Possibility theory and formal concept analysis in information systems

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Abstract— *The setting of formal concept analysis presupposes the existence of a relation between objects and properties. Knowing that an unspecified object has a given property induces a formal possibility distribution that models the set of objects known to possess this property. This view expressed in a recent work by the authors of the present paper, has led to introduce the set-valued counterpart to the four set functions evaluating potential or actual, possibility or necessity that underlie bipolar possibility theory, and to study associated notions. This framework puts formal concept analysis in a new, enlarged perspective, further explored in this article. The "actual (or guaranteed) possibility" function induces the usual Galois connexion that defines the notion of a concept as the pair of its extent and its intent. A new Galois connexion, based on the necessity measure, partitions the relation in "orthogonal" subsets of objects having distinct properties. Besides, the formal similarity between the notion of division in relational algebra and the "actual possibility" function leads to define the fuzzy set of objects having* most *properties in a set, and other related notions induced by fuzzy extensions of division. Generally speaking, the possibilistic view of formal concept analysis still applies when properties are a matter of degree, as discussed in the paper. Lastly, cases where the object / property relation is incomplete due to missing information, or more generally pervaded with possibilistic uncertainty is also discussed.*

Keywords— possibility theory; formal concept analysis.

1 Introduction

Formal concept analysis [1] exploits the duality between objects and properties in a lattice theory setting, which has led to an original and practical view of the notion of a formal concept with application in data mining. A concept is then a pair made of a set of objects and a set of properties that are in mutual correspondence. These two sets are the extent and of the intent of the concept respectively. In this framework, properties are *binary*, and *complete* information is assumed about the relation linking objects and properties.

Fuzzy set theory [2] has emphasized the idea that properties are not always all-or-nothing notions, but are rather often a matter of degree. This has led to an extension of the original formal concept analysis setting by allowing intermediate truth values for the propositions "object x has property y" [3]. However, complete information is still assumed. Namely, for any pair (object, property), it it is known to what degree the object has the property. Besides, fuzzy sets have also been the starting point for the development of a new approach for the representation of uncertainty, named possibility theory [4, 5]. Fuzzy sets then have a disjunctive reading and represent states of incomplete information, i.e. (soft) restrictions on the mutually exclusive possible values of a single-valued variable.

The authors of the present paper have recently advocated the interest of a possibilistic reading of formal concept analysis where the possibility theory set-functions are shown to be meaningful in formal concept analysis [6]. Under this view, the set of objects known to possess a given property plays the role of a formal possibility distribution that restricts the possible value (identity) of an unspecified object only described as having the given property. This leads to an enlarged setting that we continue to investigate in this paper. Besides, this enlarged setting can be itself extended either by allowing properties to be non-Boolean, or by considering that the relation between objects and properties may be incompletely known.

The next section provides the background on a possibility theory-inspired view of formal concept analysis. Four basic operators that are the counterparts of the four basic set-functions in bipolar possibility theory are introduced, leading to consider another Galois connexion distinct from the usual one that gives birth to the notion of concept. Section 3 briefly describes the extension of the enlarged setting to fuzzy properties, i.e., when the relation linking objects and properties becomes fuzzy. Section 4 relates the operator underlying classical formal concept analysis and the notions of division (and quotient) in relational algebra. This provides a basis for defining the fuzzy set of objects having $most$ properties in a set (where $most$ is a fuzzy quantifier) and other related notions. Section 5 deals with the situation where information is incomplete or uncertain. Due to the lack of space, the paper only outlines new ideas, without developing them.

2 A possibility theory view of concept analysis

A formal information system is viewed here as a binary relation R between a set Obj of objects and a set $Prop$ of Boolean properties. Some authors speak of 'attribute' instead of 'property'. As we shall see in Section 5, this distinction only matters for attributes with non-binary domains. R is called *context* in formal concept analysis. If $X \subseteq Obj$, $\overline{X}$ is its complement $Obj \setminus X$. The notation $(x, y) \in R$ means that object x has property y. $R(x) = \{y \in Prop | (x, y) \in R\}$ is the set of properties of object x. Similarly, $R^{-1}(y) = \{x \in Obj | (x, y) \in R\}$ is the set of objects having property y. Its characteristic function induces a two-valued possibility distribution π:

$$\forall x \in Obj, \quad \pi(x) = \begin{cases} 1 & \text{if } x \in R^{-1}(y) \\ 0 & \text{otherwise,} \end{cases}$$

Intuitively speaking, if all we know about an unknown object is that it has property y then this object may be any x such that $\pi(x) = 1$ in context R. Thus, the relation R for a particular property y plays the role of a possibility distribution π encoding a set of possible values for x.

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2.1 Possibility theory

In possibility theory [5], two "measures" are associated with a possibility distribution π defined on a universe U as the characteristic (membership) function of a fuzzy set E representing the available information (in the above case $U = Obj$, and $E = R^{-1}(y)$ is an ordinary subset of U). Namely

i) a possibility measure Π (or "potential possibility"):

$$\Pi(A) = \max_{x \in A} \pi(x).$$

It estimates to what extent event A is *consistent with* the information represented by π. $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ is the characteristic property of possibility measures [4].

ii) a dual measure of necessity N, expressing that an event is all the more necessarily (certainly) true as the opposite event is more impossible. N thus reflects an "actual necessity":

$$N(A) = 1 - \Pi(\overline{A}) = 1 - \max_{x \notin A} \pi(x),$$

where $\overline{A} = U \setminus A$. $N(A)$ estimates to what extent event A is *implied* by the information E represented by π (inasmuch as this information entails that any realization of $\overline{A}$ is more or less impossible). Necessity measures are characterized by the decomposability property $N(A \cap B) = \min(N(A), N(B))$.

Π and N are based on the $maximum$ of π over A and $\overline{A}$ respectively; two other set-functions [5] use the $minimum$:

iii) a measure of "actual (or guaranteed) possibility"

$$\Delta(A) = \min_{x \in A} \pi(x),$$

which estimates to what extent all elements in A are possible. Δ can be also termed "$sufficiency\ measure$" since $\Delta(A) = 1$ is enough for ensuring that all realizations of A are actually possible. Clearly, $\Delta \leq \Pi$. Note also that $\Delta(A)$ and $N(A)$ are unrelated. $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$ is the characteristic property of guaranteed possibility measures.

iv) a dual measure of "potential necessity or certainty"

$$\nabla(A) = 1 - \Delta(\overline{A}) = 1 - \min_{x \notin A}(\pi(x))$$

which estimates to what extent there exists at least one value in the complement of A that has a zero (or more generally a low) degree of possibility. This is clearly a $necessary$ condition for having "$x \in A$" somewhat certain. Property $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$ characterizes these measures.

2.2 An enlarged formal concept analysis setting

These four set functions make sense in the formal concept analysis. Namely, four remarkable sets can be defined:

$$R^{\Pi}(X) = \{y \in Prop | R^{-1}(y) \cap X \neq \emptyset\}$$
$$R^{N}(X) = \{y \in Prop | R^{-1}(y) \subseteq X\}$$
$$R^{\Delta}(X) = \{y \in Prop | R^{-1}(y) \supseteq X\}$$
$$R^{\nabla}(X) = \{y \in Prop | R^{-1}(y) \cup X \neq Obj\}$$

whose respective characteristic functions are $\Pi(X)$, $N(X)$, $\Delta(X)$, and $\nabla(X)$. Their meanings are as follows w. r. t. a subset of objects X in context R.

- $R^{\Pi}(X)$ is the set of properties that are associated with at least one object in X. Formally, we have

  $$R^{\Pi}(X) = \cup_{x \in X} R(x).$$

  $R^{\Pi}(X)$ is such that any object that satisfies one of them is possibly in X. In other words, if an object has none of the properties in $R^{\Pi}(X)$ then it cannot belong to X. Moreover, we have $R^{\Pi}(X_1 \cup X_2) = R^{\Pi}(X_1) \cup R^{\Pi}(X_2)$.

- $R^{N}(X)$ is the set of properties s. t. any object that satisfies one of them is necessarily in X. Having any property in $R^{N}(X)$ is a sufficient condition for belonging to X, and $R^{N}(X) = \overline{R^{\Pi}(\overline{X})} = Prop \setminus R^{\Pi}(\overline{X})$. Thus,

  $$R^{N}(X) = \cap_{x \notin X} \overline{R(x)}.$$

  and $R^{N}(X_1 \cap X_2) = R^{N}(X_1) \cap R^{N}(X_2)$.

- $R^{\Delta}(X)$, set of properties shared by all objects in X is

  $$R^{\Delta}(X) = \cap_{x \in X} R(x).$$

  In other words, satisfying all properties in $R^{\Delta}(X)$ is a necessary condition for an object to belong to X. $R^{\Delta}(X)$ is a partial conceptual characterization of objects in X: objects in X should have all the properties of $R^{\Delta}(X)$ and may have some others (that are not shared by all objects in X). It is worth noticing that $\overline{R^{\Pi}(X)}$ provides a negative conceptual characterization of objects in X since it gathers all the properties that are never satisfied by any object in X. Besides, it can be checked that $R^{N}(X) \cap R^{\Delta}(X)$ is the set of properties possessed by all objects in X and only by them. Moreover, we have $R^{\Delta}(X_1 \cup X_2) = R^{\Delta}(X_1) \cap R^{\Delta}(X_2)$.

- Note that $R^{\nabla}(X) = \overline{R^{\Delta}(\overline{X})} = Prop \setminus R^{\Delta}(\overline{X})$. Thus $R^{\nabla}(X)$ is the set of properties in $Prop$ that are not satisfied by at least one object in $\overline{X}$, i.e. $R^{\nabla}(X)$ is the set of properties that some object in $\overline{X}$ misses. In other words, in context R, for any property in $R^{\nabla}(X)$, there exists at least one object outside X that misses it. We have

  $$R^{\nabla}(X) = \cup_{x \notin X} \overline{R(x)}.$$

  and the following decomposability property holds $R^{\nabla}(X_1 \cap X_2) = R^{\nabla}(X_1) \cup R^{\nabla}(X_2)$.

Note that $R^{\Pi}(X)$ and $R^{N}(X)$ get larger when X increases, while $R^{\Delta}(X)$ and $R^{\nabla}(X)$ get smaller. The four modal-like operators R^{Π}, R^{N}, R^{Δ}, and R^{∇} have been considered by Düntsch and Orlowska [7] in the Boolean algebra setting, where R^{Δ} is called *sufficiency* operator, and its representation capabilities are studied. Taking inspiration as the previous authors from rough sets [8], Yao [9] also lays bare these four subsets. In both cases, the four operators were introduced without any mention of possibility theory.

Results in possibility theory have their counterparts in the enlarged formal concept analysis setting, as, e.g., [6]: If R^{-1} is s. t. $\forall y \in Prop, R^{-1}(y) \neq \emptyset$ and $R^{-1}(y) \neq Obj$, then

$$\forall X \subseteq Obj, \quad R^{N}(X) \cup R^{\Delta}(X) \subseteq R^{\Pi}(X) \cap R^{\nabla}(X). \quad (1)$$

Assuming that the property y is non trivial with respect to the set of objects Obj, i.e. $R^{-1}(y) \neq \emptyset$ (at least one object has property y) and $R^{-1}(y) \neq Obj$ (at least one object has not property y), then the four sets $R^{\Pi}(X)$, $R^{\Delta}(X)$, $R^{N}(X)$,

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Situation	$y \in R^{\Pi}(X)$	$y \in R^{\Delta}(X)$	$y \in R^{N}(X)$	$y \in R^{\nabla}(X)$
1. $X = \overline{R^{-1}(y)}$	0	0	0	0
2. $X \subset \overline{R^{-1}(y)}$	0	0	0	1
3. $\overline{R^{-1}(y)} \subset X$	1	0	0	0
4. $R^{-1}(y) \cap X \neq \emptyset$, $R^{-1}(y) \cap \overline{X} \neq \emptyset$	1	0	0	1
5. $R^{-1}(y) \subset X$	1	0	1	1
6. $X \subset R^{-1}(y)$	1	1	0	1
7. $R^{-1}(y) = X$	1	1	1	1

Figure 1: The seven possible positions of X and $R^{-1}(y)$

	objects							
	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c						×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h		×	×	×				
i				×				

Figure 2: R: a relation objects/properties a, b, c, ..., i

$R^{\nabla}(X)$ are necessary and sufficient for describing (and distinguishing between) the seven relative possible positions of X and $R^{-1}(y)$, as shown in Table 1, where 1 (resp. 0) stands for $y \in A$ (resp $y \notin A$) where A is the set $R^{*}(X)$ associated to the column (and $*$ is Π, Δ, N, or ∇). Note that the 9 = 16 - 7 remaining binary 4-tuples are ruled out by the constraints induced by (1), namely $R^{N}(X) \subseteq R^{\Pi}(X)$, $R^{\Delta}(X) \subseteq R^{\Pi}(X)$, $R^{\Delta}(X) \subseteq R^{\nabla}(X)$, $R^{N}(X) \subseteq R^{\nabla}(X)$. For instance, the "trivial" cases $R^{-1}(y) = \emptyset$ and $R^{-1}(y) = Obj$ (ruled out by the constraints) would be captured by distinct 4-tuples in Table 1, namely $(0 \quad 0 \quad 1 \quad 1)$ and $(1 \quad 1 \quad 0 \quad 0)$ respectively.

The above characterization of remarkable sets of properties w. r. t. a set of objects can be easily adapted for defining the corresponding sets of objects associated to a set of properties $Y \in Prop$: namely $R^{-1\Pi}(Y)$, $R^{-1N}(Y)$, $R^{-1\Delta}(Y)$, and $R^{-1\nabla}(Y)$. Their definitions can be easily obtained by swapping R and R^{-1} and exchanging the roles of the sets Obj and $Prop$. Namely,

$$R^{-1\Pi}(Y) = \{x \in Obj | R(x) \cap Y \neq \emptyset\} = \cup_{y \in Y} R^{-1}(y)$$
$$R^{-1N}(Y) = \{x \in Obj | R(x) \subseteq Y\} = \cap_{y \notin Y} \overline{R^{-1}(y)}$$
$$R^{-1\Delta}(Y) = \{x \in Obj | R(x) \supseteq Y\} = \cap_{y \in Y} R^{-1}(y)$$
$$R^{-1\nabla}(Y) = \{x \in Obj | R(x) \cup Y \neq Obj\} = \cup_{y \notin Y} \overline{R^{-1}(y)}.$$

Remark The above operators can be combined together. For instance, consider an object x_0. Let $R(x_0)$ be the set of its (known) properties. Compute $R^{-1\Delta}(R(x_0))$, the set of objects that share these properties. Then get $R^{\Pi}(R^{-1\Delta}(R(x_0)))$, which is the set of properties that are associated with at least one object sharing the properties of x_0. Viewing the table $Obj \times Prop$ as the information pertaining to a repertory of cases, and x_0 as a partially known extra object (not in Obj) for which one tries to guess other properties, the expression $R^{\Pi}(R^{-1\Delta}(R(x_0)))$ may be viewed as the result of a *case-based reasoning* procedure, i.e. a set of potential properties that x_0 may also have. Besides, its subset $R^{N}(R^{-1\Delta}(R(x_0)))$ is the set of properties that alone characterize the objects sharing the properties of x_0. Thus, if one of the properties in $R^{N}(R^{-1\Delta}(R(x_0)))$ is not already among the known properties of x_0, it may be considered as a serious candidate property for x_0.

2.3 Galois connexions

In formal concept analysis, the pair of set valued functions R^{Δ} and $R^{-1\Delta}$ induces a Galois connexion [10] between 2^{Obj} and 2^{Prop}. Then, a $formal\ concept$ is a pair (X, Y) such that $X = \{x \in Obj | R(x) \supseteq Y\}$ and $Y = \{y \in Prop | R^{-1}(y) \supseteq X\}$, i.e. such that $X = R^{-1\Delta}(Y)$ and $Y = R^{\Delta}(X)$, X is called its *extent* and Y its *intent*. In other words, in a formal concept (X, Y), Y is the set of properties shared by all the objects in X, and X is the set of objects that possess all the properties in Y. Then $X \times Y \subseteq R$, i.e. $\forall x \in X, \forall y \in Y, (x, y) \in R$. A formal concept is a maximal pair that satisfies the latter condition (where maximality is taken in the sense of set inclusion).

Putting formal concept analysis in the perspective of possibility theory, it becomes then natural to also consider

- the pairs (X, Y) s. t. $X = R^{-1\Pi}(Y)$ and $Y = R^{\Pi}(X)$;
- the pairs (X, Y) s. t. t $X = R^{-1N}(Y)$ and $Y = R^{N}(X)$;
- the pairs (X, Y) s. t. $X = R^{-1\nabla}(Y)$ and $Y = R^{\nabla}(X)$.

First, observe that $X = R^{-1\nabla}(Y)$ and $Y = R^{\nabla}(X)$ holds if and only if $\overline{X} = R^{-1\Delta}(\overline{Y})$ and $\overline{Y} = R^{\Delta}(\overline{X})$ holds, i.e., if $(\overline{X}, \overline{Y})$ is a formal concept, due to the duality between operators R^{Δ} and R^{∇}. Similarly, $X = R^{-1\Pi}(Y)$ and $Y = R^{\Pi}(X)$ holds if and only if $\overline{X} = R^{-1N}(\overline{Y})$ and $\overline{Y} = R^{N}(\overline{X})$ holds. But, it can be easily seen that a pair (X, Y) such that $X = R^{-1N}(Y)$ and $Y = R^{N}(X)$, i.e. such that $X = \{x \in Obj | R(x) \subseteq Y\}$ and $Y = \{y \in Prop | R^{-1}(y) \subseteq X\}$ is not generallly a formal concept, as now exemplified. This Galois connexion has been introduced by [11] on a formal basis, but its practical meaning was apparently not really discussed.

Example 1 *We consider an example of relation R described by the table of Figure 2. This relation defines the links between eight objects $Obj = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and nine properties $Prop = \{a, b, c, d, e, f, g, h, i\}$. There is a "$\times$" in the cell corresponding to an object x and to a property y if the object x has property y, in other words the "$\times$"s describe the relation R (or context). An empty cell corresponds to the fact that $(x, y) \notin R$, i.e., it is known that object x has not property y. It can be checked that the pairs $(\{1, 2, 3, 4\}, \{g, h, i\})$, $(\{5, 6, 7, 8\}, \{a, b, c, d, e, f\})$ are pairs (X, Y) such that $X = R^{-1N}(Y)$ and $Y = R^{N}(X)$. These two pairs are not formal concepts. They are disjoint w. r. t. both Obj and $Prop$. Examples of formal concepts are $(\{2, 3, 4\}, \{g, h\})$, $(\{6, 7, 8\}, \{a, c, d\})$, or $(\{5, 6, 7, 8\}, \{a, d\})$. Note that these latter examples are here obtained by considering appropriate subsets in the previous pairs.*

A pair (X, Y) that satisfies $X = R^{-1N}(Y)$ and $Y = R^{N}(X)$

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objects

properties	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c						×	×	×
d				×	×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h		×	×	×				
i				×				

Figure 3: R': relation R modified

is such that all the objects in X possess *at least one* property in Y and the properties in Y are only (possibly) possessed by the objects in X. While the intent of a formal concept is a *conjunction* of properties, the pairs (X, Y) forming an "N-block" correspond to sets of objects defined through *disjunctions* of properties. Finding such pairs, which may not exist, aims at decomposing the relation R into independent blocks without object or property in common, as in the Figure 2 example. When such a decomposition no longer holds, as in Figure 3, pairs (X, Y) such that $X = R^{-1N}(Y)$ and $Y = R^N(X)$ no longer exist, except for the trivial pair (Obj, $Prop$), as shown in the next example.

Example 2 *Let us now consider a modified version of relation R, say R', depicted in the table of Figure 3, where object 4 has also the additional property d now. Then it can be checked that we still have $R'^N(\{1, 2, 3, 4\}) = \{g, h, i\}$, but $R'^{-1N}(\{g, h, i\}) = \{1, 2, 3\}$, since $R'(4) = \{d, g, h, i\} \not\subseteq \{g, h, i\}$. Similary, $R'^{-1N}(\{a, b, c, d, e, f\}) = \{5, 6, 7, 8\}$, but $R'^N(\{5, 6, 7, 8\}) = \{a, b, c, e, f\}$ since $R'^{-1}(d) = \{4, 5, 6, 7, 8\} \not\subseteq \{5, 6, 7, 8\}$. Thus the pairs $(\{1, 2, 3, 4\}, \{g, h, i\})$, $(\{5, 6, 7, 8\}, \{a, b, c, d, e, f\})$ are no longer pairs (X, Y) such that $X = R'^{-1N}(Y)$ and $Y = R'^N(X)$ in the new context R'.*

3 Handling fuzzy properties in the new setting

In the previous section, properties were supposed to be Boolean. Hence, when an object satisfies a property, it fully satisfies it: there is no intermediary degree of satisfaction since the property is not gradual. Thus, the relation linking objects and properties was all-or-nothing. When properties become a matter of intensity, i.e., when an object may have a property to some degree, the relation R between objects and properties becomes fuzzy. However, when relaxing the Booleanity assumption, we still assume that we have complete information. Namely, it is known to what extent α object x has property y for any pair (x, y), which is denoted $\mu_R(x, y) = \alpha$. Then, $\mu_{R^{-1}(y)}(x) = \alpha$ denotes the fact that object x satisfies property y at degree α where $\mu_{R^{-1}(y)}$ is the membership function of the fuzzy set of objects that constitutes the extension of $R^{-1}(y)$. Such an extension to standard concept analysis (based on the operator called here Δ) has been studied by Belohlavek [3]; see also [12].

Then, the four operators introduced in the previous section easily extend to the case where relation R is fuzzy. Namely,

$$\begin{aligned}
\mu_{R^\Pi(X)}(y) &= \Pi_y(X) &&= \max_{x\in X} \quad \mu_{R^{-1}(y)}(x)\\
\mu_{R^N(X)}(y) &= N_y(X) &&= \min_{x\notin X} \quad 1-\mu_{R^{-1}(y)}(x)\\
\mu_{R^\Delta(X)}(y) &= \Delta_y(X) &&= \min_{x\in X} \quad \mu_{R^{-1}(y)}(x)\\
\mu_{R^\nabla(X)}(y) &= \nabla_y(X) &&= \max_{x\notin X} \quad 1-\mu_{R^{-1}(y)}(x)
\end{aligned}$$

where Π_y, N_y, Δ_y and ∇_y are respectively potential possibility, actual necessity, actual possibility, and potential necessity measures, based on the gradual possibility distribution $\pi = \mu_{R^{-1}(y)}$. They thus enjoy the corresponding characteristic decomposability properties of these respective measures.

The following results are straightforward:

- $\mu_{R^N(X)}(y) = 1 - \mu_{R^\Pi(\overline{X})}(y)$;
- $\mu_{R^\nabla(X)}(y) = 1 - \mu_{R^\Delta(\overline{X})}(y)$
- if $\mu_{R^\Pi(X)}(y) = \alpha$ then $\exists x \in X, \mu_R(x, y) = \alpha$ and $\forall x \in X, \mu_R(x, y) \le \alpha$
- if $\mu_{R^N(X)}(y) = \alpha$ then $\mu_R(x, y) > 1 - \alpha \Rightarrow x \in X$
- if $\mu_{R^\Delta(X)}(y) = \alpha$ then $x \in X \Rightarrow \mu_R(x, y) \ge \alpha$
- if $\mu_{R^\nabla(X)}(y) = \alpha$ then $\exists x \notin X, \mu_R(x, y) = 1 - \alpha$ and $\forall x \notin X, \mu_R(x, y) \ge 1 - \alpha$

The first two results extend duality relations to the graded case. The other ones express the meaning of each fuzzy set. Thus a property belongs to $R^\Pi(X)$ to degree α inasmuch as objects in X possess this property to at most degree α. Then $h(R^\Pi(X)) = \max_{y\in Y} \mu_{R^\Pi(X)}(y) = 0$ means that no object in X possesses a property in Y to any extent. A property belongs to $R^N(X)$ to degree α if any object possessing this property to a degree greater than $1 - \alpha$ necessarily belongs to X. In particular, any object possessing this property to some positive degree belongs to X, if $\alpha = 1$. A property belongs all the more to $R^\Delta(X)$ as any object in X possesses this property to a greater degree. Lastly, $\hbar(R^\nabla(X)) = \min_{y\in Y} \mu_{R^\nabla(X)}(y) = \alpha$ means that for any property y in Y objects outside X possess this property to at most to degree $1 - \alpha$. In particular, if $\hbar(R^\nabla(X)) = 1$, for any property in Y there exists an object outside X that misses it.

Moreover, we have the following counterpart to (1):

Proposition 1 *If $R^{-1}(y)$ is such that for $y \in Prop$, $h(\mu_{R^{-1}(y)}) = 1$ and $\hbar(\mu_{R^{-1}(y)}) = 0$, then*

$$\forall X \subseteq Obj, \quad \max(N_y(X), \Delta_y(X)) \le \min(\Pi_y(X), \nabla_y(X)).$$

Lastly, it is clear that when nesting the operators as in the Remark at the end of Section 2.2, or when extending Galois connexions when R is fuzzy, we are led to extend again the above definitions to cases where X and Y become fuzzy sets themselves. From a possibility theory point of view, this means defining generalized measures for fuzzy events. Depending on the properties we want to preserve, several choices are possible here (see on this point [13], especially pages 51–64), which would lead to different extensions for the four operators. Choosing the appropriate extensions depends on the intended use and interpretation of the definition we want to generalize. We leave these questions for further research.

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4 Generalized quotient and fuzzy quantifiers

In a relational database, given an ordered set of attributes $\mathbf{A} = \{\mathcal{A}^1, ..., \mathcal{A}^n\}$, information is stored in a relational table $\mathcal{R}$ where each column corresponds to an attribute, and a row to an object belonging to *Obj*. Thus a cell in such a relational table corresponds to the value of an attribute for an object. Any row in the relational table $\mathcal{R}$ is also called a 'tuple'. A 'tuple' is thus an ordered set of attribute values pertaining to an object. Let $\mathcal{T}u(\mathcal{R})$ denote the set of tuples in $\mathcal{R}$. Quotients are relational algebra operations that aim at finding out the subrelational table $\mathcal{R} \div \mathcal{S}$ of a finite relational table $\mathcal{R}$, containing subtuples of $\mathcal{R}$ that have for complements in $\mathcal{R}$ all the tuples of a relational table $\mathcal{S}$. The quotient operation is defined by

Definition 1 *Relational quotient.*

$$\mathcal{R} \div \mathcal{S} = \{t, \forall s \in \mathcal{T}u(\mathcal{S}), (t, s) \in \mathcal{T}u(\mathcal{R})\}$$

where s denotes a tuple of $\mathcal{S}$ and t a subtuple of $\mathcal{R}$ such that (t, s) is a tuple of $\mathcal{R}$.

The definition of $R^{\Delta}(X)$ can be viewed as a particular case of such a division. Indeed $R^{\Delta}(X) = \{y \in Prop | R^{-1}(y) \supseteq X\} = \{y \in Prop | \forall x \in X, (x, y) \in R\}$. Besides, a relation $R \subseteq Prop \times Obj$ can be viewed as equivalent to a 2-attribute relational table $\mathcal{R}$ with $\mathbf{A} = \{Object - name, Property - name\}$, where (x, y) (resp. (y, x)) is a tuple in $\mathcal{R}$ (resp. $\mathcal{R}^{-1}$) if and only if $(x, y) \in R$. Then, it becomes clear that $R^{\Delta}(X) = \mathcal{R}^{-1} \div \mathcal{X}$, where $\mathcal{X}$ is the one-attribute relational table containing the object names. Similarly, $R^{-1\Delta}(Y) = \mathcal{R} \div \mathcal{Y}$ ($\mathcal{Y}$ is the one-attribute relational table associated to Y).

The fuzzy extensions of the basic operationsR^{Δ}, $R^{-1\Delta}$, which underly formal concept analysis, can thus be related to fuzzy division operations in fuzzy relational databases where tuples are weighted [14, 15]. As we shall see, this also provides a way for introducing fuzzily quantified conjunctions in order to require that tuples in R are associated with "at least k", or more generally '*most*' objects in X , rather than *all* elements in X as in formal concept analysis basic operations. In the following, we discuss these extensions in the setting of formal concept analysis.

First, the definition of $R^{\Delta}(X)$ extended to the case where R becomes fuzzy, as given in the previous section, namely $\mu_{R^{\Delta}(X)}(y) = min_{x \in X} \mu_{R^{-1}(y)}(x)$, is the exact counterpart to the fuzzy division $\mu_{\mathcal{R}^{-1} \div \mathcal{X}}(y) = \min_{x \in \mathcal{T}u(\mathcal{X})} \mu_R(x, y)$, for all pair (y, x) in $\mathcal{T}u(\mathcal{R}^{-1})$, when R is a fuzzy relation and X remains a classical set. The definition of a fuzzy division in fuzzy relation databases includes the more general case where X is also a fuzzy set. This can be done as well here, choosing an appropriate type of inclusion between fuzzy sets, i.e. an implication connective $\rightarrow$ in the expression:

$$\mu_{R^{\Delta}(X)}(y) = \min_{x \in Obj} \ \mu_X(x) \rightarrow \mu_{R^{-1}(y)}(x)$$

in relation with the intended meaning of having X fuzzy. For instance, taking Gödel implication ($a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = b$ if $a > b$) amounts to seeing $\mu_X(x)$'s as a significance threshold to which $\mu_R(x, y)$ is compared, while using Dienes implication ($a \rightarrow b = \max(1 - a, b)$) would be more in agreement with the idea of viewing $\mu_X(x)$ as a level of priority of x, just requiring the inclusion of important objects in $R^{-1}(y)$ (indeed the less important x, the greater $(1 - \mu_X(x))$, and the smaller the impact of x on the global evaluation, even when x totally fails to have property y (see [14, 15] for details).

Having both a fuzzy relation R and a fuzzy set X of objects may sound unrealistic in practice. In formal concept analysis, it is the starting point for natural weakening of the quantifier 'for all' into "at least k", or even into 'most'. The idea is to require that "at least k" (or more generally 'most') objects in X are the important objects that are the most in relation R with property y.

Let I be a fuzzy constraint on integers, defined by a membership function of the form: $\mu_I(0) = 1 \text{and} \mu_I(i) \geq \mu_I(i+1)$. For instance, "at least k objects are important" is represented by $\mu_I(i) = 1$ if $0 \leq i \leq k$ and $\mu_I(i) = 0$ for $i \geq k + 1, n$ where $n = |X|$. Let us reorder the $\mu_{R^{-1}(y)}(x_k)$'s decreasingly, so that objects (x_k)'s that are more in relation R with y are the most important ones:

$$\mu_{R^{-1}(y)}(x_{\sigma(1)}) \geq \mu_{R^{-1}(y)}(x_{\sigma(2)}) \geq ... \geq \mu_{R^{-1}(y)}(x_{\sigma(n)}).$$

Then the extent to which property y is possessed by 'at least k' objects in X can be computed as

$$\mu_{R^{\Delta}(X),I}(y) = \min_i \ \max(\mu_{R^{-1}(y)}(x_{\sigma(i)}), 1 - \mu_I(i)),$$

I being defined as above. This expression, which involves an absolute fuzzy quantifier, may be easily modified in order to introduce relative quantifiers Q like 'most', having an increasing membership function in $[0, 1]$, by changing $1 - \mu_I(i + 1)$ into $\mu_Q(i/n)$ for $i = 0, n - 1$ and $\mu_Q(1) = 1$. It gives

$$\mu_{R^{\Delta}(X),Q}(y) = \min_i \ \max(\mu_{R^{-1}(y)}(x_{\sigma(i)}), \mu_Q(\frac{i-1}{n})).$$

Clearly, if Q means 'all', $\mu_Q(t) = 0$ for $t < 1$, then the above expression reduces to $\mu_{R^{\Delta}(X)}(y) = \min_{x \in X} \ \mu_{R^{-1}(y)}(x)$.

5 Incomplete and uncertain information

Until now, it has been assumed that we have *complete* information about the existing links between properties in $Prop$ and objects in Obj. Namely, $(x, y) \in R$ means that object x satisfies property y and $(x, y) \notin R$ means that object x does not satisfy property y, rather than "we do not know if $(x, y) \in R$ or not". Clearly, this assumption may be relaxed, while Boolean properties are still assumed: One may consider that there are pairs (x, y) for which it is not known at all if x has property y or not. This case has been considered in [16]. Information may be also uncertain, i.e., we are certain at level α that x has property y, or at level β that x has not property y.

In the most general case, properties are non Boolean (i.e. $\mu_R(x, y)$ is supposed to belong to $[0, 1]$), but the extent $\mu_R(x, y)$ to which an object x has a property y may be only fuzzily known under the form of a possibility distribution $\pi_{\mu_R}^{(x,y)}$ on $[0, 1]$ that restricts its possible values. Then one may not be even sure in general that some property y is possessed by an object x at least at some degree α). Since the information about $\mu_R(x, y)$ is now represented by a fuzzy set (on $[0, 1]$), and the four measures introduced in Section 3

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$\mu_{R^{\Pi}(X)}(y) = \Pi_y(X)$, $\mu_{R^N(X)}(y) = N_y(X)$, $\mu_{R^{\Delta}(X)}(y) = \Delta_y(X)$, $\mu_{R^{\nabla}(X)}(y) = \nabla_y(X)$ can themselves only be known under the form of induced possibility distributions (using the fuzzy set extension principle [2]). Let us take the example of $\mu_{R^{\Pi}(X)}(y) = \Pi_y(X) = \max_{x \in X} \mu_R(x, y)$. The induced possibility distribution is given by
$\pi_{\Pi_y(X)}(t) = \max_{i:\max_i t_i = t} \min_{x_i \in X} \pi_{\mu_R}^{(x_i,y)}(t_i)$ i.e.
$\pi_{\Pi_y(X)}(t) =$
$\max_i \min(\pi_{\mu_R}^{(x_i,y)}(t), \min_{j \neq i}(\max_{t_j \leq t} \pi_{\mu_R}^{(x_j,y)}(t_j))$.
Such a computation may be heavy in practice, but one may at least compute an upper bound of the possibility that a property y is associated to an object x with a degree equal to α, as $\pi^*_{\Pi_y(X)}(\alpha) = \max_{x \in X} \pi_{\mu_R}^{(x,y)}(\alpha)$. One may also compute the degree of membership of property y to the fuzzy set of properties that are possibly associated with at least one object in X at least to a degree ρ as $\Pi_{y,\rho}(X) = \max_{x \in X} \max_{\{t|t \geq \rho\}} \pi_{\mu_R}^{(x,y)}(t)$.

A maybe more promising approach for dealing with incomplete information in formal concept analysis is to first slightly modify the setting we start with by accommodating directly many-valued attributes instead of binary ones. We now outline this idea. Indeed when attribute domains are two-valued, they only give birth to a binary property (and its negation), while any non empty subset of a many-valued attribute domain (different from the domain itself) defines a non-trivial property. Take the example of the $color$ attribute with domain $\{black, red, yellow, blue, green, ...\}$, $red\ or\ green$, or $red\ or\ yellow\ or\ blue$ are (imprecise) properties, beside the basic colors $black$, red, etc. Let Y now denote a set of attributes y, $dom(y)$ be the domain of y, P_y denote a non-empty subset of $dom(y)$. Let $\Gamma_y(x)$ represent the available information about the value of attribute y for object x. It is assumed that $\Gamma_y(x) \subseteq dom(y)$. Information is imprecise if $\Gamma_y(x)$ is not a singleton. For the moment, we suppose that information may be incomplete but not uncertain. $\Gamma_y(x) = \varnothing$ means that y does not apply to x, and $\Gamma_y(x) = dom(y)$ means that the value of y is unknown for x. $\Gamma_y(x) \neq \varnothing$ is now assumed.

Then $R^{-1N}(P_y) = \{x|\Gamma_y(x) \subseteq P_y\}$ is the set of objects that (certainly) have property P_y w. r. t. attribute y. Three other similar sets can be defined by reversing $\subseteq$, or replacing it by non-empty intersection, or non-covering union conditions, in the spirit of the basic definitions of Section 2. Let $R^{-1N\star}$ be the relation that expresses that *certainly* objects have some (maybe imprecise) properties; it is defined on $X \times \cup_{y \in Y} \mathcal{P}(y)$, where $\mathcal{P}(y)$ denote the power set of $dom(y)$. Since $(\mathcal{P}(y), \subseteq)$ is a Boolean lattice, $(x, P_y) \in R^{-1N\star}$ entails $(x, P'_y) \in R^{-1N\star}$ as soon as $P_y \subseteq P'_y$ (if an object is red, it is also $red\ or\ green$). Clearly, $\{x|(x, red\ or\ green) \in R^{-1N\star}\} = \{x|\Gamma_{color}(x) = red\} \cup \{x|\Gamma_{color}(x) = green\} \cup \{x|\Gamma_{color}(x) = red\ or\ green\}$. Moreover, we can also find out if there are only *possibly* $red\ or\ green$ objects, e.g. those that are known to be $red\ or\ blue$. If not, it means that there is no completion of the knowledge that can alter the extension of the set of $red\ or\ green$ objects in our example. More generally, we can look for concepts associated with sufficiently imprecise properties that remain stable under any knowledge completion. This can be extended to gradual uncertain knowledge by working with the α-cuts of the $\Gamma_y(x)$'s, i.e. pieces of information that are $(1-\alpha)$-certain. These are lines for further research.

6 Concluding remarks

Starting with a possibility-theoretic reading of concept analysis, we have reintroduced four operators, that enable us to describe all the different possible relations between a set of objects and a set of properties. Apart from retrieving the Galois connexion defining formal concepts, another Galois connexion based on the "actual necessity" operator is laid bare for decomposing the relation into independent blocks. The proposed setting extends to graded properties, leading to two kinds of Galois connected pair of fuzzy sets, whose meaning must be laid bare. Besides, the formal similarity between the actual possibility operator and relational algebra division operation, suggests a relaxation of the definition of concepts, computing the extent to which a property is *highly* possessed by *most* objects in a set. Lastly, extensions of the formal concept analysis setting to incomplete or uncertain information have been outlined. It is clear that many pending issues remain, such as e.g. the use of rough set reducts in this setting.

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