A parametric approach to solve quadratic programming problems with fuzzy environment in the set of constraints

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Abstract— *Quadratic programming can be seen both as a general approach to linear programming and a special class of nonlinear programming. Moreover, Quadratic Programming problems are of utmost importance in a variety of relevant practical fields, such as, portfolio selection. This work presents and develops a novel fuzzy-sets-based method that solves a class of quadratic programming problems with vagueness in the set of constraints. As vagueness is natural and ever-present in real-life situations requiring solutions, it makes perfect sense to attempt to address them using fuzzy quadratic programming. This kind of problem modeling is being applied in an increasing variety of practical fields especially those with logistics problems. Some illustrative numerical examples illustrating the solution approach are solved and analyzed to show the efficiency of this proposed method.*

Keywords— Fuzzy sets, decision making, fuzzy mathematical programming, quadratic optimization.

1 Introduction

In the early sixties, based on the fact that classical logic does not reflect, to the extent that it should, the omnipresent imprecision in the real world, L. A. Zadeh proposed the Theory of Fuzzy Sets and Fuzzy Logic. Nowadays Fuzzy Logic, or rather Soft Computing, is employed with great success in the conception, design, construction and utilization of a wide range of products and systems whose functioning is directly based on how human beings reason. This is specifically patent in the case of optimization problems, and particularly in so called Mathematical Programming problems. Mathematical programming is an area which solves problems that involve minimization (or maximization) of the objective function in a function domain that can be constrained or not. In this area, Quadratic Programming represents a special class where the objective function is a quadratic function and the constraints are linear. This set of problems can be formalized in the following form:

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathbf{t}}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^{\mathbf{t}}\mathbf{Q}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{1}$$

where $\mathbf{c}$ is an n vector, $\mathbf{b}$ is an m vector, $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{Q}$ is an $n \times n$ symmetric matrix.

Thus, on the one hand, it is clear that quadratic programming encompasses all linear problems, including applications in scheduling, planning and flow computations, and they may be used to solve some interesting combinatorial optimization problems. On the other hand, quadratic programming is a particular kind of nonlinear programming. There are several classes of problems that are naturally expressed as quadratic problems. Examples of such problems can be found in game theory, engineering modeling, design and control, problems involving economies of scale, facility allocation and location problems, problems in microeconomics amongst others. Several applications and test problems for quadratic programming can be found in [5, 7, 8, 16, 17]. Some traditional methods are available in the literature [2, 24] for solving such problems. An interesting web page about quadratic programming is [6].

Among the several applications, we will present a portfolio selection problem which is an important research field in modern finance. This problem was first introduced by Markowitz [14, 15], and provided a risk investment analysis. Vagueness, approximate values and lack of precision in this problem are very frequent in that context, and quadratic programming problems have shown to be extremely useful in solving a variety of portfolio models. In any case, it is important to point out that the aim of this work is not to solve portfolio models. They are only considered here for the sake of illustrating the fuzzy quadratic programming problems solution approach presented, which in fact is the goal and main aim of this contribution. Some works about portfolio selection problem by using fuzzy approaches can be found in [9, 11, 19, 21, 22].

Moreover, there are some cases where the parameters of the real-world problems are seldom known exactly and have to be estimated by the decision maker. Therefore, the application of Soft Computing, and Fuzzy Logic in particular, has shown, in recent years, great potential for modeling systems which are non-linear, complex, ill-defined and not well understood. Fuzzy Logic is a way to describe this vagueness mathematically and it has found numerous and different applications due to its easy implementation, flexibility, tolerant nature to imprecise data, low cost implementations and ability to model non-linear behavior of arbitrary complexity because of its basis in terms of natural language. The uncertainties can be found in the relation, constants, decision variables or in all parameters of the problem. Some authors have applied Soft Computing methodologies to quadratic programming as can be seen in [1, 3, 4, 12, 13, 18, 20, 23, 25, 26, 27].

With this in mind, the goal of this paper is to present a novel approach that transforms a quadratic programming problem with uncertainties in the coefficients and order relation of the

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set of constraints into a parametric problem and to obtain a set of optimal solutions of this new problem that belong to the fuzzy solution.

Thus, Problem (1), which is a classic quadratic programming problem, can be rewritten in the following way:

$$\begin{aligned} \min \quad & \mathbf{c^t x} + \tfrac{1}{2}\mathbf{x^t Q x} \\ \text{s.t.} \quad & \tilde{\mathbf{A}}\mathbf{x} \leq^f \tilde{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{2}$$

where $\mathbf{x}$ and $\mathbf{c}$ are $n \times 1$ vectores with real numbers and $\tilde{\mathbf{b}}$ is an $m \times 1$ vector with fuzzy numbers, $\mathbf{Q}$ is an $n \times n$ matrix with real numbers and $\tilde{\mathbf{A}}$ is an $m \times n$ matrix with fuzzy components, and the symbol "$\leq^f$" shall mean just that the decision-maker is willing to permit some violations in the accomplishment of the constraints. Both fuzzy numbers and these violations are measured by membership functions $\mu_i : \mathbb{R} \to [0,1],\ i = 1,\dots,m$ that for each solution provide the decision maker's degree of satisfaction with the accomplishment of each constraint.

The paper is organized as follows: Section 2 demonstrates that a fuzzy quadratic programming problem can be transformed into a parametric quadratic problem and that the solution obtained by traditional techniques is an acceptable fuzzy solution. To illustrate the approach, Section 3 presents a general portfolio selection problem formulated as a fuzzy quadratic programming; Section 4 presents numerical simulations for the proposed problems and an analysis of the results obtained. Finally, in Section 5 some conclusions are pointed out.

2 A general model for fuzzy quadratic programming

In this section, a novel approach is presented to solve quadratic programming problems with uncertainties in the order relation and coefficients in the set of constraints. This approach transforms this fuzzy quadratic problems into a parametric quadratic programming problem. A quadratic programming problem with this fuzzy environment has been formulated in (2) and the following sub-section will show how the uncertainties can be dealt with.

2.1 *Parametric ideas for solving a fuzzy quadratic programming problem*

The fuzzy coefficients of the quadratic problem (2) are defined with fuzzy nature, that is, some violations in the accomplishment of such restriction functions are permitted. Therefore these fuzzy parameters can be determined by the decision maker.

It is clear that each membership function will give the degree of membership (satisfaction) such that any $\mathbf{x} \in \mathbb{R}^n$ accomplishes the corresponding fuzzy objective function and constraint upon which it is defined. These membership functions can be formulated as follows

$$\mu_i : \mathbb{R} \to (0,1],\quad i \in I$$

where μ is an linear membership function (and formally can also be a non linear one), and I is the set that contains all fuzzy parameters.

In order to solve this problem in a two-phase method, first let us define for each fuzzy constraint, $i \in I$

$$X_i = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \tilde{A}_i \mathbf{x} \leq^f \tilde{b}_i, \mathbf{x} \geq 0 \right\}.$$

If $\mathbf{X} = \bigcap_{i \in I} X_i$ then the former fuzzy quadratic problem can be addressed in a compact form as

$$\min \{ f(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X} \}$$

It is clear that $\forall \alpha, \gamma \in (0,1]$, an (α,γ)-cut of the fuzzy constraint set will be the classical set

$$X(\alpha,\gamma) = \{ \mathbf{x} \in \mathbb{R}^n \mid \mu_X(\mathbf{x}) \geq \min\{\alpha,\gamma\} \}$$

where $\forall \mathbf{x} \in \mathbb{R}^n$,

$$\mu_X(\mathbf{x}) = \inf \mu_i(\mathbf{x}), i \in I$$

Hence an (α,γ)-cut of the i-th constraint will be denoted by $X_i(\alpha,\gamma)$. Therefore, if $\forall \alpha, \gamma \in (0,1]$,

$$S(\alpha,\gamma) = \{ \mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = \min\ f(\mathbf{y}),\ \mathbf{y} \in X(\alpha,\gamma) \}$$

the fuzzy solution to the problem will therefore be the fuzzy set defined by the following membership function

$$\mu_{S(\alpha,\gamma)}(\mathbf{x}) = \begin{cases} \sup\{\alpha,\gamma : \mathbf{x} \in S(\alpha,\gamma)\} & \mathbf{x} \in \bigcup_\alpha S(\alpha,\gamma) \\ 0 & \text{otherwise.} \end{cases}$$

Provided that $\forall \alpha \in (0,1]$,

$$X(\alpha,\gamma) = \bigcap_{i \in I} \{ \mathbf{x} \in \mathbb{R}^n \mid [(\tilde{A})_\alpha x]_i \leq r_i(\alpha,\gamma), \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{R}^n \}$$

with $r_i(\alpha,\gamma) = [(\tilde{b})_\alpha]_i + [(\tilde{d})_\alpha]_i (1-\gamma)$.

Thus, the operative solution to the former problem can be found, (α,γ)-cut by (α,γ)-cut, by means of the following auxiliary parametric quadratic programming model,

$$\begin{aligned} \min \quad & \mathbf{c^t x} + \tfrac{1}{2}\mathbf{x^t Q x} \\ \text{s.t.} \quad & \tilde{\mathbf{A}}_\alpha \mathbf{x} \leq \tilde{\mathbf{b}}_\alpha + \tilde{\mathbf{d}}_\alpha (1-\gamma) \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{3}$$

Therefore, the fuzzy quadratic programming problem was parameterized at the end of the first phase. In the second phase, the parametric quadratic programming problem is presented as a multiobjective approach which is solved for each of the different α and γ values using any multiobjective optimization technique. We must find efficient solutions to parametric problem for each α and γ that satisfy Karush-Kuhn-Tucker's necessary efficient optimality conditions.

2.1.1 *Methods to solve quadratic programming problems with fuzzy relations*

By using the parametric idea described above a general method to solve quadratic programming problems with fuzzy relations is presented here, as described in [18]. This method is an extension of the method that was developed to solve fuzzy linear programming problems;

$$\begin{aligned} \min \quad & \mathbf{c^t x} + \tfrac{1}{2}\mathbf{x^t Q x} \\ \text{s.t.} \quad & (Ax)_i \leq b_i + d_i(1-\alpha), i \in I \\ & x_j \geq 0, j \in J, \alpha \in (0,1]. \end{aligned} \tag{4}$$

where d_i is the violation permitted for each constraint i, $\mathbf{Q}$ and $\mathbf{A}$ are an $n \times n$ matrix and an $m \times n$ matrix with real numbers, and $\mathbf{c}$ and $\mathbf{b}$ are an n vector and m vector with real numbers.

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2.1.2 *Methods to solve quadratic programming problems with fuzzy coefficients in the set of constraints*

This multiobjective approach was developed in [10] to solve fuzzy nonlinear programming problems. This approach will be used to solve Problem (2) with fuzzy coefficients in the constraints set. The fuzzy solution is obtained by transforming a fuzzy nonlinear programming problem into a parametrical multiobjective nonlinear programming problem in which the parameters $\alpha, \gamma, \beta_{ij} \in [0,1]$, $i = 1,\ldots,m, j = 1,\ldots,n+1$ can be treated as new decision variable.

The goal of this parametrical multiobjective problem is to minimize $f(\mathbf{x})$, maximize γ, and maximize and minimize α, γ, β_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n+1$, simultaneously. Therefore, Problem (3) is transformed into a multiobjective quadratic programming problem that is stated as follows:

$$\begin{array}{ll} \min & \left[\mathbf{c^t x} + \frac{1}{2}\mathbf{x^t Q x}, \beta_{11}, 1-\beta_{11}, \ldots, \beta_{m,n+1}, 1-\beta_{m,n+1}\right] \\ \text{s.t.} & \sum_{j=1}^{n}\left((a_\alpha^L)_{ij} + \beta_{ij}\left((a_\alpha^U)_{ij} - (a_\alpha^L)_{ij}\right)\right)x_j \leq (b_\alpha^L + d_\alpha^L)_i + \\ & \quad +\beta_{i,n+1}\left((b_\alpha^U)_i - (b_\alpha^L)_i + \left((d_\alpha^U)_i - (d_\alpha^L)_i\right)(1-\gamma)\right) \\ & \mathbf{x} \geq \mathbf{0}, \alpha, \beta_{ij} \in [0,1],\ i = 1,\ldots,m,\ j = 1,\ldots,n+1 \end{array} \tag{5}$$

where it considers $m(n+3)$ new decision variables α, γ and β_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n+1$, to transform the intervals $I_{ij}(\alpha) = [h_{ij}^{-1}(1-\alpha), g_{ij}^{-1}(1-\alpha)]$ into functions of the form $z_{ij}(\alpha,\beta_{ij}) = h_{ij}^{-1}(1-\alpha) + \beta_{ij}(g_{ij}^{-1}(1-\alpha) - h_{ij}^{-1}(1-\alpha))$.

The obtained results for each α and γ value generate a set of solutions $S(\alpha,\gamma)$ and then the Representation Theorem can be used to integrate all these specific alpha-solutions and ending the second phase. Therefore the outlined solution to the parametric method is a valid solution to the fuzzy quadratic problem.

3 Portfolio selection problem

As said previously, in order to illustrate the above described two phases method for solving fuzzy quadratic programming problems, we now focus on general Portfolio Problems. It is important to point out that up to now we have not tried to improve other solution methods for this kind of important problems, but only to show how our solution approach performs. A description of a classical portfolio selection problem that was formulated by Markowitz as a quadratic programming problem is given in [15]. Assume that there are n securities denoted by S_j $(j = 1,\ldots,n)$, then this quadratic problem can be written in the following form:

$$\begin{array}{ll} \min & \mathbf{x^t \Sigma x} \\ \text{s.t.} & \mathbf{x^t E(R)} \geq \rho \\ & \mathbf{1x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{6}$$

where $\mathbf{x}$ is an n vector that represents the percentage of money invested in assets, i.e., the proportion of total investment funds devoted to each security; $\mathbf{E(R)}$ is the average vector of returns over m periods because $\mathbf{R} = [r_{ij}]$ is an $m \times n$ matrix that represents the random variables of the returns of assets varying in m discrete times; ρ is a parameter representing the minimal rate of return required by an investor; and $\Sigma = [\sigma_{ij}^2]$ is a covariance $n \times n$ matrix between returns of asset which can be written as:

$$\sigma_{ij}^2 = \sum_{k=1}^{m} \frac{(r_{ki} - E(r_i))(r_{kj} - E(r_j))}{m-1}. \tag{7}$$

Therefore, the objective of Problem (6) is minimizing the risk variance and the investment diversification subject to a given average return ρ.

The expected return rate, ρ, is a decision maker's value that represents an expert's knowledge, then a fuzzy approach can be used in the constraint of the portfolio selection problem. Problem (6) with the use of fuzzy sets can be formulated in the following form:

$$\begin{array}{ll} \min & \mathbf{x^t \Sigma x} \\ \text{s.t.} & \mathbf{x^t E(R)} \geq^f \rho \\ & \mathbf{1x} = 1 \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{8}$$

Thus, Problem (8) can be rewritten as

$$\begin{array}{ll} \min & \mathbf{x^t \Sigma x} \\ \text{s.t.} & \mathbf{x^t E(R)} \geq \rho - d(1-\alpha) \\ & \mathbf{1x} = 1 \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{9}$$

4 Numerical experiments

The problems we use to evaluate this method are one hypothetic mathematical formulation and one fuzzy portfolio selection problem with the fuzzy approach described in Section 3. Nevertheless, they are efficient in validating the realized study. First, in subsection 4.1 we will show the formulation of the problems. Then in subsection 4.2 the computational results and a comparative analysis of the classic methods and the iterative methods responses will be presented.

The tests were all performed on a PC with two 2.26GHZ Intel® Core(TM) 2 Duo processors, 4GB RAM running Ubuntu 8.10 operational system. All the problems presented in this work were resolved using **fminimax** function to solve constraint programming problems of *ToolBox Optimization* of MATLAB® 7.4.0 program.

4.1 *Formulation of the numerical examples*

We present some real-world and theoretical problems found in the literature with a view to validating the proposed algorithm. We simulate two quadratic programming problems both with uncertainties in the order relation and coefficients in the set of contraints.

Table 1 provides one theoretical quadratic problem that is described in [8]. The optimal solution to the problems without uncertainties is presented in the columns $\overline{\mathbf{x}}^t$ and $f(\mathbf{x})$ of table 1.

In order to show the performance of our method, we use the set of historical data shown in Table 2 introduced by Markowitz. The columns 2-10 represent American Tobacco, A.T.&T., United States Steel, General Motors, Atcheson&Topeka&Santa Fe, Coca-Cola, Borden, Firestone and Sharon Steel securities data, respectively. The returns on the nine securities, during the years 1937-54, are presented in Table 2.

This example will consider performances of portfolios with respect to "return" thus defined. This assumes that a dollar of realized or unrealized capital gains is exactly equivalent to a

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dollar of dividends, no better and no worse. This assumption is appropriate for certain investors, for example, some types of tax-free institutions. Other ways of handling capital gains and dividends, which are appropriate for other investors, can be viewed in [15].By computing the average value of all the years of each column of random variables of Table 2, we obtained the expected values of each return of the securities that are described in (10). Then, using Equation (7), we computed the covariance matrix that is presented in (11).

4.2 Results and Analysis

Here we show the results obtained for the problem by the fuzzy quadratic programming method introduced in Section 2. The problem described in this work was solved by using linear membership functions as presented by Problem (3). Table 3 shows the solution of the theoretical problem described by Table 1, while the solution of the real-world portfolio selection problem, described by Table 2, is presented in Table 4.

Table 3 presents the results of the hypothetical problem with fuzzy coefficients and relation in the constraints set. Each row describes parametrical solutions to different α-cut levels of the fuzzy coefficients and each column represents parametrical solutions to different violations of the fuzzy relation in the set of constraints. Now, by applying the Representation Theorem in these parametrical solutions, we can define a fuzzy solution that describes a satisfactory solution of the fuzzy quadratic programming problem shown in Table (1).

Table 4 presents the results of the fuzzy portfolio selection problem to different α-cut levels of the fuzzy coefficients and violations of the fuzzy relation in the set of constraints. Thus, a fuzzy solution can be defined by applying the Representation Theorem in these parametrical solutions. This fuzzy solution describes a satisfactory solution of the fuzzy portfolio selection problem.

5 Conclusions

Fuzzy quadratic programming problems are of utmost importance in an increasing variety of practical fields because real-world applications inevitably involve some degree of uncertainty or imprecision, for example in logistics management. One of these problems is the portfolio selection problem, where the imperfect knowledge of the returns on the assets and the uncertainty involved in the behaviour of financial markets may be introduced by means of fuzzy quantities and/or fuzzy constraints. In this context this paper has presented an operative and novel method for solving Fuzzy Quadratic Programming problems which is carried out by performing two phases that finally provide the user with a fuzzy solution. The method has been validated by solving a number of practical problems. The solutions obtained aid the authors to follow along this line of research to try to solve real problems in practice, in such a way that oriented Decision Support Systems involving Fuzzy Quadratic Programming problems can be built.

A parametric approach that solve a fuzzy quadratic problem was proposed. The set of optimal solutions obtained by one parametric approach to each $\alpha, \beta, \gamma \in [0, 1]$ constructs the fuzzy solution by using the Representation Theorem. The authors aim is to extend the line of investigation involving Fuzzy Quadratic Programming problems in order to try to solve other practical problems.

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Table 1: Fuzzy quadratic programming problem

$f(\mathbf{x})$	$\mathbf{x}_{initial}$	Constraints	Violation	Classic solution	
				$\overline{\mathbf{x}}^T$	$f(\mathbf{x})$
$9-8x_1-6x_3-4x_3+$ $+2x_1^2+2x_2^2+x_3^2+2x_1x_2+2x_1x_3$	$[0.5;0.5;0.5]^T$	$x_1+x_2-2x_3-3 \leq^f 0$ $x_i \geq^f 0, i=1,2,3$	$d_1=\widetilde{0.3}$	$[4/3;7/9;4/9]^T$	1/9

Table 2: Fuzzy portfolio selection problem

	#1	#2	#3	#4	#5	#6	#7	#8	#9
Year	Am.T	A.T&T.	U.S.S.	G.M.	A.T.&S.	C.C.	Bdm.	Frstn.	S.S.
1937	-0.305	-0.173	-0.318	-0.477	-0.457	-0.065	-0.319	-0.4	-0.435
1938	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
1939	0.055	0.2	-0.047	0.165	-0.424	-0.078	0.381	-0.093	-0.295
1940	-0.126	0.03	0.104	-0.043	-0.189	-0.077	-0.051	-0.09	-0.036
1941	-0.28	-0.183	-0.171	-0.277	0.637	-0.187	0.087	-0.194	-0.24
1942	-0.003	0.067	-0.039	0.476	0.865	0.156	0.262	1.113	0.126
1943	0.428	0.300	0.149	0.255	0.313	0.351	0.341	0.580	0.639
1944	0.192	0.103	0.260	0.290	0.637	0.233	0.227	0.473	0.282
1945	0.446	0.216	0.419	0.216	0.373	0.349	0.352	0.229	0.578
1946	-0.088	-0.046	-0.078	-0.272	-0.037	-0.209	0.153	-0.126	0.289
1947	-0.127	-0.071	0.169	0.144	0.026	0.355	-0.099	0.009	0.184
1948	-0.015	0.056	-0.035	0.107	0.153	-0.231	0.038	0	0.114
1949	0.305	0.030	0.133	0.321	0.067	0.246	0.273	0.223	-0.222
1950	-0.096	0.089	0.732	0.305	0.579	-0.248	0.091	0.650	0.327
1951	0.016	0.090	0.021	0.195	0.040	-0.064	0.054	-0.131	0.333
1952	0.128	0.083	0.131	0.390	0.434	0.079	0.109	0.175	0.062
1953	-0.010	0.035	0.006	-0.072	-0.027	0.067	0.21	-0.084	-0.048
1954	0.154	0.176	0.908	0.715	0.469	0.077	0.112	0.756	0.185

$$\mathbf{E}(\mathbf{R}) = \begin{bmatrix} 0.0659 & 0.0616 & 0.1461 & 0.1734 & 0.1981 & 0.0551 & 0.1276 & 0.1348 & 0.1156 \end{bmatrix} \quad (10)$$

$$\Sigma = \begin{pmatrix} 0.0565 & 0.0228 & 0.0303 & 0.0518 & 0.0172 & 0.0341 & 0.0257 & 0.0464 & 0.0383 \\ 0.0228 & 0.0155 & 0.0199 & 0.0259 & 0.0085 & 0.0106 & 0.0153 & 0.0265 & 0.0221 \\ 0.0303 & 0.0199 & 0.0905 & 0.0663 & 0.0470 & 0.0141 & 0.0111 & 0.0836 & 0.0445 \\ 0.0518 & 0.0259 & 0.0663 & 0.1011 & 0.0546 & 0.0307 & 0.0220 & 0.0775 & 0.0388 \\ 0.0172 & 0.0085 & 0.0470 & 0.0546 & 0.1354 & 0.0136 & 0.0221 & 0.0683 & 0.0476 \\ 0.0341 & 0.0106 & 0.0141 & 0.0307 & 0.0136 & 0.0437 & 0.0119 & 0.0254 & 0.0229 \\ 0.0257 & 0.0153 & 0.0111 & 0.0220 & 0.0221 & 0.0119 & 0.0305 & 0.0229 & 0.0184 \\ 0.0464 & 0.0265 & 0.0836 & 0.0775 & 0.0683 & 0.0254 & 0.0229 & 0.1024 & 0.0553 \\ 0.0383 & 0.0221 & 0.0445 & 0.0388 & 0.0476 & 0.0229 & 0.0184 & 0.0553 & 0.0839 \end{pmatrix} \quad (11)$$

Table 3: Results of the first phase of the hypothetical problem.

γ/α	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
1.0	0.1111	0.1045	0.0982	0.0920	0.0860	0.0803	0.0747	0.0693	0.0642	0.0592	0.0544
0.9	0.1037	0.0973	0.0911	0.0851	0.0794	0.0738	0.0684	0.0632	0.0583	0.0535	0.0489
0.8	0.0958	0.0896	0.0836	0.0779	0.0723	0.0669	0.0618	0.0568	0.0521	0.0475	0.0432
0.7	0.0874	0.0814	0.0757	0.0701	0.0648	0.0597	0.0548	0.0501	0.0456	0.0413	0.0372
0.6	0.0784	0.0727	0.0672	0.0620	0.0569	0.0521	0.0474	0.0430	0.0388	0.0348	0.0311
0.5	0.0689	0.0635	0.0583	0.0534	0.0486	0.0441	0.0398	0.0357	0.0319	0.0282	0.0248
0.4	0.0588	0.0538	0.0490	0.0444	0.0400	0.0359	0.0320	0.0283	0.0248	0.0216	0.0186
0.3	0.0483	0.0437	0.0393	0.0351	0.0312	0.0275	0.0240	0.0208	0.0178	0.0150	0.0125
0.2	0.0374	0.0333	0.0294	0.0258	0.0224	0.0192	0.0163	0.0136	0.0112	0.0090	0.0070
0.1	0.0265	0.0230	0.0197	0.0167	0.0139	0.0114	0.0092	0.0072	0.0054	0.0039	0.0026
0.0	0.0160	0.0132	0.0108	0.0085	0.0066	0.0048	0.0034	0.0022	0.0013	0.0006	0.0002

Table 4: Results of the first phase of the portfolio selection problem.

γ/α	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
1.0	0.0343	0.0332	0.0322	0.0313	0.0305	0.0297	0.0290	0.0284	0.0278	0.0273	0.0268
0.9	0.0305	0.0298	0.0291	0.0286	0.0281	0.0275	0.0270	0.0266	0.0261	0.0257	0.0253
0.8	0.0282	0.0277	0.0272	0.0268	0.0263	0.0259	0.0255	0.0251	0.0247	0.0243	0.0239
0.7	0.0265	0.0261	0.0257	0.0253	0.0249	0.0245	0.0241	0.0238	0.0234	0.0230	0.0227
0.6	0.0266	0.0247	0.0243	0.0240	0.0240	0.0236	0.0233	0.0226	0.0223	0.0220	0.0216
0.5	0.0252	0.0249	0.0245	0.0245	0.0229	0.0234	0.0223	0.0216	0.0213	0.0210	0.0207
0.4	0.0240	0.0237	0.0234	0.0230	0.0227	0.0225	0.0222	0.0218	0.0215	0.0202	0.0209
0.3	0.0232	0.0228	0.0225	0.0222	0.0218	0.0215	0.0212	0.0209	0.0206	0.0203	0.0200
0.2	0.0220	0.0217	0.0215	0.0212	0.0209	0.0206	0.0204	0.0201	0.0198	0.0196	0.0193
0.1	0.0212	0.0209	0.0207	0.0204	0.0201	0.0199	0.0196	0.0194	0.0192	0.0189	0.0187
0.0	0.0204	0.0202	0.0199	0.0197	0.0195	0.0192	0.0190	0.0188	0.0186	0.0184	0.0182

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