by the following membership function

$$\mu_{\Pi_t(\omega)}(x) = max\{1 - \mid (x - \widehat{\Pi}_t(\omega))/\beta_t(\omega) \mid, 0\},$$

that is, the fuzzy random variable Π_t is of the triangular type, with centre $\widehat{\Pi}_t(\omega)$, and left-width and right-width $\beta(\omega)$. The assumption of fuzziness is related to the manager's subjective belief about the future profitability of the project. The choice of a triangle-type shape is not restrictive at all and is introduced for simplicity only. Observe that the fuzziness in the process increases as $\beta(\omega)$ becomes bigger. The α-cuts of $\Pi_t(\omega)(x)$ are $\Pi^{\pm}_{t,\alpha}(\omega) = \left[\Pi^-_{t,\alpha}(\omega), \Pi^+_{t,\alpha}(\omega)\right] = \left[\widehat{\Pi}_t(\omega) - (1-\alpha)\beta(\omega), \widehat{\Pi}_t(\omega) + (1-\alpha)\beta(\omega)\right]$. It is also reasonable to assume that K is a fuzzy number. In the case of an option to defer, K is the estimated liquidation value of the firm's stock of capital and is affected by depreciation, fluctuating market evaluation and taxation regimes. In the case of an option to abandon, K denotes the investment cost and has many components which can change during the waiting period, due to various unpredictable circumstances.

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The extension principle is then applied to obtain the fuzzy Π^* and $V(\Pi^*)$ from the exact solutions given in equation (9). In the formulae (6)-(7) the vector $\widehat{x}_i$ is equal to $(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{r}_i, \widehat{K}_i)$ and some of the partial derivatives that define the slopes of the representation are nothing else than the first order Greeks, in particular, $\frac{\partial f(\widehat{\mu}_i,\widehat{\sigma}_i,\widehat{r}_i,\widehat{K}_i)}{\partial \sigma}$ is the Vega and $\frac{\partial f(\widehat{\mu}_i,\widehat{\sigma}_i,\widehat{r}_i,\widehat{K}_i)}{\partial r}$ is the Rho.

The degree of the uncertainty and the way in which it is spread from the model, play a central role in the analysis of the real option. The nonlinearities entering in the definition of $V(\Pi)$ in (9) are the main cause of such effects and they can propagate or contract uncertainty. It is very important to perceive the magnitude and the type of these effects. In particular we are interested in the analysis of how the various kinds of uncertainties inserted into the parameters will produce the corresponding uncertainties in Π^*, $V^* = V(\Pi^*)$, Π^{**} and $V^{**} = V(\Pi^{**})$.

As soon as information (on μ, σ, r, K) is modelled by fuzzy numbers, Π^* and V^* also become fuzzy and are represented by $\alpha - cuts$ $\left[{\Pi^*_\alpha}^-, {\Pi^*_\alpha}^0\right]$ and $\left[{V^*_\alpha}^-, {V^*_\alpha}^0\right]$ for all degrees of possibility α. The maximal uncertainty corresponds to the supports at $\alpha = 0$, given by the intervals $\left[{\Pi^*_0}^-, {\Pi^*_0}^0\right]$ and $\left[{V^*_0}^-, {V^*_0}^0\right]$ for Π^* and V^* respectively.

Due to the nonlinearity of Π^* and V^*, the $\alpha - cuts$ are not necessarily symmetric and, for a given uncertainty on the input values μ, σ, r and K, they have different left and right variations. Let $\widehat{\Pi}^*$ and $\widehat{V}^*$ denote the values of Π^*_α and V^*_α corresponding to $\alpha = 1$. It is immediate to argue that V^* is symmetric if and only if $\Delta {V^*_\alpha}^0 = \Delta {V^*_\alpha}^-$, $\forall \alpha \in [0, 1[$ where

$$\Delta {V^*_\alpha}^0 = {V^*_\alpha}^0 - \widehat{V}^*, \quad \Delta {V^*_\alpha}^- = \widehat{V}^* - {V^*_\alpha}^-.$$

The quantity $\Delta {V^*_\alpha}^0$ represents the possible increase in $\widehat{V}^*$ due to uncertainty and analogously, $\Delta {V^*_\alpha}^-$ measures the possible decrease. The same argument can be applied to Π^*_α and $\widehat{\Pi}^*$, defining the quantities $\Delta {\Pi^*_\alpha}^0 = {\Pi^*_\alpha}^0 - \widehat{\Pi}^*$ and $\Delta {\Pi^*_\alpha}^- = \widehat{\Pi}^* - {\Pi^*_\alpha}^-$.

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We test the fuzziness effect in the option to defer investment by running several computational experiments; the fuzzy version of the indicated parameters (say θ) are obtained as triangular symmetric fuzzy numbers, centered at the crisp values and with the support being the interval $[\theta - 0.1\theta, \theta + 0.1\theta]$, corresponding to a symmetric uncertainty of 10% of the value of the parameter. To analyze the effect of the uncertainty on the output variables, we give the plots of their membership functions and the tables including the values for $\alpha - levels$ with $\alpha = 1$ (the crisp level), $\alpha = 0.75$, $\alpha = 0.5$, $\alpha = 0.25$, $\alpha = 0$ (corresponding to the uncertainty of 10% in the parameters).

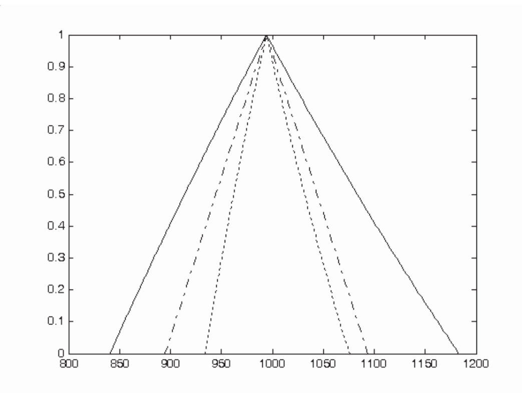
The robustness of the fuzzy model for the option to defer investment is tested with three sets of real data that we call, for short, Data1, Data2 and Data3, referring to three different industrial sectors. *Data1* refers to an investment decision in the human genome sciences project (HGSI) whose data are taken from the Human Genome project database (details in http://www.ornl.gov/sci/ techresources/ Human_Genome/ home.shtml). *Data2* refers to an investment decision in a big infrastructure, that is the Eurotunnel project (details in [2]). *Data3* deals with the case of an investment in new capacity in the public-utility sector, i.e. the electricity market (details in [8]). The values of the parameters μ, σ, r and K are the following:

	: HkH(	: HkH)	: HkH-
μ	0.01	0.025	0.03
σ	0.048	0.183	0.173
r	0.044	0.06	0.08
K	704.9	8865	600

In figures concerning the behavior of Π^* we report the three different cases that we will denote as: *Allfuzzy* (straight line) when the parameters μ, σ, r and K are fuzzy, *Kcrisp* (dotted line) when μ, σ, r are fuzzy and K is crisp and finally *Kfuzzy* (dashed line) when μ, σ, r are crisp and K is the unique source of uncertainty.

4.0.1 Results for Data1

Figure 2 shows that, as expected, the greatest uncertainty in Π^* occurs in the Allfuzzy case, when all the fuzzy quantities are considered to be fuzzy; but it is interesting to observe that in the Kcrisp case (dotted line) the generated uncertainty is less then in the Kfuzzy case (dashed line), i.e. the uncertainty in the values of only K produces more uncertainty on Π^* then the uncertainty in the values of μ, σ and r.



DbTl hR) 4 Π^* for Data1

Table 1, Table 2 and Table 3 report values of the $\alpha - cut$ for Data1 in the Allfuzzy, Kcrisp and Kfuzzy case respectively.

Table 1

α	Π^-	Π^+
1.0	994.28	994.28
0.75	953.16	1037.56
0.5	913.97	1083.26
0.25	876.50	1131.69
0	840.58	1183.25

Table 2

Π^-	Π^+
994.28	994.28
977.60	1012.25
962.07	1031.67
947.56	1052.74
933.98	1075.68

Table 3

α	Π^-	Π^+
1.0	994.28	994.28
0.75	969.43	1019.14
0.5	944.57	1043.99
0.25	919.71	1068.86
0	894.86	1093.71

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Observe that in the Kfuzzy case (Table 3) the threshold value Π^* displays a symmetric shape in all analyzed projects because Π^* depends linearly on K. In the Allfuzzy and Kcrisp cases, instead, we can observe an asymmetric pattern, due to the nonlinear dependence of Π^* with respect to the other variables.

At level 0.5 the average values are 998.615 in Allfuzzy and 996.87 in Kcrisp, which are larger than the crisp value 994.28. Since on average the fuzzy threshold value is larger then without fuzziness, just considering the crisp value the decision to invest would be too early. Figure 3 shows the graphical behavior of the fuzzy function $V(\Pi)$ in the Allfuzzy case; the little crosses point the optimal values of Π corresponding to the levels of Π^* for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

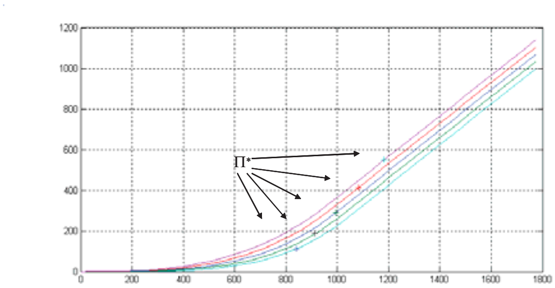

Figure 3: Data1 when parameters μ, σ, r and K are fuzzy.

It is evident that fuzziness implies a certain degree of freedom in the choice of Π^*. Figure 4 illustrates $V(\Pi)$ as a fuzzy function (a sequence of fuzzy numbers).

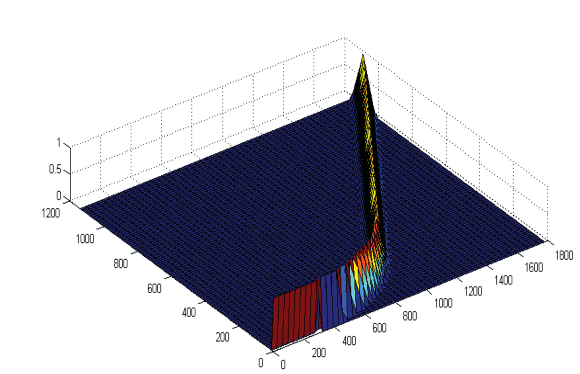

Figure 4: $V(\Pi)$ in the Allfuzzy case

4.0.2 *Results for Data2*

Figure 5 shows the behavior of Π^* in the three different cases: again the biggest uncertainty occurs in the Allfuzzy case, when all the quantities are fuzzy; but we observe that in the Kcrisp case (dotted line), when μ, σ, r are the sources of uncertainty and K is the unique crisp value, the uncertainty in Π^* is bigger then in the Kfuzzy case, i.e. the same level of uncertainty in K produces less uncertainty on Π^* then the uncertainty in the other parameters. With respect to Data1, there is here an inversion.

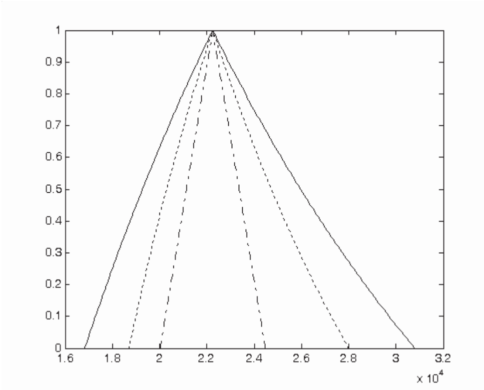

Figure 5: Π^* for Test 2

Table 4 and Table 5 report the values of the $\alpha - cut$ of Π^* in the Allfuzzy and Kcrisp case for Data2.

Table 4:

α	Π^-	Π^+
1.0	22249.62	22249.62
0.75	20686.18	23997.55
0.5	19277.60	25967.74
0.25	18000.33	28209.16
0	16835.43	30786.48

Table 5:

α	Π^-	Π^+
1.0	22249.62	22249.62
0.75	21216.59	23412.24
0.5	20292.21	24731.18
0.25	19459.82	26241.08
0	18706.04	27987.71

If we compute again the average values at level 0.5, they are 22622.67 in Allfuzzy and 22511.695 in Kcrisp, which are larger than the crisp value 22249.62. It follows that in the Data2 project it is confirmed the suggestion to wait for the decision to invest.

4.0.3 *Results for Data3*

The last project we consider for an option to defer investment is Data3; the relative values of Π^* are reported in Figure 6:

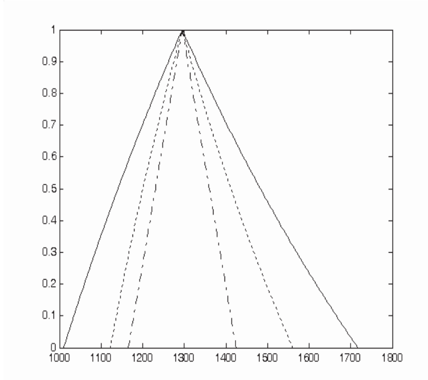

Figure 6: Π^* for Test4

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Table 6, Table 7 report values of the $\alpha - cut$ for Data3 in the Allfuzzy and Kcrisp case respectively.

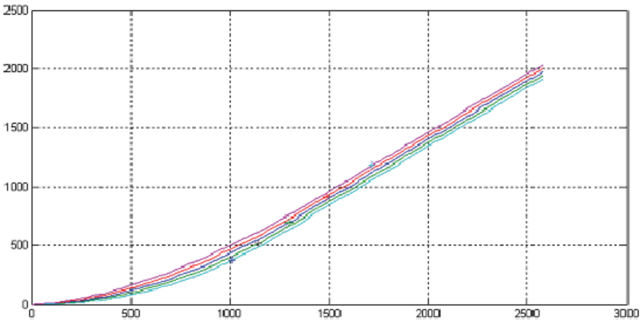
GHI dR2

α	Π^-	Π^+
1.0	1295.31	1295.31
0.75	1214.47	1384.31
0.5	1140.61	1482.98
0.25	1072.75	1593.18
0	1010.11	1717.32

GHI dR3

Π^-	Π^+
1295.31	1295.31
1245.61	1350.55
1200.64	1412.36
1159.73	1482.03
1122.34	1561.20

The graphical representation of $V(\Pi)$ in Allfuzzy case is in Figure 7.



DbTI hR34 Data3 in the Allfuzzy case

Some further considerations concerning the $\alpha - cut$ values in all the data set enable us to state that our model allows us to describe how the investment decision is actually affected by a perceived increase in "fuzziness". For a pessimistic (optimistic) firm an increase in fuzziness decreases (increases) the perceived value of the project in comparison with the crisp value. On average - for most decision makers- an increase in fuzziness has a positive impact on the investment opportunity, i.e. it increases the perceived value of the project. As a consequence, the decision to invest is delayed in comparison with the absence of fuzziness. However, for pessimistic decision-makers imprecise information about the project value becomes available over time, which makes waiting with investment less valuable. Thus, for pessimistic firms higher fuzziness erodes the subjective value of the investment opportunity. Notice that this result is in keeping with the literature on real options and ambiguity aversion it contrasts with the impact of volatility in the standard real option theory.

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The powerful contribution of fuzzy modelling can be shown in many fields and, especially in human sciences like economics, it can provide rigorous models (a detailed motivation of its use is in Zadeh [13]). In tis paper, we model the uncertainty involved in real options theory through fuzzy numbers represented in the LU model; when including fuzziness, the decision rule moves away from the original one and the choice to delay or not the investment becomes a key feature of the fuzzy model.

Our model allows us to quantify how the threshold values change with fuzzy parameters. Practically, we solve the investment/disinvestment decision in terms of a problem of fuzzy stopping time - that is, a fuzzification of the classical optimal stopping time - and our computation experiments provide the sensitivity analysis of the decision variable with respect to the relevant parameters. Our main results are that the decision to invest is delayed in comparison with the case of absence of fuzziness, the decision to disinvest is anticipated in a fuzzy environment. The fuzzy model we use provides the set of values within which the decision is taken by the manager. Our empirical validation, which is based on case studies from real world, seems to confirm the validity of the model and to open up further ways of research in finance in a fuzzy environment.

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