Preservation of properties of interval-valued fuzzy relations

B. Pękala
Institute of Mathematics, University of Rzeszów
Rejtana 16a, 35-310 Rzeszów, Poland
Email: bpekala@univ.rzeszow.pl

Abstract— *The goal of this paper is to consider properties of the composition of interval-valued fuzzy relations which were introduced by L.A. Zadeh in 1975. Fuzzy set theory turned out to be a useful tool to describe situations in which the data are imprecise or vague. Interval-valued fuzzy set theory is a generalization of fuzzy set theory which was introduced also by Zadeh in 1965. This paper generalizes some properties of interval matrices considered by Pękala (2007) on these of interval-valued fuzzy relations.*

Keywords— Fuzzy relations, interval-valued fuzzy relations, properties of interval-valued fuzzy relations

1 Introduction

The idea of a fuzzy relation was defined in [28]. An extension of fuzzy set theory is interval-valued fuzzy set theory. Any interval-valued fuzzy set is defined by an interval-valued membership function: a mapping from the given universe to the set of all closed subintervals of [0,1] (it means that information is incomplete). In this paper we study properties of the composition of interval-valued fuzzy relations. Consideration of diverse properties of the composition is interesting not only from a theoretical point of view but also for the applications, since the composition of interval-valued fuzzy relations has proved to be useful in several fields, see for example, [20] (performance evaluation), [27] (genetic algorithm), [19] (approximate reasoning) or in other (see [1, 16]). In Section 2, we recall elementary properties of the composition of interval-valued fuzzy relations. Next, we consider preservation of properties of interval-valued fuzzy relations by the composition and lattice operations.

We give the following definition. Let X, Y, Z be crisp finite non-empty sets.

Definition 1 (cf. [29, 26]). Let $Int([0,1])$ be the set of all closed subintervals of $[0,1]$. An interval-valued fuzzy relation R in a universe X, Y is a mapping $R : X \times Y \to Int([0,1])$ such that $R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in Int([0,1])$, for all pairs $(x,y) \in (X \times Y)$. The class of interval-valued fuzzy relations in a universe $X \times Y$ will be denoted by *IVFR*$(X \times Y)$ or *IVFR*(X) for $X = Y$.

Interval-valued fuzzy relations reflect the idea that membership grades are often not precise and the intervals represent such uncertainty.
The boundary elements in *IVFR*$(X \times Y)$ are $\mathbf{1} = [1,1]$ and $\mathbf{0} = [0,0]$. The relation R^{-1} between Y and X is defined by $R^{-1}(y,x) = R(x,y)$ for all $(x,y) \in (X \times Y)$ which we will call the inverse relation of R.
Let us look at immediate properties for interval-valued fuzzy relations: Let $P, R \in$ *IVFR*$(X \times Y)$.
Then for every $(x,y) \in (X \times Y)$ we can define

$$P(x,y) \le R(x,y) \Leftrightarrow \underline{P}(x,y) \le \underline{R}(x,y), \overline{P}(x,y) \le \overline{R}(x,y),$$

$$(P \vee R)(x,y) = [\max(\underline{P}(x,y), \underline{R}(x,y)), \max(\overline{P}(x,y), \overline{R}(x,y))],$$

$$(P \wedge R)(x,y) = [\min(\underline{P}(x,y), \underline{R}(x,y)), \min(\overline{P}(x,y), \overline{R}(x,y))],$$

where operations $\vee$ and $\wedge$ are the supremum and the infimum in *IVFR*$(X \times Y)$, respectively. For arbitrary set $T \neq \emptyset$ similarly we use

$$(\bigvee_{t \in T} R_t)(x,y) = [\bigvee_{t \in T} \underline{R_t}(x,y), \bigvee_{t \in T} \overline{R_t}(x,y)],$$

$$(\bigwedge_{t \in T} R_t)(x,y) = [\bigwedge_{t \in T} \underline{R_t}(x,y), \bigwedge_{t \in T} \overline{R_t}(x,y)].$$

We know that $([0,1], \max, \min)$ is a complete, distributive lattice, and therefore (*IVFR*$(X \times Y), \vee, \wedge)$ is also a complete, distributive lattice. So it is a particular case of lattices considered by Goguen in [18].

Interval-valued fuzzy relations (sets) are equivalent to some other extensions of fuzzy relations (sets) (see [12]). Among others, interval-valued fuzzy relations are isomorphic to Atanassov's fuzzy relations. This fact was noticed by several authors [5, 11, 12]. An Atanassov's fuzzy relation is a pair of fuzzy relations, namely a membership and a nonmembership functions, which represent positive and negative aspects of the given information. This objects introduced by Atanassov and originally called intuitionistic fuzzy relations were recently suggested to be called Atanassov's intuitionistic fuzzy relations or just bipolar fuzzy relations [15]. An Atanassov's fuzzy set theory is also widely applied in solving real-life problems. An example of such application is the optimization in Atanassov's intuitionistic fuzzy environment (an extension of fuzzy optimization and an application of bipolar fuzzy sets) where by applying this concept it is possible to reformulate the optimization problem by using degrees of rejection of constraints and values of the objective which are non-admissible. This concept allows one to define a degree of rejection which cannot be simply a complement of the degree of acceptance [2]. The idea of a positive and negative information was confirmed by psychological investigations [9]. Moreover, multiattribute decision making using Atanassov's intuitionistic fuzzy sets is possible (see [22, 23]).

Definition 2 (cf. [3]). Let $X \neq \emptyset, \mathfrak{R}, \mathfrak{R}^d : X \times Y \to [0,1]$ be fuzzy relations fulfilling the condition

$$\mathfrak{R}(x,y) + \mathfrak{R}^d(x,y) \leqslant 1, \ \ (x,y) \in (X \times Y).$$

A pair $\rho = (\mathfrak{R}, \mathfrak{R}^d)$ is called an Atanassov's intuitionistic fuzzy relation. The family of all Atanassov's intuitionistic fuzzy relations described in a given sets X, Y is denoted by $AIFR(X \times Y)$.

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Basic operations on Atanassov's intuitionistic fuzzy relations $\rho = (\mathfrak{R}, \mathfrak{R}^d)$, $\sigma = (\mathfrak{S}, \mathfrak{S}^d)$ are defined in the following way

$$\rho \cup \sigma = (\max(\mathfrak{R}, \mathfrak{S}), \min(\mathfrak{R}^d, \mathfrak{S}^d)),$$

$$\rho \cap \sigma = (\min(\mathfrak{R}, \mathfrak{S}), \max(\mathfrak{R}^d, \mathfrak{S}^d)),$$

$$\rho \leqslant \sigma \iff (\mathfrak{R} \le \mathfrak{S},\ \mathfrak{S}^d \le \mathfrak{R}^d).$$

The isomorphism which proves the equivalence between Atanassov's intuitionistic fuzzy relations and interval-valued fuzzy relations is the following

Theorem 1 (cf. [11]). *The mapping $\psi : IVFR(X \times Y) \to AIFR(X \times Y)$, such that $R \to \rho$ is an isomorphism between the lattices $(IVFR(X \times Y), \vee, \wedge)$ and $(AIFR(X \times Y), \cup, \cap)$, where $R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)]$, $R \in IVFR(X \times Y)$ and $\rho(x,y) = (\underline{R}(x,y), 1 - \overline{R}(x,y))$, $(x,y) \in (X \times Y)$.*

2 Composition of interval-valued fuzzy relations

Now, we consider the composition of interval-valued fuzzy relations.

Definition 3 (cf. [8]). Let $* : [0,1]^2 \to [0,1]$, $P \in IVFR(X \times Y)$, $R \in IVFR(Y \times Z)$.
By the $\sup -*$ composition of relations P and R we call the relation $P \circ R \in IVFR(X \times Z)$,

$$(P \circ R)(x,z) = [(\underline{P} \circ \underline{R})(x,z), (\overline{P} \circ \overline{R})(x,z)], \quad (1)$$

where

$$(\underline{P} \circ \underline{R})(x,z) = \bigvee_{y \in Y} (\underline{P}(x,y) * \underline{R}(y,z)),$$

$$(\overline{P} \circ \overline{R})(x,z) = \bigvee_{y \in Y} (\overline{P}(x,y) * \overline{R}(y,z))$$

and $(\underline{P} \circ \underline{R})(x,z) \le (\overline{P} \circ \overline{R})(x,z)$.

Other types of compositions may be also considered. For example, the composition of interval-valued fuzzy relations with supremum and $*$ being a triangular norm or a triangular conorm is examined in [8].

Definition 4 (cf. [24]). A triangular norm T (conorm S) is an increasing, commutative, associative operation $T(S) : [0,1]^2 \to [0,1]$ with a neutral element 1 (0).

In [8] it was proved, for finite sets X, Y, Z, that compositions mentioned there are associative if and only if the first operation in composition is $\vee$ or $\wedge$ and the second is an arbitrary triangular norm or conorm. This is why the choice of the first operation in the composition (1) is reasonable.

For our further considerations we need the following properties

Definition 5 (cf. [18]). Let $* : [0,1]^2 \to [0,1]$. Operation $*$ is infinitely sup-distributive, if

$$\bigvee_{t \in T} (x_t * y) = (\bigvee_{t \in T} x_t) * y, \quad \bigvee_{t \in T} (y * x_t) = y * (\bigvee_{t \in T} x_t).$$

Definition 6 (cf. [6]). Operation $* : [0,1]^2 \to [0,1]$ is isotone if it fulfils the condition:

$$\underset{x,y,z \in [0,1]}{\forall} x \le y \Rightarrow x * z \le y * z, z * x \le z * y.$$

By generalization of the results of the papers [14, 25] and using [13] we obtain

Lemma 1. *If $*$ is isotonic, then $\sup -*$ composition is also isotonic.*

Proof. Let $*$ be left side isotonic.
For $P, R \in IVFR(X \times Y), Q \in IVFR(Y \times Z)$

$$P \le R \Leftrightarrow [\underline{P}, \overline{P}] \le [\underline{R}, \overline{R}] \Leftrightarrow$$

$$\underset{x,y \in X \times Y}{\forall} \underline{P}(x,y) \le \underline{R}(x,y), \overline{P}(x,y) \le \overline{R}(x,y)$$

by isotonicity of the $*$ and supremum we have for $x \in X, y \in Y, z \in Z$

$$\bigvee_{y \in X} (\underline{P}(x,y) * \underline{Q}(y,z)) \le \bigvee_{y \in X} (\underline{R}(x,y) * \underline{Q}(y,z)),$$

$$\bigvee_{y \in X} (\overline{P}(x,y) * \overline{Q}(y,z) \le \bigvee_{y \in X} (\overline{R}(x,y) * \overline{Q}(y,z)) \Leftrightarrow$$

$$\underline{P} \circ \underline{Q} \le \underline{R} \circ \underline{Q}, \overline{P} \circ \overline{Q} \le \overline{R} \circ \overline{Q} \Leftrightarrow P \circ Q \le R \circ Q.$$

The right side isotonicity of $\circ$ may be proved similarly. □

Lemma 2. *Let the operation $*$ have a zero element z. If $*$ has a neutral element e, then $\sup -*$ composition has a neutral element $\boldsymbol{S}_e = [S_e, S_e]$,*

$$S_e(x,y) = \begin{cases} e & if\ x = y \\ z & if\ x \ne y \end{cases} \quad \textit{for } x \in X, y \in Y.$$

Proof. Let $*$ operation have a zero element z and a neutral element e.

$$(\mathbf{S}_e \circ R)(x,y) =$$

$$[\bigvee_{z \in X} (S_e(x,z) * \underline{R}(z,y)), \bigvee_{z \in X} (S_e(x,z) * \overline{R}(z,y))] =$$

$$[(e*\underline{R}(x,y)) \vee \bigvee_{x \ne z} (z*\underline{R}(z,y)), (e*\overline{R}(x,y)) \vee \bigvee_{x \ne z} (z*\overline{R}(z,y))] =$$

$$[e * \underline{R}(x,y), e * \overline{R}(x,y)] = [\underline{R}(x,y), \overline{R}(x,y)] = R(x,y).$$

The proof for $R \circ \mathbf{S}_e = R$ is similar. □

If $*$ is an isotonic operation, then we may prove

$$\bigvee_{t \in T} (P_t \circ R) \le (\bigvee_{t \in T} P_t) \circ R, \quad \bigwedge_{t \in T} (P_t \circ R) \ge (\bigwedge_{t \in T} P_t) \circ R. \quad (2)$$

Now we examine the problem of sup-distributivity. Authors [8] in Theorem 6 present condition for sup-distributivity by $*$ equal to any t-norm or t-conorm and by finite non-empty X, Y. By generalization we obtain

Lemma 3. *If an operation $*$ is infinitely sup-distributive, then $\sup -*$ composition is also infinitely sup-distributive i.e., for $P_t \in IVFR(X \times Y), R \in IVFR(Y \times Z)$*

$$\bigvee_{t \in T} (P_t \circ R) = (\bigvee_{t \in T} P_t) \circ R. \quad (3)$$

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Proof. Let $*$ be sup-distributive, $x \in X, y \in Y, z \in Z$. Then

$$((\bigvee_{t\in T} P_t) \circ R)(x,z) =$$

$$[\bigvee_{y\in Y}((\bigvee_{t\in T}\underline{P_t})(x,y)*\underline{R}(y,z)), \bigvee_{y\in Y}((\bigvee_{t\in T}\overline{P_t})(x,y)*\overline{R}(y,z))] =$$

$$[\bigvee_{y\in Y}(\bigvee_{t\in T}(\underline{P_t}(x,y)*\underline{R}(y,z))), \bigvee_{y\in Y}(\bigvee_{t\in T}(\overline{P_t}(x,y)*\overline{R}(y,z)))] =$$

$$[\bigvee_{t\in T}(\bigvee_{y\in Y}(\underline{P_t}(x,y)*\underline{R}(y,z))), \bigvee_{t\in T}(\bigvee_{y\in Y}(\overline{P_t}(x,y)*\overline{R}(y,z)))] =$$

$$[\bigvee_{t\in T}(\underline{P_t}\circ\underline{R})(x,z), \bigvee_{t\in T}(\overline{P_t}\circ\overline{R})(x,z)] = \bigvee_{t\in T}(P_t\circ R)(x,z).$$

□

Here we discuss the most important property of binary operations, i.e. the associativity. The associativity of $*$ is not sufficient for the associativity of the $\sup -*$ composition. Some results of this problem we see in ([8], Theorem 9) for t-norms or t-conorms on finite sets. As a result the question about the associativity of the composition of interval-valued fuzzy relations we get for arbitrary sets X, Y, Z, U the following lemmas.

Lemma 4. *If an operation $*$ is associative and infinitely sup-distributive, then $\sup -*$ composition is associative.*

Proof. Let operation $*$ be associative and infinitely sup-distributive, $P \in IVFR(X \times Y), R \in IVFR(Y \times Z)$, $Q \in IVFR(Z \times U)$ and $x \in X, y \in Y, z \in Z, u \in U$.

$$((P\circ R)\circ Q)(x,u) = [((\underline{P}\circ\underline{R})\circ\underline{Q})(x,u), ((\overline{P}\circ\overline{R})\circ\overline{Q})(x,u)] =$$

$$[\bigvee_z(\underline{P}\circ\underline{R})(x,z)*\underline{Q}(z,u), \bigvee_z(\overline{P}\circ\overline{R})(x,z)*\overline{Q}(z,u)] =$$

$$[\bigvee_z(\bigvee_y(\underline{P}(x,y)*\underline{R}(y,z))*\underline{Q}(z,u)),$$

$$\bigvee_z(\bigvee_y(\overline{P}(x,y)*\overline{R}(y,z))*\overline{Q}(z,u))] =$$

$$[\bigvee_z\bigvee_y(\underline{P}(x,y)*(\underline{R}(y,z)*\underline{Q}(z,u))),$$

$$\bigvee_z\bigvee_y(\overline{P}(x,y)*(\overline{R}(y,z)*\overline{Q}(z,u)))] =$$

$$[\bigvee_y(\underline{P}(x,y)*\bigvee_z(\underline{R}(y,z)*\underline{Q}(z,u))),$$

$$\bigvee_y(\overline{P}(x,y)*\bigvee_z(\overline{R}(y,z)*\overline{Q}(z,u)))] =$$

$$[(\underline{P}\circ(\underline{R}\circ\underline{Q}))(x,u), (\overline{P}\circ(\overline{R}\circ\overline{Q}))(x,u)] = (P\circ(R\circ Q))(x,u).$$

□

As a direct consequence of the above lemmas we observed that set of all interval-valued fuzzy relations with the composition (1) create a semigroup.

Proposition 1. *If $*$ is associative, infinitely sup-distributive operation with a zero element z=0 and a neutral element e=1, then $(IVFR(X), \circ)$ is an ordered semigroup with the identity $\boldsymbol{I} = [I, I]$.*

In the sequel we denote by D the set of all binary operations $*: [0,1]^2 \to [0,1]$ which are associative and infinitely sup-distributive (these conditions imply that $*$ is isotonic). As a result a special case of $*$ may be a left-continous triangular norm or conorm. If $* \in D$, then in a semigroup $(IVFR(X), \circ)$ we can consider the powers of its elements, i.e. relations R^n for $R \in IVFR(X)$, $n \in \mathbb{N}$. By analogy to [21] we define

Definition 7. By the powers of a relation $R \in IVFR(X)$ we call interval-valued fuzzy relations

$$R^1 = R,\ R^{m+1} = R^m \circ R, \text{ where } m = 1, 2, \ldots .$$

By the upper closure $R^{\vee}$ and the lower closure $R^{\wedge}$ of the relation R we call, respectively

$$R^{\vee} = \bigvee_{k=1}^{\infty} R^k,\ \ R^{\wedge} = \bigwedge_{k=1}^{\infty} R^k, \text{ where } R^k = [\underline{R}^k, \overline{R}^k]. \quad (4)$$

Proposition 2. *If $* \in D$ and $P, R \in IVFR(X)$, then*

$$\forall_{n\in\mathbb{N}}\ (P\vee R)^n \geq P^n \vee R^n,\ \ (P\wedge R)^n \leq P^n \wedge R^n, \quad (5)$$

$$(P\vee R)^{\vee} \geq P^{\vee} \vee R^{\vee},\ \ (P\wedge R)^{\vee} \leq P^{\vee}\wedge R^{\vee}, \quad (6)$$

$$(P\vee R)^{\wedge} \geq P^{\wedge} \vee R^{\wedge},\ \ (P\wedge R)^{\wedge} \leq P^{\wedge}\wedge R^{\wedge}. \quad (7)$$

Proof. Let $n \in \mathbb{N}$. From the isotonicity of $*$ and the Lemma 1 we know that the $\sup -*$ composition is also isotonic. As a result we obtain the isotonicity for powers. Then by

$$P\vee R \geq P,\ P\vee R \geq R$$

$$(\underline{P}\vee\underline{R} \geq \underline{P},\ \underline{P}\vee\underline{R} \geq \underline{R},\ \overline{P}\vee\overline{R} \geq \overline{P},\ \overline{P}\vee\overline{R} \geq \overline{R})$$

we have

$$(\underline{P}\vee\underline{R})^n \geq \underline{P}^n,\ (\underline{P}\vee\underline{R})^n \geq \underline{R}^n \text{ and}$$

$$(\overline{P}\vee\overline{R})^n \geq \overline{P}^n,\ (\overline{P}\vee\overline{R})^n \geq \overline{R}^n,$$

so we obtain

$$(P\vee R)^n \geq P^n,\ (P\vee R)^n \geq R^n \Rightarrow (P\vee R)^n \geq P^n \vee R^n.$$

Similarly, we can prove $(P \wedge R)^n \leq P^n \wedge R^n$. Moreover, supremum and infimum are isotonic, so from the condition (5) also closures have this property
$P^{\vee} \leq (P\vee R)^{\vee}$, $R^{\vee} \leq (P\vee R)^{\vee} \Rightarrow P^{\vee}\vee R^{\vee} \leq (P\vee R)^{\vee}$,
$(P\wedge R)^{\vee} \leq P^{\vee}$, $(P\wedge R)^{\vee} \leq R^{\vee} \Rightarrow (P\wedge R)^{\vee} \leq P^{\vee}\wedge R^{\vee}$.
Similarly, we may prove the inequalities in (7). □

Proposition 3. *Let $* \in D$ and $P, R \in IVFR(X)$.
If $P \circ R = R \circ P$, then*

$$\forall_{n\in\mathbb{N}}\ (P\circ R)^n = P^n \circ R^n, \quad (8)$$

$$(P\circ R)^{\vee} \leq P^{\vee}\circ R^{\vee},\ \ (P\circ R)^{\wedge} \geq P^{\wedge}\circ R^{\wedge}. \quad (9)$$

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Proof. The given equality in (8) may be proved by the mathematical induction and associativity of $*$ and commutativity of powers, which is implied by commutativity of P and R. Then

$$(P \circ R)^{\vee} = \bigvee_{k=1}^{\infty} (P \circ R)^k = [\bigvee_{k=1}^{\infty} (\underline{P}^k \circ \underline{R}^k), \bigvee_{k=1}^{\infty} (\overline{P}^k \circ \overline{R}^k)],$$

by (3), (2) and isotonicity of the operation $\sup -*$ we have

$$[\bigvee_{k=1}^{\infty} (\underline{P}^k \circ \underline{R}^k), \bigvee_{k=1}^{\infty} (\overline{P}^k \circ \overline{R}^k)] \leq [\bigvee_{k=1}^{\infty} (\underline{P}^{\vee} \circ \underline{R}^k), \bigvee_{k=1}^{\infty} (\overline{P}^{\vee} \circ \overline{R}^k)]$$

$$\leq [\underline{P}^{\vee} \circ \bigvee_{k=1}^{\infty} \underline{R}^k, \overline{P}^{\vee} \circ \bigvee_{k=1}^{\infty} \overline{R}^k] = [\underline{P}^{\vee} \circ \underline{R}^{\vee}, \overline{P}^{\vee} \circ \overline{R}^{\vee}] = P^{\vee} \circ R^{\vee},$$

which proves the first condition in (9). The second condition in (9) one may be justified in a similar way. □

3 Properties of interval-valued fuzzy relations

Now, we will examine whether the given properties are preserved by the composition of interval-valued fuzzy relations. We see that many properties of the $*$ operation are transposed to the operation of $\sup -*$ composition but not all of them. Namely, if $* = \min$, then $\sup - \min$ composition is not commutative. Now, we examine very interesting properties, namely subidempotency and superidempotency. These properties are of the large interest for example in economy where they are applied in valuation of supply and demand. Similarly to definitions of properties of fuzzy relations in [21] we have

Definition 8. Let $R \in IVFR(X)$. The relation R is called idempotent, subidempotent (transitive) or superidempotent if $R^2 = R,\ R^2 \leq R,\ R^2 \geq R$, respectively.

For the Boolean matrices of dimension $n \times n$ the following computations hold true

Table 1: The subidempotent relations in the family of all relations.

n	all	subidempotent	%
2	16	13	81,25000
3	512	171	33,39844
4	65536	3994	6,09436
5	33554432	154303	0,45986

A similar situation takes place in any distributive and bounded lattice, it means that the percentage of the subidempotent matrices is rapidly decreasing with the growth of n. This is why the consideration and determination of the upper closure $R^{\vee}$ is very important.

Theorem 2. *Let $* \in D$ and $R \in IVFR(X)$. $R^{\vee}$ is the least subidempotent relation greater than or equal to R. Moreover, the relation R is subidempotent if and only if $R = R^{\vee}$.*

Proof. Let $R, S, Q \in IVFR(X)$. If R is subidempotent, then

$$\underset{n\in\mathbb{N}}{\forall} R^n \leq R \text{ and } R \leq R^{\vee} = \bigvee_{n\in\mathbb{N}} R^n \leq R,$$

so we obtain $R^{\vee} = R$.
If $R^{\vee} = R$, then for $S = R^{\vee}$ we have

$$S^2 = \bigvee_{k=2}^{\infty} R^k \leq R^{\vee} = S$$

and $R^{\vee}$ is subidempotent.
We show that if there exists $Q = [\underline{Q}, \overline{Q}]$ such that

$$[\underline{R}, \overline{R}] \leq [\underline{Q}, \overline{Q}] \text{ and } [\underline{Q}^2, \overline{Q}^2] \leq [\underline{Q}, \overline{Q}],$$

then by isotonicity the $\sup -*$ composition (Lemma 1) we obtain

$$[\underline{R}^2, \overline{R}^2] \leq [\underline{R} \circ \underline{Q}, \overline{R} \circ \overline{Q}] \leq [\underline{Q}^2, \overline{Q}^2]$$

thus

$$[\underline{R}^k, \overline{R}^k] \leq [\underline{Q}^k, \overline{Q}^k] \leq [\underline{Q}, \overline{Q}] \text{ for } k \in \mathbb{N}.$$

So by isotonicity of the supremum $R^{\vee} \leq Q$. □

The closures and powers of interval-valued fuzzy relations also preserve some properties of such relations.

Theorem 3. *Let an operation $* \in D$ be commutative and $R \in IVFR(X)$. If R is subidempotent (superidempotent), then R^n, $R^{\wedge}$ ($R^{\vee}$) are subidempotent (superidempotent).*

Proof. Let $R^2 \leq R$. Then R^n are also subidempotent, $n \in \mathbb{N}$. By (2) and subidempotency of R we have

$$(R^{\wedge})^2 \leq (R^2)^{\wedge} \leq R^{\wedge},$$

so $R^{\wedge}$ is subidempotent. The property of superidempotency may be proved analogously. □

Theorem 4. *Let $T \neq \emptyset$ and $* \in D$ and $R_t \in IVFR(X)$, $t \in T$. If $(R_t)_{t\in T}$ is a family of subidempotent relations, then the relation $R = \bigwedge_{t\in T} R_t$ is subidempotent.*

Proof. Let $R_t^2 \leq R_t$, $t \in T$, then by (2)

$$R^2 = (\bigwedge_{t\in T} R_t)^2 = (\bigwedge_{s\in T} R_s) \circ (\bigwedge_{t\in T} R_t) \leq$$

$$\bigwedge_{t\in T}((\bigwedge_{s\in T} R_s) \circ R_t) \leq \bigwedge_{s,t\in T} (R_s \circ R_t) \leq \bigwedge_{t\in T} R_t = R$$

i.e., the relation R is subidempotent. □

Theorem 5. *Let $* \in D$ and $P, R \in IVFR(X)$. If P, R are subidempotent, then $P \vee R$ is subidempotent if and only if $P \circ R \vee R \circ P \leq P \vee R$.*

Proof. Since $P^2 \leq P$, $R^2 \leq R$ so by (3) we obtain

$$(P \vee R)^2 = [(\underline{P} \vee \underline{R}) \circ (\underline{P} \vee \underline{R}), (\overline{P} \vee \overline{R}) \circ (\overline{P} \vee \overline{R})] =$$

$$[(\underline{P} \circ \underline{P}) \vee (\underline{R} \circ \underline{P}) \vee (\underline{P} \circ \underline{R}) \vee (\underline{R} \circ \underline{R}),$$

$$(\overline{P} \circ \overline{P}) \vee (\overline{R} \circ \overline{P}) \vee (\overline{P} \circ \overline{R}) \vee (\overline{R} \circ \overline{R})] \leq$$

$$[\underline{P} \vee (\underline{R} \circ \underline{P}) \vee (\underline{P} \circ \underline{R}) \vee \underline{R}, \overline{P} \vee (\overline{R} \circ \overline{P}) \vee (\overline{P} \circ \overline{R}) \vee \overline{R}] =$$

$$[\underline{P} \vee \underline{R} \vee (\underline{R} \circ \underline{P}) \vee (\underline{P} \circ \underline{R}), \overline{P} \vee \overline{R} \vee (\overline{R} \circ \overline{P}) \vee (\overline{P} \circ \overline{R})],$$

as a result

$$(P \vee R)^2 \leq P \vee R \Leftrightarrow P \circ R \vee R \circ P \leq P \vee R,$$

because

$$P \leq R \Leftrightarrow P \vee R = R.$$

□

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Theorem 6. *Let* $* \in D$ *and* $P, R \in IVFR(X)$. *If* P, R *are subidempotent (superidempotent) and* $P \circ R = R \circ P$, *then* $P \circ R$ *is subidempotent (superidempotent).*

Proof. If $\underline{P} \circ \underline{R} = \underline{R} \circ \underline{P}$ and $\overline{P} \circ \overline{R} = \overline{R} \circ \overline{P}$, $P^2 \leq P$, $R^2 \leq R$, then by the associativity and monotonicity of the $\sup -*$ composition we have

$$(P \circ R)^2 = [(\underline{P} \circ \underline{R})^2, (\overline{P} \circ \overline{R})^2] =$$

$$[\underline{P} \circ (\underline{R} \circ \underline{P}) \circ \underline{R}, \overline{P} \circ (\overline{R} \circ \overline{P}) \circ \overline{R}] =$$

$$[\underline{P} \circ \underline{P} \circ \underline{R} \circ \underline{R}, \overline{P} \circ \overline{P} \circ \overline{R} \circ \overline{R}] = [\underline{P}^2 \circ \underline{R}^2, \overline{P}^2 \circ \overline{R}^2] \leq$$

$$[\underline{P} \circ \underline{R}, \overline{P} \circ \overline{R}] = P \circ R.$$

As a result $P \circ R$ is subidempotent. The proof for superidempotency is similar. □

4 Conclusion

In this work we present only some problems connected with the preservation of interval-valued fuzzy relation properties by the $\sup -*$ composition and related to it operation. We can also consider preservation of other properties (for example, symmetry, asymmetry, antisymmetry, reflexivity, irreflexivity, connectedness) by the composition of interval-valued fuzzy relations and also by its powers and lattice operations.
We may consider dual composition to the one defined in (1). This is the $\inf -*'$ composition with the dual binary operation $*'$, where $x *' y = 1 - (1 - x) * (1 - y)$ for $x, y \in [0, 1]$. The properties of this composition may be deduced from the $\sup -*$ composition.
Moreover, we may study interval-valued t-norms and t-conorms as operations on *Int*([0,1]) which are important functions because they are useful in approximate reasoning, in medical diagnosis and information retrieval. For example, the authors of [7, 10] examine the construction of t-norms and t-conorms in the lattice $(Int[0, 1], \vee, \wedge)$ and analyze some properties of them.

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