Membership dependent stability analysis of TS fuzzy controlled systems using coupling attenuation

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Abstract— *The use of Linear Matrix Inequalities and Common Quadratic Lyapunov Functions is a powerful and commonplace tool for Takagi-Sugeno fuzzy controlled system analysis and synthesis. However, in practice, few practical and performing results are available when the subsystems exhibit different input matrices, because of the strong coupling between the subsystems/subcontrollers. In this paper, this coupling is demonstrated and a method is proposed which allows to synthesize, for a number of subsystems higher than 2, the local gains of a Parallel Distributed Controller. It is shown that the controller gains depend on the values of the input matrices and of the membership functions, and are thus able to relax classical stability conditions by embedding information on the fuzzy premises.*

Keywords— Fuzzy Control, Takagi-Sugeno Fuzzy Systems, Stability, Parallel Distributed Control.

1 Introduction

T-S fuzzy model is a fuzzy dynamic model which embeds a set of fuzzy rules in order to describe a global nonlinear system as a set of local linear models, these being smoothly connected by fuzzy membership functions [1]. As T-S fuzzy models are equivalent to polytopic linear models, systematic approaches to stability analysis and controller design can be developed using powerful conventional (linear and nonlinear) control theory with the help of optimization techniques such as Linear Matrix Inequalities solvers [2].

Control of T-S fuzzy system is most of the time achieved by the so-called Parallel Distributed Compensation method (PDC) for which the fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts, and, thus, each control rule is distributively designed for the corresponding rule of a T-S fuzzy model. Since the consequent parts of T-S fuzzy models are described by linear state equations, linear control theory and corresponding stability tools can be used to design the fuzzy control local gains [3].

Among the mathematical tools used for assessing the stability of a T-S fuzzy system (with or without PDC control), the use of a Common Quadratic Lyapunov Function (CQLF) for all subsystems is the most popular, because it allows to derive easily explicit conditions [2] [4] [5]. It has been noted that Common Quadratic Lyapunov functions tend to be conservative, and even worse, might not exist for many complex highly nonlinear systems. Several works have allowed to relax stability conditions [4] [6] [7], and as an alternative, such tools as piecewise quadratic Lyapunov functions or fuzzy Lyapunov functions have been proposed (e.g. [8]). The drawbacks of these methods lie in the difficulty of synthesizing control gains - and not only analyze closed-loop stability with fixed controllers -, and that the results are quite independent of the membership functions, leading sometimes to poor overall performances. However, extended results have been obtained when considering additional bounds or shapes on membership functions considering linear consequents and PDC [9] [10] or nonlinear / non PDC controllers [11] [12] [13].

In short, there still remains a need for designing further tools to synthesize PDC control for T-S fuzzy systems, which appears to be very difficult when the input matrices of the systems of the fuzzy consequents are not identical or not proportional. In this case, the closed-loop system is not a parallel distribution of the individual closed-loop subsystems, because additional coupling terms appear, as the control gain designed for one subsystem is distributed over the remaining subsystems. Whereas most papers embed the general case, the examples are nearly always proposed for proportional or identical input matrices, or for a very limited number of input matrices.

When input matrices are different, it is not only necessary that all subsystems be stable, but also that any subsystem should be at least stable (or performing) under every other local controller, and, moreover, that all these closed-loop controlled subsystems should share a CQLF. A solution to this problem consists of solving a Riccati equation which has a positive definite solution that corresponds to a Lyapunov function, using some results from linear control theory [14]. In this case, the set of coupling terms is represented by a product of matrices involving a single uncertain matrix with a norm smaller than one, leading to a global Riccati equation [2].
Whereas this method is powerful and depends on the amplitude and variations of the membership functions, finding a global bounding matrix for the coupling terms is often not easy to work out, because these terms depend on the control gains themselves; in the case of control synthesis, when the gains are a priori unknown, this task can be impossible. The cancelation of such coupling terms has been tackled only for large-scale systems [15]. To sum up, practical and membership function-dependent of PDC control synthesis has not yet be performed when subsystems exhibit different input matrices as the explicit cancelation of coupling terms has not been proposed.

In this paper, the closed-loop T-S system under PDC control is rewritten in such a way that the cross-coupled terms are clearly

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seen as a weighted sum of the pairwise products involving the difference of two input matrices and their corresponding control gains. This allows, first, to determine easily a Riccati equation for each subsystem, which does not depend on the other subsystems control gains, and that takes the variation of fuzzy membership functions into account. Moreover, adding a little more conservativeness, allowing the control gains to be bounded, turns the stability conditions into a set of very simple Lyapunov equations, which might be a good help for control synthesis. Finally, it is shown that, when the number of subsystems is sufficient, the cross-coupled terms can be canceled by proposing nonlinear control gains.

2 Takagi-Sugeno fuzzy model and controller

2.1 Closed-loop model decomposition

The fuzzy model proposed by Takagi and Sugeno consists of a set of r fuzzy IF...THEN rules for which the consequents are linear models:

Plant Rule R_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $\dot{x} = A_i x + B_i u$;
where $x(t)$, $u(t)$ are state and input vectors, $z_i(t)$, M_{ij} are the premise variables and corresponding fuzzy rules, the final output of the fuzzy system being inferred as follows:

$$\dot{x} = \sum_{i=1}^{r} \mu_i \big(A_i x + B_i u\big), \tag{1}$$

where $\mu_i = \frac{\omega_i}{\sum_{i=1}^{r} \omega_i}$, ω_i is the grade of membership function of rule R_i.

For every subsystem S_i, a local controller can be defined as $u = K_i x$, where K_i is a control gain (possibly nonlinear). It is intuitive to build a fuzzy controller using a set of rules which share the same premises as the fuzzy model, hence distributing the local controllers within the global controllers according to their systems' weights.

Controller C_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $u = K_i x$, yielding:

$$u = \sum_{i=1}^{r} \mu_i K_i x. \tag{2}$$

Lemma 2.1. *Let the system $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$ with PDC control $u = \sum_{i=1}^{r} \mu_i K_i x$ such that $A_i + B_i K_i = G_i$ and $\sum_{i=1}^{r} \mu_i \leq 1$, $\mu_i \geq 0$. The closed-loop system is:*

$$\dot{x} = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j G_i + \sum_{i=1}^{r} A_i \Big(1 - \sum_{j=1}^{r} \mu_j\Big) + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x. \tag{3}$$

Proof. One has

$$\dot{x} = \sum_{i=1}^{r} \mu_i \Big(A_i x + B_i \sum_{j=1}^{r} \mu_j K_j x \Big)$$
$$= \sum_{i=1}^{r} \mu_i \Big(A_i + \mu_i B_i K_i + B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j \Big) x.$$
$$\dot{x} = \sum_{i=1}^{r} \left(\mu_i^2 G_i + \mu_i A_i (1 - \mu_i) + \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j \right) x.$$

Moreover,

$$\sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j = \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j (G_i - A_i) + \sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j B_i K_i.$$

In this equation, one can rearrange the two last sums into a sum of pairwise terms:

$$\sum_{i,j=1, j\neq i}^{r} \mu_j B_j \mu_i K_i + \mu_i B_i \mu_j K_j - \mu_i \mu_j B_i K_i - \mu_j \mu_i B_j K_j$$
$$= \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i).$$

Hence,

$$\sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j = \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j (G_i - A_i) + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i).$$

One has now:

$$\dot{x} = \left(\sum_{i=1}^{r} \Big(\mu_i^2 G_i + \mu_i A_i (1 - \mu_i) \Big) + \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j (G_i - A_i) + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

As $\sum_{i=1}^{r} \mu_i^2 G_i + \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j G_i = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j G_i$, and

$$\sum_{i=1}^{r} \mu_i A_i (1 - \mu_i) - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j A_i = \sum_{i=1}^{r} \mu_i A_i \Big(1 - \sum_{j=1}^{r} \mu_j\Big).$$

We demonstrate the final result:

$$\dot{x} = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j G_i + \sum_{i=1}^{r} \mu_i A_i \Big(1 - \sum_{j=1}^{r} \mu_j\Big) + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

□

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Remark: When $\sum_{i=1}^{r} \mu_i = 1$, formula (3) is reduced to:

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

2.2 Global Stability Verification

It should be recalled that a Linear Matrix Inequality (LMI) is a set of equations which can be put under the form $F_O + F_1 x_1 + \cdots F_n x_n \prec 0$, where $F_i^T = F_i$. A matrix F is said negative definite, which is noted $F \prec 0$ if $\forall x \neq 0, x^T F x < 0$.
A Bilinear Matrix Inequality involves bilinear terms and cannot be solved in a straightformard way as LMIs are.

Theorem 2.2. *The system $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$, under PDC control $u = \sum_{i=1}^{r} \mu_i K_i x$, such that $A_i + B_i K_i = G_i$ and $A_i + B_i K_j = G_{ij}$, is stable if there exists a common positive definite matrix P such that:*

$$\begin{aligned} &\forall\, i = 1 \cdots r, PG_i + G_i^T P \prec 0, \\ &\forall\, i < j, P\left(G_{ij} + G_{ji}\right) + \left(G_{ij} + G_{ji}\right)^T P \prec 0. \end{aligned} \tag{4}$$

Remark: Theorem (2.2) allows the determination of both the Lyapunov matrix and the controller gain, using a change of variable $N_i = K_i P^{-1}$, when being replaced in the stability conditions, leads to a set of LMIs in N_i and in P, the PDC controller being provided by $K_i = N_i P$. The conservativeness of the result comes from several reasons: of course, the global system can be stable without sharing a Common Quadratic Lyapunov function. The conditions are independent of the membership functions, and those roughly mean that system *i* is considered to behave "well" with the corresponding local controller $u = K_i x$, but also with any other local controller $u = K_j x, j \neq i$. Of course, it cannot be expected that a system with a controller designed for another plant has a "good" behavior, and hence, the PDC controller is designed according to the "worst" case among the pairs {Plant *i*, Controller *j*}.

3 Coupling terms attenuation

Theorem 3.1. *[14] First, we consider the linear uncertain system for which $\dot{x} = A + \sum_{i=1}^{r} D_i \delta_i E_i, \|\delta_i\| \leq 1$, and the elements of δ are Lebesgue measurable. Then the positive-definite matrix P is a common Lyapunov matrix for this system if there exists r positive scalars η_i such that:*

$$PA + A^T P + \sum_{i=1}^{r} \eta_i P D_i D_i^T P + \eta_i^- 1 E_i^T E_i \prec 0,$$

or simply, if $\forall i, \eta_i = 1$

$$PA + A^T P + \sum_{i=1}^{r} P D_i D_i^T P + E_i^T E_i \prec 0. \tag{5}$$

Remark: This Theorem was applied first by Tanaka [2] and then by numerous authors to the whole coupling term. Whereas this method provides for a rather non-conservative solution, it is clear that finding individual uncertain matrices might be a tedious task, because the rate of variation and thus the bounds on the uncertain matrix depend on the control gains themselves. It can thus be applied to analyze an existing solution (the gains are fixed) but not for gain synthesis considering models/controllers coupling. The following Theorem proposes a different application of this method to every individual component of the coupling terms.

Theorem 3.2. *Consider the system $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$, under PDC controller:*

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

Let us suppose that: $\forall\, i$, there exists b_i such that:

$$\sum_{j / B_i \neq B_j} \mu_j (B_j - B_i) = b_i \delta_i, \text{ where } \|\delta_i\| \leq 1.$$

The matrices δ_i thus depend on membership functions and other input matrices μ_j and B_j; as μ_j may vary with time, δ_i is a matrix which may vary with time or with the state space x. The corresponding norm is the Euclidean norm.
The closed-loop system is quadratically stable if:

$$\forall\, i = 1 \cdots r, PG_i + G_i^T P + P b_i b_i^T P + K_i^T K_i \prec 0. \tag{6}$$

This can be turned into:

$$\forall\, i = 1 \cdots r, \begin{pmatrix} PG_i + G_i^T P & Pb_i & K_i^T \\ b_i^T P & -I & 0 \\ K_i & 0 & -I \end{pmatrix} \prec 0. \tag{7}$$

Proof.

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \left(\mu_i G_i + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x \\ &= \sum_{i=1}^{r} \mu_i \sum_{j=1, j \neq i}^{r} G_i + \mu_j (B_i - B_j) K_i. \end{aligned}$$

One has now: $G_i + \mu_j (B_i - B_j) K_i = G_i + b_i \delta_i K_i$, and one can apply the Theorem (3.1). □

Remark: Uncertain matrices δ_i do not depend anymore on the control gains but only on input matrices and membership functions which are known a priori. Their determination is thus quite easy and the membership functions are indeed embedded in the control synthesis. Note also that the corresponding *i* Riccati equations in (6) are decoupled, i.e. the *i*th equation only depends on the *i*th control gain, the influence of the other subsystems being lumped into the matrix $b_i \delta_i$. However, one can realize that equation (7) is a BMI in P and K_i, and thus less tractable for control gain synthesis, which motivates the following Corollary.

Corollary 3.3. *Let us suppose that:*

$$\forall\, i = 1 \cdots r,\ \exists Q_i \succ 0,\ K_i^T K_i - Q_i \prec 0.$$

Then, condition (7) can be expressed as:

$$\exists\, P \prec 0, \forall\, i = 1 \cdots r, PG_i + G_i^T P + Q_i' \prec 0, \tag{8}$$

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where $Q'_i = Pb_i b_i^T P + Q_i$, *with* $K_i^T K_i - Q_i \prec 0$,
which can be turned into:

$$\forall i = 1 \cdots r, \begin{cases} \begin{pmatrix} PG_i + G_i^T P + Q_i & Pb_i \\ b_i^T P & -I \end{pmatrix} \prec 0, \\ K_i^T K_i - Q_i \prec 0. \end{cases}$$

The Corollary simply reduces the search for a common Lyapunov matrix to a series of r Lyapunov equations and thus r LMIs. This is really a drastic improvement to other methods because, now, control gains can nearly be selected independently without the need to taking care of coupling terms, at the expense of a priori gain limitation.

4 Example

Let us take the 3 following systems:

$$A_1 = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$
$$A_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$
$$A_3 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, B_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

along with local gains:

$$K_1 = \begin{pmatrix} 2 & 1 \end{pmatrix}, K_2 = \begin{pmatrix} -2 & 1 \end{pmatrix}, K_3 = \begin{pmatrix} 2 & 0 \end{pmatrix}.$$

The premisses corresponding to systems 1, 2 and 3 are:
$\mu_1 = z$, $\mu_2 = 1 - z$ and $\mu_3 = z$ where $z \in [-1 \cdots 1]$.

For every subsystem i, it is quite easy to compute the matrices b_i such that $\sum_{B_i \neq B_j} \mu_i (B_j - B_i) = b_i \delta_i$, since the upper bound depends on the fuzzy variable z.
One finds:
${b_1}^T = \begin{pmatrix} 1 & 0.25 \end{pmatrix}$, ${b_2}^T = \begin{pmatrix} 0.75 & 0.25 \end{pmatrix}$,
${b_3}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

The application of Theorem (3.2) allows to find a common positive definite matrix $P = \begin{pmatrix} 1.28 & -0.37 \\ -0.37 & 0.87 \end{pmatrix}$ whereas it is impossible to find one by the classical method; it is easy to check that the gain K_2 is unable to stabilize matrix A_1 and the converse for K_1 and A_2. It is quite interesting to note that the result is quite tied to the value of the matrices b_i.
When all other variables keep the same values, but $b_2^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$, then the Theorem (3.2) is no more applicable because a positive definite CQLF cannot be found. Thus, Theorem (3.2) is able to relax stability conditions, depending strongly on the membership functions and input matrices values.

5 Conclusion

In this paper, the stability of a Takagi Sugeno fuzzy system under the Parallel Distributed Compensation controller has been studied. This control strategy allocates the same weight to a local controller that the one in the fuzzy combination of the local submodels.
The influence of the coupling between any local subsystem and any local controller (different from the corresponding local controller designed from the local subsystem considered) in the closed-loop response has been highlighted, and it has been shown to be effective when the input matrices of the subsystems are different. It has been subsequently shown that a controller synthesis based on an analysis of each local subsystem controlled by any local compensator, would lead to conservative results.

A new approach has been proposed which, for every local subsystem, takes the coupling term coming from other subsystems into account, and proposes to choose the gain in order to cope with the effect of the coupling terms. This strategy allows to minimize the number of Linear Matrix Inequalities to be solved for controller synthesis and to take into account the shape of the membership functions.

References

[1] T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modelling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, 15:116–132, 1985.

[2] K. Tanaka, T. Ikeda, and H. O. Wang. Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, H∞ control theory, and linear matrix inequalities. *IEEE Transactions on Fuzzy Systems*, 4:1–13, 1996.

[3] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *The LMI Control Toolbox*. The Mathworks, Inc., Natick, MA, 1995.

[4] K. Tanaka, T. Ikeda, and H. O. Wang. Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, 6:250–265, 1998.

[5] J. Yoneyama. Robust H∞ control analysis and synthesis for Takagi-Sugeno general uncertain fuzzy systems. *Fuzzy Sets and Systems*, 157:2205–2223, 2006.

[6] C.H. Fang, Y.S. Liu, S.W. Kau, L. Hong, and C.H. Lee. A new LMI-based approach to relaxed quadratic stabilization of T-S fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 14:286–297, 2006.

[7] E. Kim and H. Lee. New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 8:523–534, 2000.

[8] G. Feng, C. L. Chen, D. Sun, and X. P. Guan. H∞ controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities. *IEEE Transactions on Fuzzy Systems*, 13:94–103, 2005.

[9] A. Sala and C. Ariño. Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes. *IEEE Transactions on Fuzzy Systems*, 16:1328–1336, 2008.

[10] A. Sala and C. Ariño. Relaxed stability and performance conditions for Takagi-Sugeno fuzzy systems with knowledge on membership function overlap. *IEEE Transactions On Systems, Man, And Cybernetics-Part B*, 37:727–732, 2007.

[11] M. Abbaszadeh and H. J. Marquez. LMI optimization approach to robust H∞ observer design and static output feedback stabilization for discrete-time nonlinear uncertain systems. *IEEE Int. J. Robust Nonlinear Control*, 19:313–340, 2009.

[12] H.K. Lam and F.H.F. Leung. Stability analysis of fuzzy control systems subject to uncertain grades of membership. *IEEE Transactions on Systems, Man and Cybernetics - Part B*, 35:1322–1325, 2005.

ISBN: 978-989-95079-6-8

[13] H.K. Lam and F.H.F. Leung. LMI-based stability and performance conditions for continuous-time nonlinear systems in Takagi-Sugeno's form. *IEEE Transactions on Systems, Man and Cybernetics - Part B*, 37:1396–1406, 2007.

[14] P.P. Khargonekar, I.R. Petersen, and K. Zhou. Robust stabilization of uncertain linear systems: Relations between quadratic stabilizability and H1 control theory. *IEEE Transactions on Automatic Control*, 37:356–61, 1990.

[15] W.J. Wang and W.W. Lin. PDC Synthesis for T-S fuzzy large-scale systems. *IEEE Transactions on Fuzzy Systems*, 12:309–315, 2004.

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