A Generalized Numerical Solution for Fuzzy Relation Equations

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Abstract— *In this paper, line search based on Sequential Quadratic Programming is implemented in order to find a solution to Fuzzy Relation Equations. Sequential Quadratic Programming is a gradient-based method that uses a quadratic estimation of the objective function in each iteration's neighborhood. Unlike analytical approaches, the method can handle equations with any combinations of t-norms and t-conorms and at any dimensions. It is assumed that the FRE problem has at least one solution.*

Keywords— Fuzzy Relation Equation (FRE), Fuzzy Triangular Norms, Line Search, Numerical Solution, Sequential Quadratic Programming (SQP).

1 Introduction

Fuzzy Relation Equations were first introduced by Sanchez [1] in 1976. Most of the works done on fuzzy relation equations are focused on specific non-parametric t-norms (min and product) and t-conorms (max and probabilistic sum). Markovskii [2], Shieh [3-5], Molai and Khorram [6], Hana et al. [7], and Perfilieva and Nosková [8] have presented some algorithms for FREs with such compositions. In special cases, an exact solution can be obtained through direct analytical methods. However, for parametric operators like *Schweizer-Sklar*, *Frank*, *Yager*, *Sugeno-Weber*, *Dombi*, and *Dubois-Prade* there is no direct solution available. This is mostly because of the complex non-linear nature of these operators.

An FRE can be formed as:

$$P \circ Q = R \tag{1}$$

where $\circ$ denotes the S-T composition. The problem involves finding *P* from *Q* and *R*.

Stamou and Tzafestas [9] developed a criterion for the existence of minimal solutions, and interpreted a *Fuzzy Inference System* (FIS) as an FRE, indicating that FREs are appropriate for FIS implementation. Shieh [3, 4] recently extended the discussion to a general condition using continuous t-norms. In such cases, when the solution set for (1) is not empty, it can be completely determined by a unique maximum solution and a finite number of minimal solutions.

However, the current models mostly discuss analytical approaches to solve the FRE problem in case-specific conditions—such as max-min or max-product compositions and sizes of $1\times m$ and $1\times n$ for binary fuzzy relations *P* and *R*.

In this paper, we propose a new numerical method based on *Sequential Quadratic Programming* (SQP) line search. The proposed model's advantages are solving FREs for any set of parametric operators and capability of handling sizes of $p\times m$ and $p\times n$ for *P* and *R*.

It should be noted that the fundamental assumption of this effort is that there exists at least one solution to the FRE problem.

The rest of this paper is organized as follows. In section 2, basic concepts of Fuzzy Relation Equations and Sequential Quadratic Programming are presented. Section 3 describes the method in detail. In section 4, some examples are provided for demonstration of the model's performance. Finally, section 5 gives the conclusions and suggests ideas for future research.

2 Basic Concepts

2.1 *Parametric Fuzzy Operators*

The most familiar fuzzy operators are max and algebraic sum (as t-conorms or s-norms) and min and product (as t-norms). But they are not limited to these special cases, and can be parametrically constructed.

The advantages of using parametric operators are:

- Most of the parametric operators have the capability to reduce to the aforementioned non-parametric operators, with specific values for the parameters. For example, the Frank t-norm reduces to min when $p = 0$.
- The parameters can be tuned according to the specific requirements of the system in question.

On the other hand, the disadvantage is that the increasing complexity leads to longer running times and more sensitivity to the tuning mechanism.

2.2 *Fuzzy Relation Equations*

An FRE is as presented in (1), in which $\circ$ acts as the S-T composition, meaning:

$$\mathop{S}_{j=1}^{m}\left[T\left(p_{ij}, q_{jk}\right)\right] = r_{ik} \tag{2}$$

where *P*, *Q*, and *R* are fuzzy binary relations and *p*, *q*, and *r* are their elements respectively [10]. The Yager series of t-norms and t-conorms are well-known fuzzy operators:

$$T_w(a,b) = 1 - \min\left\{1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right\} \tag{3}$$

$$S_w(a,b) = \min\left\{1, (a^w + b^w)^{1/w}\right\} \tag{4}$$

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Equation (3) represents the t-norm and (4) is the t-conorm of Yager operators.

2.3 Sequential Quadratic Programming

SQP methods represent the state of the art in nonlinear programming methods. Schittkowski [11], for example, has implemented and tested a version that outperforms every other tested method in terms of efficiency, accuracy, and percentage of successful solutions, over a large number of test problems.

The method closely mimics Newton's method for constrained optimization just as is done for unconstrained optimization. At each major iteration, an approximation is made of the Hessian of the Lagrangian function using a quasi-Newton updating method. This is then used to generate a Quadratic Programming sub-problem whose solution is used to form a search direction for a line search procedure.

Consider the following general non-linear minimization problem (*P*):

$$\textit{Minimize } f(x)$$
$$\textit{subject to} \begin{cases} h_i(x) = 0; & i = 1,...,l \\ h_j(x) \le 0; & j = l+1,...,m \end{cases}$$

where $x \in R^n, f : R^n \to R, h : R^n \to R^m$.

At *k*th iteration, the SQP algorithm generates a search direction (*d*) solving the definite quadratic sub-problem (*Q*) below:

$$\textit{Minimize } \nabla f_k^T d + \tfrac{1}{2} d^T B_k d$$
$$\textit{subject to} \begin{cases} h_i(x_k) + \nabla h_i^T(x_k) d = 0; & i = 1,...,l \\ h_i(x_k) + \nabla h_i^T(x_k) d \le 0; & j = l+1,...,m \end{cases}$$

where B_k is a positive definite approximation to the Hessian matrix of the Lagrangian function of the original problem (*P*):

$$L(x,\mu) = f(x) + \sum_{i=1}^{m} \mu_i h_i(x) \tag{5}$$

At each major iteration a positive definite quasi-Newton approximation of the Hessian of the Lagrangian function, *L*, is calculated using Broyden–Fletcher–Goldfarb–Shanno (BFGS) method [12].

A detailed description of the SQP method for nonlinear optimization can be found in Fletcher [13], Gill et al. [14], Powell [15], Hock and Schittkowski [16], Nocedal and Wright [17], and Bartholomew-Biggs [18].

3 The Method

SQP, as any other line search method, requires an initial solution to start from. A fairly large random search of the solution space can provide us with this initial solution. Random solutions are generated as matrices of *p*×*m*—dimensions of *P* in (1)—with elements uniformly distributed between 0 and 1.

There are various line search methods for solving nonlinear problems, the most prominent of which are gradient descent, the Newton method, quasi-Newton methods, and Sequential Quadratic Programming. These four were thoroughly tested for our problem and SQP turned out to be the most superior in terms of running time and error measure. So SQP is the line search algorithm of choice for our method.

Since our goal is to minimize the final error, the objective function would rationally be *RMSE*, that is, the root-mean-square of the errors obtained by the difference between the actual *R* and the one calculated from the composition below.

$$R' = P' \circ Q \tag{6}$$

where *P′* is the solution in search of *P*. Thus, the objective function will be as follows.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{p}\sum_{k=1}^{n}(r_{ik} - r'_{ik})^2}{p \times n}} \tag{7}$$

Using (2), we arrive at this formula:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{p}\sum_{k=1}^{n}\left\{\mathop{S}_{j=1}^{m}\left[T(p_{ij}, q_{jk})\right] - \mathop{S}_{j=1}^{m}\left[T(p'_{ij}, q_{jk})\right]\right\}^2}{p \times n}} \tag{8}$$

Constraints for *P′* elements are obviously the lower band of 0 and the upper band of 1.

This approach guarantees that regardless of the operators chosen, the difference between *R* and *R′* will be minimized. Moreover, continuity and differentiability are necessary conditions for the line search. Specifically, operators containing such terms as $\max\{c, f(a,b)\}$ render SQP unusable. In situations like this, the following substitutions will be used:

$$\max\{c, f(a,b)\} = \frac{c + f(a,b)}{2} + \frac{|c - f(a,b)|}{2} \tag{9}$$

$$\min\{c, f(a,b)\} = \frac{c + f(a,b)}{2} - \frac{|c - f(a,b)|}{2} \tag{10}$$

It's worth mentioning that unlike the "minimum" and "maximum" functions, the "absolute value" function is differentiable using symbolic math—across all its domain except where $u(x) = 0$.

$$\frac{d}{dx}|u(x)| = \frac{|u(x)|}{u(x)} u'_x(x) \tag{11}$$

4 Sample Results

The MATLAB 2008a environment was chosen to implement the proposed method, because of its efficiency in matrix operations and also the built-in functions and methods. The sample M code for Yager's operators can be found in the appendix.

For each run, *P* and *Q* are generated randomly and *R* is obtained through $P \circ Q$ with predetermined t-norm and t-conorm parameters to guarantee the existence of at least one solution for *P*. This initial value of *P* is then discarded so as not to interfere with the search process.

P, *Q*, and *R* are all 5×5 in the samples, since dimensions do not cause any significant change in the method's behavior. The initial random search produces 2000 solutions and chooses the one with the least *RMSE* for the SQP initialization. The stopping criterion for SQP is the number of *RMSE* evaluations, which was determined to be 1500.

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Operators of the same type (e.g. Dombi) were chosen in each run. Sample results are summarized in Table 1 for the following triangular norms and conorms:

- Frank
- Dombi
- Schweizer & Sklar 1
- Schweizer & Sklar 2
- Schweizer & Sklar 3
- Schweizer & Sklar 4
- Yager

More information on parametric fuzzy operators can be found in [10].

The method was repeated 10 times for each operator, with the best, worst, and average results for *RMSE* reported in the Table 1. Also included is the average running time.

Table 1 shows the performance of the proposed method with various parametric operators. The results imply that the Frank operators provide the most suitable search space for this method.

Table 1: test results for various fuzzy operators

Fuzzy Operator	Worst RMSE	Best RMSE	Average RMSE	Average Running Time
Frank	0.0008	0.00005	0.000313	222.48
Dombi	0.064	0.0009	0.026438	88.11
Schweizer & Sklar 1	0.014	0.0003	0.003795	300.98
Schweizer & Sklar 2	0.064	0.0050	0.018533	188.55
Schweizer & Sklar 3	0.046	0.0006	0.016644	166.08
Schweizer & Sklar 4	0.036	0.0003	0.008164	363.53
Yager	0.072	0.0001	0.020254	1006.42

For the sake of clarity, a numerical example will be fully featured here .Given the following values for Q and R, we want to estimate P such that

$$P \circ Q = R$$

using the Frank norm.

$$Q = \begin{bmatrix} 0.3500 & 0.3500 & 0.2900 & 0.0800 & 0.1300 \\ 0.2000 & 0.8300 & 0.7600 & 0.0500 & 0.5700 \\ 0.2500 & 0.5900 & 0.7500 & 0.5300 & 0.4700 \\ 0.6200 & 0.5500 & 0.3800 & 0.7800 & 0.0100 \\ 0.4700 & 0.9200 & 0.5700 & 0.9300 & 0.3400 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.8551 & 0.9785 & 0.9356 & 0.9769 & 0.6889 \\ 0.6664 & 0.9419 & 0.8942 & 0.6672 & 0.6712 \\ 0.6423 & 0.9227 & 0.8260 & 0.8592 & 0.5874 \\ 0.4230 & 0.8127 & 0.8054 & 0.5878 & 0.5892 \\ 0.6713 & 0.9630 & 0.9023 & 0.9533 & 0.6681 \end{bmatrix}$$

After 57 iterations (1500 evaluations), P' is the estimated value for P.

$$P' = \begin{bmatrix} 0.2921 & 0.4785 & 0.7671 & 0.9538 & 0.8433 \\ 0.6804 & 0.9593 & 0.2627 & 0.5497 & 0.2491 \\ 0.2617 & 0.5455 & 0.3835 & 0.4141 & 0.7563 \\ 0.1811 & 0.5821 & 0.7026 & 0.1342 & 0.2487 \\ 0.0139 & 0.2564 & 0.8796 & 0.3088 & 0.9254 \end{bmatrix}$$

Then, R' is calculated according to (6):

$$R' = P' \circ Q$$

$$R' = \begin{bmatrix} 0.8549 & 0.9785 & 0.9360 & 0.9772 & 0.6888 \\ 0.6664 & 0.9418 & 0.8942 & 0.6672 & 0.6713 \\ 0.6417 & 0.9236 & 0.8279 & 0.8590 & 0.5859 \\ 0.4231 & 0.8125 & 0.8052 & 0.5878 & 0.5894 \\ 0.6715 & 0.9629 & 0.9021 & 0.9536 & 0.6682 \end{bmatrix}$$

And here's the difference between the actual R and the one obtained by the estimation of P (P'), which will lead to our error measure:

$$R' - R = \begin{bmatrix} -0.0002 & 0.0000 & 0.0005 & 0.0003 & -0.0001 \\ 0.0000 & -0.0001 & 0.0000 & 0.0000 & 0.0001 \\ -0.0005 & 0.0009 & 0.0019 & -0.0002 & -0.0015 \\ 0.0001 & -0.0002 & -0.0002 & 0.0000 & 0.0002 \\ 0.0002 & -0.0001 & -0.0002 & 0.0003 & 0.0000 \end{bmatrix}$$

Finally, by applying (7), we arrive at the value of *RMSE* as a universal error measure.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{5}\sum_{k=1}^{5}(r_{ik} - r'_{ik})^2}{5 \times 5}} = 5.5246\text{E-}04$$

5 Conclusions and Future Research

In this paper, we have proposed a new numerical method based on Sequential Quadratic Programming for solving fuzzy relation equations. This work is distinguished from previous methods in that:

- it views the FRE problem in the most general sense, being capable of handling sizes of $p \times m$ and $p \times n$ for P and R; and
- unlike analytical approaches, it can handle equations with any combinations of t-norms and t-conorms and at any dimensions.

However, the proposed method is not capable of tuning the parameters of t-norms and t-conorms. Also, it can be extended to be able to handle systems of Fuzzy Relation Equations. Such improvements can be considered for future research.

Appendix

The sample M file only for Yager's fuzzy operators is available through the following URL:
http://h1.ripway.com/commonlove1985/SQP.txt

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