Non-additive robust ordinal regression with Choquet integral, bipolar and level dependent Choquet integrals

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Abstract— *Choquet integral has proved to be an effective aggregation model in multiple criteria decision analysis when interactions between criteria have to be taken into consideration. Recently, some generalizations of Choquet integral have been proposed to take into account more complex forms of interaction. This is the case of the bipolar Choquet integral and of the level dependent Choquet integral. To apply Choquet integral and its generalizations in decision problems it is necessary to determine one capacity permitting to represent the preferences of the Decision Maker (DM). In general the capacities are determined on the basis of some exemplary decisions supplied by the DM. It has been observed that effectively there is not only one capacity compatible with the DM's preferences, but rather a whole set of capacities. The determination of the whole set of compatible capacities and the consequent definition of proper preference relations is the domain of the non-additive robust ordinal regression. The authors have already proposed a methodology for non-additive robust ordinal regression when dealing with classical Choquet integral in ranking or choice decision problems. In this presentation, we want to give the basis of a general methodology for non-additive robust ordinal regression for Choquet integral and its generalizations (therefore also the bipolar Choquet integral and the level dependent Choquet integral) in the whole spectrum of decision problems (i.e. not only ranking and choice, but also multicriteria classification).*

Keywords— Choquet integral; Bi-Capacity; Bipolar Choquet integral; Level dependent Choquet integral; Non-additive robust ordinal regression.

1 Introduction

In the field of Multiple Criteria Decision Aid (MCDA), the main purpose of the Multi-Attribute Utility Theory (MAUT) [1] is to represent the preferences of the Decision Maker (DM) on a set of alternatives, taking in account different and conflicting points of view, called *criteria*, to get an overall utility function of every alternative.

The principal assumption underlying MAUT is the independence of criteria, not well suited to many real decision problems in which some interactions between criteria should be considered. In this last direction, Choquet integral [2] has proved to be an effective aggregation model in multiple criteria decision analysis when interactions between criteria have to be taken into consideration. Recently, according to some studies in psychology [3], many real decision problems are often based on *affect*. It seems natural to consider the criteria evaluations on a scale going from negative (bad) to positive (good) values, with a central neutral value. Consequently, the criteria are evaluated on a *bipolar* scale, *i.e.* some complex interactions among criteria arise depending on their good or positive values. To handle bipolar scales of criteria, the notion of capacity has been extended to that of bi-capacity by Grabisch and Labreuche in [4] and independently, to the bipolar capacity by Greco, Matarazzo and Słowiński in [5]. As a result, the Choquet integral has been generalized with the bipolar Choquet integral [6].

In this work we also consider a recent generalization of the Choquet integral: the level dependent Choquet integral [7]. The level dependent Choquet integral handles either unipolar scales or bipolar scales and takes into account the fact that importance of criteria depends also on the level of their evaluation.

In this paper, we propose a general framework for *non-additive robust ordinal regression* (see [8]) permitting to determine the capacities of the Choquet integral and of its above generalizations being compatible with the DM's preferences. On the basis of these sets of compatible capacities, we can define appropriate preference relations permitting to give a recommendation in the whole spectrum of decision problems (*i.e.* not only ranking and choice, but also multicriteria classification). Moreover, we propose to help the DM by identifying one capacity being the most representative among the many compatible capacities for the decision problem at hand.

The paper is organized as follows. In Section 2, we present the basic concepts relative to the Choquet integral and its generalizations, the bi-polar Choquet integral and the level dependent Choquet integral. In Section 3, the non-additive robust ordinal regression is extended to such generalizations. Section 4 contains some conclusions.

2 Choquet integral and its generalizations

2.1 *Preliminary notation*

In a multiple criteria decision analysis, let $X = X_1 \times X_2 \times \cdots X_n$ with $X_1, \cdots, X_i, \cdots, X_n \subseteq \mathbb{R}^n$ be the possible values taken by n criteria describing a finite set X of m alternatives. We denote every alternative $\mathbf{x} \in X$ by the evaluation vector $\mathbf{x} = (x_1, \cdots, x_i, \cdots, x_n) \in X$ and the index set of criteria by $N = \{1, \cdots, i, \cdots n\}$.

2.2 *Choquet integral*

A *fuzzy measure* (called also capacity) on N is a set function

$$\mu : 2^N \rightarrow [0, 1]$$

with $\mu(\emptyset) = 0$, $\mu(N) = 1$ (*boundary conditions*) and $\forall\, A \subseteq B \subseteq N$, $\mu(A) \leq \mu(B)$ (*monotonicity condition*).

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In the framework of multicriteria decision problems, the value $\mu(A)$ on the set of criteria A can be interpreted as the importance weight given by the DM to the set of criteria A.

A fuzzy measure is *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$, for any $A, B \subseteq N$ such that $A \cap B = \emptyset$.

In case of additive fuzzy measures, $\mu(A)$ is simply obtained by $\mu(A) = \sum_{i \in A} \mu(\{i\}), \forall\ A \subseteq N$. In the other cases, we have to define a value $\mu(A)$ for every subset A of N, obtaining 2^n coefficients values.

Given $\mathbf{x} \in X \subseteq \mathbb{R}^n_+$ and μ being a fuzzy measure on N, then the *Choquet integral* [2] is defined by:

$$\begin{aligned} \mathcal{C}_\mu(\mathbf{x}) &= \textstyle\sum_{i=1}^n \left[x_{(i)} - x_{(i-1)}\right] \mu\left(A_i\right) = \\ &= \textstyle\sum_{i=1}^n x_{(i)} \Big(\mu\left(A_i\right) - \mu\left(A_{i+1}\right)\Big) \end{aligned}$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that:

$$x_{(1)} \leq x_{(2)} \leq x_{(3)} \leq ... \leq x_{(n)}, \tag{1}$$

with $A_i = \{(i), ..., (n)\}$ where $A_{n+1} = \{\emptyset\}$ $(i = 1, .., n)$ and $x_{(0)} = 0$.

2.3 Bipolar Choquet integral

Let $\mathcal{S}(N) = \{(C, D)\ :\ C \subseteq N,\ D \subseteq N,\ C \cap D = \emptyset\}$ be the set of pairs of subsets of N.

A *bi-polar capacity*, defined in [5], is a function

$$\check{\mu}\ :\ \mathcal{S}(N) \rightarrow [0, 1] \times [0, 1]$$

such that,

1. $\check{\mu}(A, \emptyset) = (a, 0)$ and $\check{\mu}(\emptyset, B) = (0, b)$, with $A, B \in \mathcal{S}(N)$ and $a, b \in [0, 1]$;
2. $\check{\mu}(N, \emptyset) = (1, 0)$ and $\check{\mu}(\emptyset, N) = (0, 1)$;
3. For each (C, D), $(E, F) \in \mathcal{S}(N)$, such that $C \supseteq E$ and $D \subseteq F$, we have $\check{\mu}(C, D) = (c, d)$ and $\check{\mu}(E, F) = (e, f)$, $c, d, e, f \in [0, 1]$, with $c \geq e$ and $d \leq f$.

The properties 1) and 2) are the *boundary conditions*, while the property 3) is the *monotonicity condition*.

Given $(C, D) \in \mathcal{S}(N)$ with $\check{\mu}(C, D) = (c, d)$, we use the following notation, $\check{\mu}^+(C, D) = c$ and $\check{\mu}^-(C, D) = d$.

A *bi-capacity*, defined in [4], is a function, $\hat{\mu}\ :\ \mathcal{S}(N) \rightarrow [-1, 1]$ such that,

1. $\hat{\mu}(\emptyset, \emptyset) = 0$ and $\hat{\mu}(N, \emptyset) = 1$ and $\hat{\mu}(\emptyset, N) = -1$ *(boundary conditions)*;
2. If $C \supseteq E$ and $D \subseteq F$, then $\hat{\mu}(C, D) \geq \hat{\mu}(E, F)$ *(monotonicity conditions)*.

From each bi-polar capacity, a corresponding bi-capacity is obtained by

$$\hat{\mu}(C, D) = \check{\mu}^+(C, D) - \check{\mu}^-(C, D),\ \ \forall\ \ (C, D) \in \mathcal{S}(N).$$

Let be $_{(\cdot)}$ a permutation of the elements of N such that,

$$|x_{(1)}| \leq |x_{(2)}| \leq \ldots \leq |x_{(i)}| \leq \ldots \leq |x_{(n)}|,$$

and $|x_{(0)}| = 0$.

Let be the following two subsets of N, $A_i^+ = \{j \in N\ :\ x_j \geq |x_{(i)}|\}$ and $A_i^- = \{j \in N\ : x_j < 0,\ -x_j \geq |x_{(i)}|\}$.

The bi-polar Choquet integral of the positive part is defined as,

$$\mathcal{C}^+(\mathbf{x}, \check{\mu}) = \sum_{i \in N} \Big(|x_{(i)}|\ - |x_{(i-1)}|\Big) \check{\mu}^+(A_i^+, A_i^-).$$

Analogously, *the bi-polar Choquet integral* of the negative part is defined as,

$$\mathcal{C}^-(\mathbf{x}, \check{\mu}) = \sum_{i \in N} \Big(|x_{(i)}| - |x_{(i-1)}|\Big) \check{\mu}^-(A_i^+, A_i^-).$$

The bi-polar Choquet integral is defined (see [5]) as:

$$\mathcal{BC}(\mathbf{x}, \hat{\mu}) = \mathcal{C}^+(\mathbf{x}, \check{\mu}) - \mathcal{C}^-(\mathbf{x}, \check{\mu}).$$

$\mathcal{BC}(\mathbf{x}, \hat{\mu})$ can be also formulated in terms of bi-capacities (see [6]), as follows,

$$\mathcal{BC}(\mathbf{x}, \hat{\mu}) = \sum_{i \in N} \Big(|x_{(i)}| - |x_{(i-1)}|\Big) \hat{\mu}(A_i^+, A_i^-).$$

2.4 Some particular submodels

In the bipolar decision making setting, one of the main drawbacks is the huge number of parameters to be elicitated from the DM in order to define the bi-capacity or the bipolar capacity. In fact, since $\mathcal{S}(N)$ is isomorphic to the set of functions from $\mathbb{N}$ to $\{-1, 0, 1\}$, then $|\mathcal{S}(N)| = 3^n$. Let us remark that in this contest the non-additive ordinal regression is particularly useful, because it does not require the elicitation of all the parameters, but it is mainly based on some holistic preferences on a reference set of alternatives from which the whole set of bi-capacities or bipolar capacities compatible with the DM's preferences. Nevertheless, dealing with a model with smaller number of parameters is always useful and for this reason, in the following sections we recall some well-known bipolar submodels, that are interesting from the point of view of applications and a decomposition of the bi-polar capacities, introduced in [9], more meaningful from a DM's point of view.

2.4.1 Decomposable bi-polar measures

We define a $2-$order decomposable bi-polar measure (see [9]) such that

$$\begin{aligned} \mu^+(C, D) &= \sum_{i \in C} a^+(\{i\}, \emptyset) + \sum_{\{i,j\} \subseteq C} a^+(\{i, j\}, \emptyset) + \\ &\quad + \sum_{i \in C,\ j \in D} a^+(\{i\}, \{j\}) \end{aligned}$$

$$\begin{aligned} \mu^-(C, D) &= \sum_{j \in D} a^-(\emptyset, \{j\}) + \sum_{\{i,j\} \subseteq D} a^-(\emptyset, \{i, j\}) + \\ &\quad \sum_{i \in C,\ j \in D} a^-(\{i\}, \{j\}) \end{aligned}$$

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The above decomposition of the bi-polar capacity has a more manageable and meaningful interpretation according to the DM's preferences.

In fact, $a^{\pm}(\cdot)$ can be interpreted in the following way:

- $a^+(\{i\},\emptyset)$, represents the power of the criterion i by itself; this value is always positive.
- $a^+(\{i,j\},\emptyset)$, represents the interaction between i and j, when their values are both positive; when its value is zero there is no interaction; on the contrary, when the value is positive there is a synergy effect when putting together i and j; a negative value means that the two criteria are redundant.
- $a^+(\{i\},\{j\})$, represents the power of the criterion j against the criterion i, when the criterion i has a positive value and j has a negative value; this provokes always a reduction or no effect on the value of μ^+ since this value is always non-positive.

Analogous interpretation can be applied to the value of $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{i,j\})$, and $a^-(\{i\},\{j\})$.

In what follows, for the sake of simplicity, we will use a_i^+, a_{ij}^+, $a_{i|j}^+$, instead of $a^+(\{i\},\emptyset)$, $a^+(\{i,j\},\emptyset)$, and, $a^+(\{i\},\{j\})$, respectively; and a_j^-, a_{ij}^-, $a_{i|j}^-$, instead of $a^-(\emptyset,\{j\})$, $a^-(\emptyset,\{i,j\})$, and $a^-(\{i\},\{j\})$, respectively.

2.5 The level dependent Choquet integral

Let us also recall a recent further generalization of the bi-capacity (see [7]): the generalized capacity. Such measures take in account the fact that importance of criteria depends also on the level of their evaluation.

In particular, we define a *generalized capacity* a function $\mu^{\mathcal{G}}: 2^N \times \mathbb{R}_+ \to [0,1]$ such that

1. for all $t \in \mathbb{R}_+$ and $A,B \subseteq N$, $\mu^{\mathcal{G}}(A,t) \le \mu^{\mathcal{G}}(B,t)$;
2. for all $t \in \mathbb{R}_+$, $\mu^{\mathcal{G}}(\emptyset,t) = 0$ and $\mu^{\mathcal{G}}(N,t) = 1$;

We define the generalized Choquet integral of $\mathbf{x} = [x_1,\cdots,x_n] \in \mathbb{R}_+^n$, with respect to the generalized capacity $\mu^{\mathcal{G}}$ as follows:

$$\mathcal{GC}(\mathbf{x},\mu^{\mathcal{G}}) = \int_0^{+\infty} \mu^{\mathcal{G}}(A(\mathbf{x},t),t)\,dt$$

where $A(\mathbf{x},t) = \{i \in N : x_i \ge t\}$.

Let us remark that the generalized Choquet integral can always be written as:

$$\mathcal{GC}(\mathbf{x},\mu^{\mathcal{G}}) = \sum_{i=1}^{n} \int_{x_{(i-1)}}^{x_{(i)}} \mu^{\mathcal{G}}(A(\mathbf{x},t),t)\,dt.$$

The level dependent Choquet integral can be defined also with respect to a generalized bi-capacity.

In particular, we define a *generalized bi-capacity* a function $\mu^{\mathcal{G}}: \mathcal{S}(N) \times \mathbb{R} \to \mathbb{R}$ such that

1. for all $t \in \mathbb{R}$ and $(A,B),(C,D) \in \mathcal{S}(N)$, $A \subseteq C$, $B \supseteq D$, $\mu^{\mathcal{G}}(A,B,t) \le \mu^{\mathcal{G}}(C,D,t)$;
2. for all $t \in \mathbb{R}$, $\mu^{\mathcal{G}}(\emptyset,N,t) = 0$ and $\mu^{\mathcal{G}}(N,\emptyset,t) = 1$;

We define the generalized bipolar Choquet integral of $\mathbf{x} = [x_1,\cdots,x_n] \in \mathbb{R}^n$, with respect to the generalized bi-capacity $\mu^{\mathcal{G}}$ as follows:

$$\mathcal{GC}(\mathbf{x},\mu^{\mathcal{G}}) = \int_0^{+\infty} \mu^{\mathcal{G}}(A^+(\mathbf{x},t),A^-(\mathbf{x}t),t)\,dt$$

where $A^+(\mathbf{x},t) = \{i \in N : x_i \ge t\}$ and $A^-(\mathbf{x},t) = \{i \in N : -x_i \ge t\}$.

2.5.1 Interval level dependent capacity

The generalized Choquet integral is quite difficult to calculate and consequently, to be applied since a capacity $\mu^{\mathcal{G}}(A,t)$ is needed for each level $t \in \mathbb{R}$. For this reason in [7], a manageable class of generalized capacities are proposed, namely the *interval level dependent capacities*. For the sake of simplicity, but without loss of the generality, in the following we consider $\mathbf{x} \in [0,1]^n$.

A generalized capacity $\mu^{\mathcal{G}}$ is defined *interval level dependent capacity* if there exist

1. $a_0, a_1, a_2, , a_{m-1}, a_m \in [0,1]$, with m a positive integer such that

$$0 = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = 1$$

2. m capacities $\mu_1,\cdots,\mu_m$ on N.

such that, for all $A \subseteq N$ and all $t \in [0,1]$, $\mu^{\mathcal{G}}(A,t) = \mu_j(A)$ if $t \in]a_{j-1},a_j[$, $j = 1,\cdots,m$. In this case, $\mu^{\mathcal{G}}$ is an interval level dependent capacity relative to the breakpoints $a_0,a_1,a_2,,a_{m-1},a_m \in]0,1[$ and to the capacities $\mu_1,\cdots,\mu_m$.

Finally, we recall an useful theorem, proposed in [7], which permits to split $\mathcal{GC}(\mathbf{x},\mu^{\mathcal{G}})$ in the sum of a finite number of classical Choquet integrals, more precisely one Choquet integral for each interval $]a_{j-1},a_j[$, $j = 1,\cdots,m$.

Theorem 1 *If $\mu^{\mathcal{G}}$ is the interval level dependent capacity relative to the breakpoints $a_0,a_1,a_2,,a_{m-1},a_m \in]0,1[$ and to the capacities $\mu_1,\cdots,\mu_m$ then for each $\mathbf{x} \in [0,1]^n$*

$$\mathcal{GC}(\mathbf{x},\mu^{\mathcal{G}}) = \sum_{j=1}^{n} \mathcal{C}_\mu(\mathbf{x}^j,\mu_j)$$

where $\mathbf{x}^j \in [0,1]^n$ is the vector having its elements x_i^j, $i = 1,\cdots,n$, defined as follows:

$$x_i^j = \begin{cases} 0 & \text{if } x_i < a_{j-1} \\ x_i - a_{j-1} & \text{if } a_{j-1} \le x_i \le a_j \\ a_j - a_{j-1} & \text{if } x_i \ge a_j. \end{cases}$$

3 Non-additive robust ordinal regression

3.1 Sorting problems

Within the multicriteria aggregation-disaggregation framework, *ordinal regression* aims at inducing the parameters of a decision model, for example those of a utility function, which have to represent some holistic preference comparisons of the DM. Usually, among the many utility functions representing the DM's preference information, only one utility function is

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selected (for example, in the UTA method [10]). In this context we also remember some UTA like-methods within the Choquet integral framework, proposed in [11]. Since such a choice is arbitrary to some extent, recently *additive robust ordinal regression* has been proposed with the purpose of taking into account all the sets of parameters *compatible* with the DM's preference information (for more details, see the multicriteria methodologies UTA$^{\text{GMS}}$ and GRIP proposed, respectively, in [12] and [13]). Let us remark that the principles of the *additive robust ordinal regression* have been applied also in sorting problems (see the UTADIS$^{\text{GMS}}$ method [14]).

Until now, robust ordinal regression has been implemented to additive utility functions under the assumption of criteria independence. In [15], the authors have proposed a *non-additive robust ordinal regression* on a set of alternatives X, whose utility is evaluated in terms of the Choquet integral which permits to represent the interaction among criteria, modeled by the fuzzy measures, parameterizing their approach.

In [15], besides holistic pairwise preference comparisons of alternatives from a subset of reference alternatives $X' \subseteq X$, the DM is also requested to express the intensity of preference on pairs of alternatives from X' and to supply pairwise comparisons on the importance of criteria, and the sign and intensity of interaction among pairs of criteria.

In the following, we recall the binary preference relations on the set of reference alternatives defined in [15].

Let us suppose that the preference of the DM is given by a partial pre-order $\succsim$ on $X' \subseteq X$.

The preference relation $\succsim$ can be decomposed into its symmetric part $\sim$ and into its asymmetric part $\succ$, whose semantics are, respectively:

$$\mathbf{x} \sim \mathbf{y} \;\Leftrightarrow\; \mathbf{x} \text{ is indifferent to } \mathbf{y},$$
$$\mathbf{x} \succ \mathbf{y} \;\Leftrightarrow\; \mathbf{x} \text{ is preferred to } \mathbf{y}, \text{ with } \mathbf{x}, \mathbf{y} \in X'.$$

The relation on the intensity of preference on pairs alternatives is represented by a partial pre-order $\succsim^*$ on $X' \times X'$, whose semantics is: for $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in X'$

$$(\mathbf{x}, \mathbf{y}) \succsim^* (\mathbf{z}, \mathbf{t}) \;\Leftrightarrow\; \mathbf{x} \text{ is preferred to } \mathbf{y} \text{ at least as much as } \mathbf{z} \text{ is preferred to } \mathbf{t}.$$

The following system of linear constraints synthesizes the DM's preference information expressed in the approach proposed in [15].

$$\begin{cases} \mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \mathcal{C}_\mu(\mathbf{x}) \geq \mathcal{C}_\mu(\mathbf{y}), \text{ with } \mathbf{x}, \mathbf{y} \in X', \\ (\mathbf{x}, \mathbf{y}) \succsim^* (\mathbf{z}, \mathbf{t}) \;\Leftrightarrow\; \mathcal{C}_\mu(\mathbf{x}) - \mathcal{C}_\mu(\mathbf{y}) \geq \mathcal{C}_\mu(\mathbf{z}) - \mathcal{C}(\mathbf{t}), \\ \text{ with } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in X', \\ \vdots \\ \text{Constraints on the importance and interaction of criteria} \\ \vdots \\ \text{Boundary, monotonicity conditions} \end{cases}$$

The output of the approach defines a set of fuzzy measures (capacities) μ defined *compatible* with the DM's preference information if the Choquet integral, calculated with respect to it, restores the DM's ranking on X', *i.e.*

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \mathcal{C}_\mu(\mathbf{x}) \geq \mathcal{C}_\mu(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in X'.$$

Moreover, using linear programming, our decision model establishes two preference relations:

- for any $\mathbf{r}, \mathbf{s} \in X$, the *necessary* weak preference relation $\succsim^N$, if for all *compatible* fuzzy measures the utility of $\mathbf{r}$ is not smaller than the utility of $\mathbf{s}$, *i.e.* $\mathbf{r} \succsim^N \mathbf{s} \;\Leftrightarrow\; \mathcal{C}_\mu(\mathbf{r}) \geq \mathcal{C}_\mu(\mathbf{s})$,
- for any $\mathbf{r}, \mathbf{s} \in X$, the *possible* weak preference relation $\succsim^P$, if for at least one *compatible* fuzzy measure the utility of $\mathbf{r}$ is not smaller than the utility of $\mathbf{s}$, *i.e.* $\mathbf{r} \succsim^P \mathbf{s} \;\Leftrightarrow\; \mathcal{C}_\mu(\mathbf{r}) \geq \mathcal{C}_\mu(\mathbf{s})$.

Since as it is shown in literature, bi-capacities are the right tools to represent many decision bipolar behaviors, in this work we suggest to extend the *non-additive robust ordinal regression*, described in [15], to the bipolar decision setting and to the level dependent Choquet integral.

However, the greater flexibility of decision strategies with the criteria on a bipolar scale and level dependent Choquet integral is offset by the huge number of parameters to be elicitated by the DM and the not easy interpretation of the bi-capacities and level dependent capacity for the DM.

Let us remark that in our approach the DM is not compelled to give preference information on all criteria.

Moreover, not all criteria are compulsory on a bipolar scale; some criteria could be on the usual unipolar scale (see for a result in this topic the concept of partially symmetric bi-capacities, introduced in [16]).

The set of constraints relative to the bi-polar Choquet integral is as follows:

$$\textbf{2)} \begin{cases} \mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \mathcal{BC}(\mathbf{x}, \hat{\mu}) \geq \mathcal{BC}(\mathbf{y}, \hat{\mu}), \text{ with } \mathbf{x}, \mathbf{y} \in X', \\ (\mathbf{x}, \mathbf{y}) \succsim^* (\mathbf{z}, \mathbf{t}) \;\Leftrightarrow \\ \mathcal{BC}(\mathbf{x}, \hat{\mu}) - \mathcal{BC}(\mathbf{y}, \hat{\mu}) \geq \mathcal{BC}(\mathbf{z}, \hat{\mu}) - \mathcal{BC}(\mathbf{t}, \mu), \\ \text{ with } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in X', \\ \vdots \\ \text{Constraints on the bipolar measures,} \\ \vdots \\ \text{Boundary, monotonicity conditions.} \end{cases}$$

The first set constraints are interpreted in the same way of the first set of constraints of system **1)**, with the only difference that every alternative is evaluated by the bi-polar Choquet integral.

Concerning the constraints on the bipolar measures, we suggest to adopt the decomposition explained in subsection 2.4.1. Such decomposition is more easy for the DM to be interpreted and to be elicitated.

The set of constraints relative to the level dependent Choquet integral is as follows:

$$\textbf{3)} \begin{cases} \mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow \\ \mathcal{GC}(\mathbf{x}, \mu^{\mathcal{G}}) \geq \mathcal{GC}(\mathbf{y}, \mu^{\mathcal{G}}), \text{ with } \mathbf{x}, \mathbf{y} \in X', \\ (\mathbf{x}, \mathbf{y}) \succsim^* (\mathbf{z}, \mathbf{t}) \;\Leftrightarrow \\ \mathcal{GC}(\mathbf{x}, \mu^{\mathcal{G}}) - \mathcal{GC}(\mathbf{y}, \mu^{\mathcal{G}}) \geq \mathcal{GC}(\mathbf{z}, \mu^{\mathcal{G}}) - \mathcal{GC}(\mathbf{t}, \mu^{\mathcal{G}}), \\ \text{ with } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in X', \\ \vdots \\ \text{Constraints on the level dependent capacity.} \end{cases}$$

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In this case, we adopt the interval level dependent capacity defined in Section 2.5.1, since it is more easy for the DM to express some ordinal constraints on such capacities. For example, if $X \subseteq [0,1]^n$ and we split the interval $]0,1[$ in two equal subintervals $I_1 =]0, \frac{1}{2}[$, $I_2 =]\frac{1}{2}, 1[$, DM's preference statement could be that criterion i is more important if the level of evaluation of every alternative is in I_2 than if it is in I_1.

We denote the system of constraints (types **1)**, **2)** and **3)**) on the reference alternatives X' by $E_\varepsilon(X')$.

Since linear programming is not able to handle strict inequalities in $E_\varepsilon(X')$, we put the constraints in the form of weak inequalities, by adding a small arbitrary positive value ε (see [17] for a result on this topic).

$$\textbf{4)} \begin{cases} U(\mathbf{x}) \geq U(\mathbf{y}) + \varepsilon, \text{ with } \mathbf{x}, \mathbf{y} \in X', \\ U(\mathbf{x}) - U(\mathbf{y}) \geq U(\mathbf{z}) - U(\mathbf{t}) + \varepsilon, \\ \text{with } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in X', \\ \vdots \\ \text{Constraints on the bipolar measures,} \\ \vdots \\ \text{Boundary, monotonicity conditions,} \end{cases}$$

where U stands for the aggregation operator than can be the bipolar Choquet integral or the level dependent Choquet integral. A similar approach can be used with the bipolar level dependent Choquet integral.

Once the decision making setting on the set of reference alternatives X' is chosen by the DM (types **1)**, **2)** and **3)**), two different optimization problems arise to establish a *necessary* and *possible* preference relation for any $\mathbf{r}, \mathbf{s} \in X$:

$$\begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & E_\varepsilon^{X'} \quad \text{plus the constraint } U(\mathbf{s}) \geq U(\mathbf{r}) + \varepsilon \end{array} \tag{2}$$

and

$$\begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & E_\varepsilon^{X'} \quad \text{plus the constraint } U(\mathbf{r}) \geq U(\mathbf{s}). \end{array} \tag{3}$$

If the problem (2) finds a solution with $\varepsilon \leq 0$, then $U(\mathbf{r}) \geq U(\mathbf{s})$ for all compatible sets of fuzzy measures, that implies $\mathbf{r} \succsim^N \mathbf{s}$ with $\mathbf{r}, \mathbf{s} \in X$.

On the contrary, if a positive ε solves the linear program indicated in (3), then there exists at least one *compatible* fuzzy measure such that $U(\mathbf{r}) \geq U(\mathbf{s})$, that implies $\mathbf{r} \succsim^P \mathbf{s}$ with $\mathbf{r}, \mathbf{s} \in X$.

3.2 The most representative value function

To consider the whole set of fuzzy measures compatible with the preference expressed by the DM reduces arbitrariness in the decision process. However to take into account one specific fuzzy measure can help the DM in understanding the decision process. As already proposed for robust ordinal regression with additive value functions [18], the most representative fuzzy measure is that one which better represents the necessary ranking maximizing the difference of evaluations between alternatives for which there is a preference in the necessary ranking. As secondary objective, one can consider minimizing the difference of evaluations between actions for which there is not a preference in the necessary ranking.

This comprehensive "most representative" value function can be determined through the following procedure:

1. Determine the necessary and the possible rankings in the considered set of actions.
2. For all pairs of alternatives $(\mathbf{x}, \mathbf{y})$, such that $\mathbf{x}$ is necessarily preferred to $\mathbf{x}$, add the following constraints to the linear programming constraints of types **1)**, **2)** or **3)**: $U(\mathbf{x}) \geq U(\mathbf{y}) + \varepsilon$.
3. Maximize the objective function ε.
4. Add the constraint $\varepsilon = \varepsilon^*$, with $\varepsilon^* = \max \varepsilon$ in the previous point, to the linear programming constraints of robust ordinal regression of types **1)**, **2)** and **3)**.
5. For all pairs of actions $(\mathbf{x}, \mathbf{y})$, such that neither $\mathbf{x}$ is necessarily preferred to $\mathbf{y}$ nor $\mathbf{y}$ is necessarily preferred to $\mathbf{x}$, add the following constraints to the linear programming constraints of of types **1)**, **2)** and **3)** and to the constraints considered in above point 4): $U(\mathbf{x}) - U(\mathbf{y}) \leq \delta$ and $U(\mathbf{y}) - U(\mathbf{x}) \leq \delta$.
6. Minimize the objective function δ.

3.3 Sorting problems

In this section, we briefly illustrate the principles of non-additive robust ordinal regression to sorting problems in case of nonadditive value function represented by Choquet integral or some of its generalizations.

In MCDA, the sorting problem consists in the assignment of m alternatives of a finite set X into $h = 1, \cdots, k$ homogeneous classes $C_1, \cdots C_h, \cdots C_k$, which are increasingly ordered with respect to preference, i.e. all the elements in the class C_h have a better evaluation than the elements in class C_{h-1}. Let $b_1, b_2, \cdots, b_h$ be a set of thresholds relative to the h classes. We have that $\mathbf{x}$ belongs to class h if $U(\mathbf{x}) \geq b_{h-1}$ and $U(\mathbf{x}) < b_h$, where U stands for the aggregation operator than can be the Choquet integral or one of its generalizations. In this case, the DM's preference information will consist in the assignment of some reference alternatives to some classes. Taking into account the Choquet integral, the proposed approach determines the sets of pairs $(\mu, \mathbf{b})$, where μ is a capacity and $\mathbf{b} = [b_1, b_2, \cdots, b_h]$, compatible with the DM's assignment of the reference alternatives [19]. An alternative $\mathbf{x} \in X$ is said that *possibly belongs to class* C_h if there is at least one pair $(\mu, \mathbf{b})$ for which $\mathcal{C}_\mu(\mathbf{x}) \geq b_{h-1}$ and $\mathcal{C}_\mu(\mathbf{x}) < b_h$. One can deal analogously with the above mentioned extensions of Choquet integral.

As proposed in [20] with respect to robust ordinal regression for sorting problems based on additive value function through the UTADIS$^{\text{GMS}}$ method [14], also in this case we can help the DM with the concept of the "most representative" value function.

The idea is to select among compatible fuzzy measure that one which better highlights the possible sorting which is considered the most stable part of the robust sorting obtained by UTADIS$^{\text{GMS}}$. Thus it is selected the fuzzy measure that maximizes the difference of evaluations between alternatives for

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which the intervals of possible sorting are disjoints. As secondary objective, one can consider maximize the minimal difference between values of actions **x** and **y**, such that for any compatible fuzzy measure, **x** is assigned to a class not worse than the class of **y**, and for at least one compatible value function, **x** is assigned to a class which is better than the class of **y**. In case there is still more than one such fuzzy measure, the most representative fuzzy measure minimizes the maximal difference between values of alternatives being in the same class for all compatible fuzzy measures, and between values of alternatives for which the order of classes is not univocal.

4 Conclusions

Nonadditive value functions and bipolar decision making are one of the underpinning directions of research within MCDA, as many studies have shown that in a multicriteria decision problem the criteria under consideration could be on a bipolar scale. In such context, we have proposed a multicriteria methodology taking inspiration from some recent approaches based on the principle of the *robust ordinal regression*: the UTA^{GMS} [12] and GRIP [13] in the context of choice and ranking problems, the $\text{UTADIS}^{\text{GMS}}$ method relative to sorting problems [14]. On this basis, with the aim of representing interactions between criteria, we proposed the *non-additive robust ordinal regression* which consists in the extension of the idea of robust ordinal regression to non-additive decision. More precisely, the Choquet integral and its generalizations have been adopted as utility function in different decision problems such as ranking, choice and sorting (for the specific case of Choquet integral applied to ranking and choice problems see [15]). The non-additive robust ordinal regression seems very useful because it permits to take into account the whole set of capacities compatible with the DM's preferences which are expressed through very simple questions such as:

- "is the representative alternative a better than the representative alternative b"?
- "to what class the representative alternative a belongs"?

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