Continuous OWA operator and its calculation

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Abstract— *In this paper, we study the OWA operator on the real line, which corresponds to the Continuous OWA operator (COWA). After defining it, we introduce some properties and fundamental formulas for its computation. Among them, we give as an example, a differential equation for the COWA operator.* **Keywords:** Fuzzy measures, Order weighted averaging operator, Choquet integral, Continuous OWA operator

1 Introduction

Aggregation operators [1, 3, 8] are used to combine information to obtain a datum of better quality. In recent years there is an increasing interest in these topics for their application in decision problems and artificial intelligence.

Among aggregation operators, one of the most well known and useful one is the Ordered Weighted Averaging operator (OWA) introduced by Yager [12, 13, 14]. The OWA is regarded as a Choquet integral with respect to a fuzzy measure[6, 9].

In classical statistics, when the amount of data is large, it is usual to approximate a discrete distribution, such as a binomial distribution, by a continuous distribution. A continuous distribution, such as the normal distribution, is a Lebesgue integral on the real line and is based on lots of results of classical integral theory. At present, there are a very few theoretical results about the Choquet integral on the real line. This paper is the first step for a theory of Choquet integral on the real line.

The structure of this paper is as follows. In Section 2 we review fuzzy measures and the OWA operator, and introduce a few results related to the OWA operator.

In Section 3, we define the OWA operator on the real line, called the Continuous OWA operator (COWA) operator and introduce some fundamental formulas and expressions for their computation.

In Section 4, we show an example of differential equation, which will be a hint to find a weighting function for the COWA operator. The paper finishes with some conclusions.

2 Preliminaries

In this section, we define fuzzy measures, the Choquet integral and the OWA operator, and show their basic properties.

To introduce both a discrete space and a non-discrete space in a unified way, we use the terms in general topology in this section. Let X be a locally compact Hausdorff space and $\mathcal{B}$ be a class of Borel sets, that is, the smallest $\sigma-$algebra which includes the class of all closed sets. We say that $(X, \mathcal{B})$ is a measurable space.

Example 1 *We consider two examples of Hausdorff spaces:*

(1) *The set of all real numbers R is a locally compact Hausdorff space. If $X = R$, $\mathcal{B}$ is the smallest $\sigma-$ algebra which includes the class of all closed intervals.*

(2) *Let $X := \{1, 2, \ldots, N\}$. X is a compact Hausdorff space with a discrete topology. Then we have $\mathcal{B} = 2^X$.*

Definition 1 [7] Let $(X, \mathcal{B})$ be a measurable space. A fuzzy measure (or a non-additive measure) μ is a real valued set function, $\mu : \mathcal{B} \longrightarrow [0, 1]$ with the following properties;

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{B}$.

We say that the triplet $(X, \mathcal{B}, \mu)$ is a fuzzy measure space if μ is a fuzzy measure.

A fuzzy measure is said to be continuous if $A_n \uparrow A$ implies $\mu(A_n) \uparrow \mu(A)$ and $A_n \downarrow A$ implies $\mu(A_n) \downarrow \mu(A)$.

Definition 2 Let $(X, \mathcal{B})$ be a measurable space. A function $f : X \to R$ is said to be measurable if $\{x|f(x) \geq \alpha\} \in \mathcal{B}$ for all $\alpha \in R$.

Example 2 *Let f be a continuous function. Then, for all $\alpha \in R$, $\{f \geq \alpha\}$ is a closed set. Therefore, f is measurable.*

$\mathcal{F}(X)$ denotes the class of non-negative measurable functions, that is,

$$\mathcal{F}(X) = \{f|f : X \to R^+, f : \text{measurable}\}$$

Definition 3 [2, 5] Let $(X, \mathcal{B}, \mu)$ be a fuzzy measure space. The Choquet integral of $f \in \mathcal{F}(X)$ with respect to μ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu_f(r)dr,$$

where $\mu_f(r) = \mu(\{x|f(x) \geq r\})$.

Let $A \subset X$. The Choquet integral restricted on A is defined by

$$(C)\int_A f d\mu := (C)\int f \cdot 1_A d\mu.$$

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Definition 4 *Let $D \subset R^N$. An aggregation operator Ag is a function $Ag : D \to R$ with the following properties;*

(1) *(Unanimity or idempotency)*

$Ag(a,\ldots,a) = a$ *if* $(a,\ldots,a) \in D$

(2) *(Monotonicity)*

If $a_i \le b_i$ for all $i = 1,\ldots,n$, $\mathbf{a} = (a_1,\ldots,a_N), \mathbf{b} = (b_1,\ldots,b_N)$ $\mathbf{a},\mathbf{b} \in D$, then $Ag(\mathbf{a}) \le Ag(\mathbf{b})$.

Yager introduced the Ordered Weighted Averaging operator in [12].

Definition 5 *[12] Given a weighting vector $\mathbf{w}$ with weights $(w_1,\ldots,w_N)$, the Ordered Weighted Averaging operator is defined as follows:*

$$OWA_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^{N} w_i a_{\sigma(i)}$$

where σ defines a permutation of $\{1,\ldots,N\}$ such that $a_{\sigma(i)} \ge a_{\sigma(i+1)}$, $\mathbf{a} = (a_1,\ldots,a_n)$.

A fuzzy measure μ on $\mathcal{B}$ is said to be symmetric [4] if $\mu(A) = \mu(B)$ for $|A| = |B|$, $A,B \in \mathcal{B}$. Symmetric fuzzy measures on $\{1,\ldots,N\}$ can be represented in terms of N weights w_i for $i = 1,\ldots,N$ so that $\mu(A) = \sum_{i=1}^{|A|} w_i$. Using a symmetric fuzzy measure, we can represent any OWA operator as a Choquet integral.

Proposition 6 *Let $X := \{1,2,\ldots,N\}$; then, for every $OWA_{\mathbf{w}}$, there exists a symmetric fuzzy measure satisfying $\mu(\{1\}) := w_1$ and $\mu(\{1,\ldots,i\}) := w_1 + \cdots + w_i$ for $i = 1,2,\ldots,N$, such that*

$$OWA_{\mathbf{w}}(\mathbf{a}) = (C)\int \mathbf{a}\,d\mu$$

for $\mathbf{a} \in R_+^N$.

3 Continuous OWA operator

In the following we consider aggregation operators on the real line. Let λ be a Lebesgue measure on $[0,1]$, that is, $\lambda([a,b]) = b - a$ for $[a,b] \subset [0,1]$.

Definition 7 *Let $\mathcal{F}_b([0,1])$ be a class of bounded measurable function on $[0,1]$. A continuous aggregation operator Ag on the real line is a functional $Ag : \mathcal{F}_b([0,1]) \to R$ with the following properties;*

(1) *(Unanimity or idempotency)*

$Ag(a) = a$ *if* $a(x) = a$ *for all* $x \in [0,1]$

(2) *(Monotonicity)*

If $a(x) \le b(x)$ for all $x \in [0,1]$, then $Ag(a) \le Ag(b)$.

(3) *(Continuity) Let a_n, $a \in \mathcal{F}_b([0,1])$ for $n = 1,2,3,\ldots$ and $\lim_{n\to\infty} a_n = a$. Then, $\lim_{n\to\infty} Ag(a_n) = Ag(a)$.*

Let $D \subset R^N$. For every $a := (a_1,\ldots,a_n) \in D$, we can define a function $f \in \mathcal{F}_b([0,1])$ by $f(x) := a_k$ if $(k-1)/n \le x < k/n$. Therefore the definition above is one of the generalization of aggregation operators on D.

Since the Choquet integral with respect to a continuous fuzzy measure satisfies all the conditions above, the Choquet integral with respect to a continuous fuzzy measure is a continuous aggregation operator.

Using the Choquet integral we can define the continuous OWA (COWA) operator.

Definition 8 *Let μ be a fuzzy measure on $([0,1],\mathcal{B})$. μ is said to be symmetric, if $\lambda(A) = \lambda(B)$ implies $\mu(A) = \mu(B)$.*

Definition 9 *Let $a \in \mathcal{F}_\lfloor([0,1])$. The continuous OWA operator is defined by*

$$COWA_\mu(\mathbf{a}) = (C)\int \mathbf{a}\,d\mu$$

where μ is a symmetric fuzzy measure.

Let μ be a symmetric fuzzy measure on $([0,1],\mathcal{B})$. Suppose that $\lambda(A) < \lambda(B)$. Then there exists $B' \in \mathcal{B}$ such that $\lambda(B) = \lambda(B')$ and $A \subset B'$. Then we have

$$\mu(A) < \mu(B') = \mu(B).$$

Therefore we have the next proposition.

Proposition 10 *Let μ be a symmetric fuzzy measure on $([0,1],\mathcal{B})$. Then there exists a monotone function $\varphi : [0,1] \to [0,1]$ such that $\mu = \varphi \circ \lambda$.*

It follows from the proposition above that we can consider the Choquet integral with respect to $\varphi \circ \lambda$ as the COWA operator. We will write $COWA_\varphi$ instead of $COWA_{\varphi\circ\lambda}$ and we will say that φ is the weight for the COWA operator.

Let $f : [0,1] \to R$ be monotone increasing with $f(0) = 0$ and differentiable. We define the sequence of functions $\{f_k\}$ by $f_1 = f$, $f_{k+1} = \int_0^x f_k d\lambda$ for $x \in [0,1]$, $k = 1,2,\ldots$. Then we have

$$\begin{aligned}(C)\int_{[0,1]} f d\lambda^n &= \int_0^\infty \lambda^n(f \cdot 1_{[0,x]\ge\alpha})d\alpha \\ &= \int_0^{f(x)} (x - f^{-1}(\alpha))^n d\alpha\end{aligned}$$

Let $t := x - f^{-1}(\alpha)$, Since we have $d\alpha = -f'(x-t)dt$, $t = x$ if $\alpha = 0$ and $t = 0$ if $\alpha = f(x)$, then

$$\begin{aligned}(C)\int_{[0,1]} f d\lambda^n &= \int_x^0 t^n \cdot (-f'(x-t))dt \\ &= \int_0^x t^n f'(x-t)dt\end{aligned}$$

Next let $s := x - t$, we have

$$\begin{aligned}(C)\int_{[0,1]} f d\lambda^n &= \int_0^x (x-s)^n f'(s)ds \\ &= [(x-s)^n f(s)]_0^x + n\int_0^x (x-s)^{n-1} f_1(s)ds\end{aligned}$$

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Since $f(x) = 0$, we have

$$(C)\int_{[0,1]} fd\lambda^n = n\int_0^x (x-s)^{n-1}f_1(s)ds.$$

It follows from integration by part again that

$$\int_0^x (x-s)^{n-1}f_1(s)ds = (n-1)\int_0^x (x-s)^{n-2}f_2(s)ds.$$

Repeating above calculation, we have the next lemma.

Lemma 11 *Let $f : [0,1] \to R$ be monotone increasing with $f(0) = 0$ and differentiable. We define the sequence of functions $\{f_k\}$ by $f_1 = f$, $f_{k+1} = \int_0^x f_k d\lambda$ for $x \in [0,1]$, $k = 1, 2, \ldots$. Then we have*

$$(C)\int_{[0,x]} fd\lambda^n = n!f_n(x)$$

for $x \in [0,1]$.

Example 3 *Let $f(t) = t$, we have $f_1 = \frac{1}{2}x^2, \cdots, f_n = \frac{1}{(n+1)!}x^{n+1}$.*

$$(C)\int_{[0,x]} td\lambda^n(t) = \frac{1}{n+1}x^{n+1}$$

for $x \in [0,1]$.

Let the weight w be ∞-order differentiable. Then we can express w by

$$w(x) := \sum_{k=1}^{\infty} a_k x^k.$$

Since the Choquet integral is linear with respect to the fuzzy measures. We have

$$(C)\int_{[0,x]} fdw\circ\lambda = \sum_{k=1}^{\infty} k!a_k f_k(x)$$

for $x \in [0,1]$. Therefore we have the next theorem.

Theorem 12 *Let $f : [0,1] \to R$ be monotone increasing with $f(0) = 0$ and differentiable. We define the sequence of functions $\{f_k\}$ by $f_1 = f$, $f_{k+1} = \int_0^x f_k d\lambda$ for $x \in [0,1]$, $k = 1, 2, \ldots$.*

$$COWA_{\mathbf{w}}(f) = \sum_{k=1}^{\infty} k!a_k f_k(1)$$

for $x \in [0,1]$.

4 Differential equation for COWA operator

In this section we consider the definition of a weight for the COWA. As we will see below, a differential equation helps in this definition.

Let us define a weight $w(x) := \dfrac{e^x - 1}{e-1}$. Then we have

$$w(x) = \frac{1}{e-1}\sum_{k=1}^{\infty}\frac{1}{k!}x^k.$$

Since $a_k = \dfrac{1}{(e-1)k!}$, we have

$$(C)\int_{[0,x]} fdw\circ\lambda = \frac{1}{e-1}\sum_{k=1}^{\infty} k!f_k(x)$$

for $x \in [0,1]$.

Then we have the next differential equation for a weighting function.

$$\frac{d}{dx}(C)\int_{[0,x]} fdw\circ\lambda = f(x) + \frac{1}{e-1}\sum_{k=1}^{\infty} k!f_k(x)$$

for $x \in [0,1]$. Therefore we have the next differential equation.

$$\frac{d}{dx}(C)\int_{[0,x]} fdw\circ\lambda = f(x) + (C)\int_{[0,x]} fdw\circ\lambda$$

for $x \in [0,1]$.

Proposition 13 *Let f be a monotone increasing function with $f(0) = 0$ and let f be differentiable. If the weighting function satisfies the next equation:*

$$\frac{d}{dx}(C)\int_{[0,x]} fdw\circ\lambda = f(x) + (C)\int_{[0,x]} fdw\circ\lambda$$

for $x \in [0,1]$, then we have

$$w(x) = \frac{e^x - 1}{e-1}$$

5 Conclusion

We defined the OWA operator on the real line (COWA operator), and show some fundamental formulas for its calculation. We give an example of differential equation to find a weighting function for COWA operator. This methods will be applicable to the Weighted OWA (WOWA) operator introduced by Torra [10, 11], which is one of the generalization of OWA operator. We expect to have some new results related to the WOWA operator in the near future.

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References

[1] Calvo, T., Mayor, G., Mesiar, R. (eds.) (2002) Aggregation Operators, Physica-Verlag.

[2] Choquet, G. (1955) Theory of capacities. *Ann. Inst. Fourier, Grenoble.* 5,131-295.

[3] Grabisch, M., Murofushi, T., Sugeno, M. (eds.) (2000) Fuzzy Measures and Integrals: Theory and Applications, Physica-Verlag.

[4] Miranda, P., Grabisch, M. (2002) p-symmetric fuzzy measures, Proc. of the IPMU 2002 Conference, 545-552, Annecy, France.

[5] Murofushi, T., Sugeno, M. (1989) An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems 29 201-227.

[6] Ralescu, Anca L. , Ralescu, Dan A. (1997) Extensions of fuzzy aggregation, Fuzzy Sets and Systems, 86, 321-330.

[7] Sugeno, M. (1974) *Theory of fuzzy integrals and its applications,* Doctoral Thesis, Tokyo Institute of Technology.

[8] Torra, V., Narukawa, Y. (2007) Modeling decisions: information fusion and aggregation operators, Springer.

[9] Torra, V. Narukawa, Y. (2008) Choquet Stieltjes Integral, Losonczi's Means and OWA Operators. In V. Torra, Y. Narukawa (Eds.): 5th International Conference, MDAI 2008, Sabadell, Spain, October 30-31, 2008. Proceedings. Lecture Notes in Computer Science 5285 Springer, 62-73.

[10] Torra, V. (1996) Weighted OWA operators for synthesis of information, Fifth IEEE International Conference on Fuzzy Systems (IEEE-FUZZ'96) (ISBN 0-7803-3645-3), 966-971, New Orleans, USA.

[11] Torra, V. (1997) The weighted OWA operator, Int. J. of Intel. Syst. 12 153-166.

[12] Yager, R. R. (1988) On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. on Systems, Man and Cybernetics, 18 183-190.

[13] Yager, R. R., Filev, D. P. (1994) Parameterized and-like and or-like OWA operators, Int. J. of General Systems 22 297-316.

[14] Yager, R.R. (1993) Families of OWA operators, Fuzzy Sets and Systems, 59 125-148.

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