Definition of fuzzy Pareto-optimality by using possibility theory

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Abstract— *Pareto-optimality conditions are crucial when dealing with classic multi-objective optimization problems because we need to find out a set of optimal solutions rather than only one optimal solution to optimization problem with a single objective. Extensions of these conditions to the fuzzy domain have been discussed and addressed in recent literature. This work presents a novel approach based on the use of possibility theory as a comparison index to define a fuzzily ordered set with a view to generating the necessary conditions for the Pareto-optimality of candidate solutions in the fuzzy domain. Making use of the conditions generated, one can characterize fuzzy efficient solutions by means of carefully chosen single-objective problems. The uncertainties are inserted into the formulation of the studied fuzzy multi-objective optimization problem by means of fuzzy coefficients in the objective function. Some numerical examples are analytically solved to illustrate the efficiency of the proposed approach.*

Keywords— Possibility theory, multi-objective optimization, fuzzy Pareto-optimality conditions, fuzzy mathematical programming.

1 Introduction

One of the most significant characteristics of human beings is the decision making of day-by-day problems. This characteristics is used to solve several practical problems including economic, environmental, social and technical. These problems are multidimensional and have multiple objectives that are often non-commensurable and conflict with each other. Thus, they are inserted in the set of problems that are solved by using the theory of multi-objective optimization which is a generalization of traditional single objective optimization. Although multi-objective optimization problems differ from single objective problems only in the plurality of objective functions, it is important to realize that the notion of optimality condition change because now the decision maker must find solutions that satisfy or create a compromise among the multiple objectives. These solutions are called Pareto optimal or efficient or non-dominated solutions.

Optimization is a procedure of finding and comparing feasible solutions until no better solution can be found. These solutions are defined good or bad in terms of one or several objectives when is used any optimization model. The optimization models often use classical mathematical programming, which attempts to develop an exact model to the optimization problem of interest. Such modeling may overlook ambiguities that all too frequently exist in actual optimization operations. In recent years, Fuzzy Logic [16] has showed great potential for modeling systems which are non-linear, complex, ill-defined and not well understood. Fuzzy Logic has found numerous applications due to its ease of implementation, flexibility, tolerant nature to imprecise data, and ability to model non-linear behavior of arbitrary complexity because of its basis in terms of natural language. In [18] is discussed the use of fuzzy logic that is a precise logic of imprecision and approximate reasoning. We refer to [6, 10, 19], for some applications in the fields of pattern recognition, data analysis, optimal control, economics and operational research, among others.

The representation and arithmetic manipulation of uncertain numerical quantities can be defined by means of fuzzy sets. Unfortunately, the comparison among two or more fuzzy numbers, intervals and/or sets is not easy. Some approaches to compare them (see some examples in [2, 6, 9, 10, 14]) were developed, each one being based upon a different point of view. The possibility theory, which is analogous to the probability theory, was proposed by Zadeh [17] to aggregate the concept of a possibility distribution to the theory of fuzzy sets. Comparison indexes to rank fuzzy numbers and intervals employing possibility theory were proposed in [7]. Their importance stems from the fact that much of the information on which human decisions rely upon have a possibilistic, rather than a probabilistic nature. On the other hand, some works describe a fuzzy optimization problem in a classical problem and they use the classical theory to find the Pareto optimal set. [8] transform a fuzzy single objective problem into a classical multiple objective one where the number of objectives is defined by fuzzy coefficients from the fuzzy problem.

Possibility theory emerged from the notion of fuzzy sets and his concept tries to take account of the fact that an object may more or less correspond to a certain category in which one attempts to place it. In the calculus of degree of possibility emphasizes the double relationship between possibility theory and set theory and the concept of measure, respectively. One merit of possibility theory is to represent imprecision and to quantify uncertainty at the same time.

This work is organized as follows. Section 2 presents an overview about the formulation of classical multi-objective programming problems and classical concepts to obtain the set of Pareto optimal solutions. Also, it is shown an extension of these concepts to Pareto optimal solutions of fuzzy multi-objective programming problems. Section 3 introduces a novel approach to fuzzy Pareto-optimality. A fuzzily ordered set is defined by using a possibility distribution function as a comparison measure. This section also presents the characterization of the efficient solutions through the use of well defined scalar problems. This relation between efficient solutions and scalar problems can be determined by certain theorems. To clarify the above developments, two numerical

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examples are analyzed in section 4. Finally, conclusions are presented in Section 5.

2 Multiobjective programming problem formulation and concepts

Choosing the goal to be optimized is a critical step in the process of modeling real-world problems. The local or global optimal solution depends totally upon this choice. In the vast majority of real-world problems, various objective functions could be defined. Many times these functions are conflicting and/or non-measurable. Multi-objective optimization is the branch of mathematical optimization theory devoted to developing methods to solve problems with various objective functions. A classical multi-objective problem can be formulated as follows:

$$\begin{array}{ll} \min & F(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \Omega \end{array} \quad (1)$$

where $F = (f_1, f_2, \dots, f_m),\ (m \geq 2)$ is a vector of objectives and $\Omega \subset \mathbb{R}^n$ is the set of feasible solutions.

Due to the issue of conflicting objectives, the classical concept of optimality that are used to obtain the optimal solution to optimization problems with an objective function does not fit into the multi-objective framework. Hence, in such a framework, one settles for the so-called efficient, non-dominated or Pareto-optimal solution.

The classical works [12] and [13], by Vilfredo Pareto, introduced the concept of Pareto-optimality and started the field of multi-objective optimization. A solution $x^* \in \Omega$ is said to be *Pareto-optimal* or *non-dominated* if there exists no alternative solution in Ω that improves some of the objective functions without degrading at least another objective function. Then, we can define mathematically a non-dominated solution as

Definition 1 (Pareto optimal solution) $\mathbf{x}^* \in \Omega$ *is said to be a* non-dominated solution *of Problem (1) if there exists no other feasible* $\mathbf{x} \in \Omega$ *such that* $f_i(\mathbf{x}) \leq f_i(\mathbf{x}^*),\ \forall\ i = 1, \dots, m$ *with strict inequality for at least one* i.

However, when non-linear programming problems with single objective are solved by any non-linear programming method, only local optimal solutions are guaranteed in practical. Then, the concept of local non-dominated solution can be defined in the following way:

Definition 2 (local Pareto optimal solution) $\mathbf{x}^* \in \Omega$ *is said to be a* local non-dominated solution *of Problem (1) if and only if there exists a real number* $\delta > 0$ *such that* $\mathbf{x}^*$ *is non-dominated in* $\Omega \bigcap \mathcal{N}(\mathbf{x}^*, \delta)$*, i.e., there does not exist another feasible* $\mathbf{x} \in \Omega \bigcap \mathcal{N}(\mathbf{x}^*, \delta)$ *such that* $f_i(\mathbf{x}) \leq f_i(\mathbf{x}^*),\ \forall\ i = 1, \dots, m$ *with strict inequality for at least one* i.

where $\mathcal{N}(\mathbf{x}^*, \delta)$ denotes the δ neighbourhood of $\mathbf{x}^*$ defined by $\{\mathbf{x} \in \mathbb{R} | \|\mathbf{x} - \mathbf{x}^*\| < \delta\}$. Thus, it is possible to see by these definitions that the solution of a multi-objective programming problem consists of an infinite number of points.

Many methods to solve multi-objective programming problems were proposed and some specific methods can be found in [4]. These methods are classified according to the instant the decision maker applies their criteria. Three methods are proposed: (i) *a-Priori method*, where the decision maker assigns weights to the objective functions a-priori, thus obtaining a single mono-objective criterion; (ii) *a-Posteriori method*, where the decision maker strives to create some type of efficient solutions to chose from a-posteriori; and (iii) *interactive methods*, where the decision maker informs their preferences during the search process of an efficient solution.

Although the mathematical formulation of optimization problems with multiple objectives be well defined, some values of the real-world problems have vagueness, imprecision and uncertainty. These values which have been estimated by decision maker are parameters in the set of constraints and in one or several objective functions. These uncertainties can be formulated by logic fuzzy which is a way to describe this vagueness mathematically and it has found numerous and different applications due to its easy implementation, flexibility, tolerant nature to imprecise data, low cost implementations and ability to model non-linear behavior of arbitrary complexity because of its basis in terms of natural language. The concept of fuzzy decision to obtain a solution to fuzzy programming problems was introduced by Bellman and Zadeh[1] which proved that the fuzzy programming problems can be reduced to a conventional programming problem under some assumptions determined by decision maker.

Based on this work, many researches developed methods that solve optimization problems under fuzzy environment and in [15] is used α-cut sets to define a non-dominated solution to multi-objective programming problems with fuzzy parameters which is called α-Pareto optimal solution. In this case, the fuzzy parameters can be inserted in Problem (1) and it is transformed in the following way:

$$\begin{array}{ll} \min & F(\tilde{\mathbf{a}}; \mathbf{x}) = (f_1(\tilde{\mathbf{a}}_1; \mathbf{x}), \dots, f_m(\tilde{\mathbf{a}}_m; \mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in X(\tilde{\mathbf{b}}) \triangleq \{x \in \mathbb{R}^n | g_i(\tilde{\mathbf{b}}_i; \mathbf{x}) \leq 0, i = 1, \dots, p\} \end{array} \quad (2)$$

where $\tilde{\mathbf{a}}_i$ and $\tilde{\mathbf{b}}_i$ represent a vector of fuzzy parameters involved in the objective functions and in the functions that form the set of constraints, respectively. These fuzzy parameters which reflect the expert's ambiguous understanding of the nature of the parameters in the problem formulation process, are assumed to be characterized as fuzzy numbers.

Definition 3 (α-level set) *The* α*-level set of the fuzzy numbers* $\tilde{c}$ *and* $\tilde{d}$ *is defined as the ordinary set* $(\tilde{c}, \tilde{d})_\alpha$ *for which the degree of their membership functions exceeds the level* α*:*

$$(\tilde{c}, \tilde{d})_\alpha = \{(c, d) | \mu_{\tilde{c}}(c) \geq \alpha \text{ and } \mu_{\tilde{d}}(d) \geq \alpha\}.$$

Now, suppose that the decision maker considers that the degree of all of the membership functions of the fuzzy numbers involved in Problem (2) should be greater than or equal to a certain value of α. Then, for such a degree α, Problem (2) can be interpreted as a conventional multi-objective programming problems in the following way:

$$\begin{array}{ll} \min & F(\mathbf{a}; \mathbf{x}) = (f_1(\mathbf{a}_1; \mathbf{x}), \dots, f_m(\mathbf{a}_m; \mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in X(\mathbf{b}) \triangleq \{x \in \mathbb{R}^n | g_i(x, b_i) \leq 0, i = 1, \dots, p\} \\ & (\mathbf{a}, \mathbf{b}) \in (\tilde{\mathbf{a}}, \tilde{\mathbf{b}})_\alpha \end{array} \quad (3)$$

where the coefficient vector $(\mathbf{a}, \mathbf{b}) \in (\tilde{\mathbf{a}}, \tilde{\mathbf{b}})_\alpha$ and $(\mathbf{a}, \mathbf{b})$ are arbitrary for any value in $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})_\alpha$. This fuzzy subset is formatted for all the vectors whose degree of each membership function exceeds the level α. Here, it is possible to observe that the parameters $(\mathbf{a}, \mathbf{b})$ are treated as decision variables of Problem (3) rather than constants of Problem (2). Through the

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description of Problem (3), the concepts of (local) α-Pareto optimality is defined as follows.

Definition 4 ((Local) α-Pareto optimal solution) *$x^* \in X(b)$ is said to be a (local) α-Pareto-optimal solution to the α-MONLP(Multi-Objective Non-Linear programming) if and only if there exists no other $x \in X(b)(\bigcap \mathcal{N}(x^*, r))$ and $(a, b) \in (A, B)_\alpha(\bigcap \mathcal{N}(a^*, b^*, r'))$ such that $f_i(x, a_i) \leq f_i(x^*, a_i^*)$, $i = 1, 2, \ldots, k$, with strict inequality holding for at least one i, where the corresponding values of parameters a^* and b^* are called α-level optimal parameters (and $\mathcal{N}(x^*, r) \triangleq \{x \in \mathbb{R}^n | \|x - x^*\| < r\}$ denote the r neighborhood of x^*).*

The (local) α-Pareto-optimal solutions can be obtained through a direct application of the scalarization methods, which transform a multi-objective programming problem into a single-objective programming problem, and the optimal solution obtained is formed by point $(\mathbf{x}, \mathbf{a}, \mathbf{b})$. These set of solutions, however, generally comprises an infinite number of points and the decision should select a single (local) solution based on a subjective criterion.

3 The use of possibility theory in multi-objective optimization under fuzzy environment

Normally, when a multi-objective optimization problems is formulated, many parameters need to be assigned by the decision maker and they may be described by possible values. In most practical situations, it is natural to consider that the possible values of these parameters are often only vaguely known and it is appropriated to interpret them by the decision maker's understanding. This parameters can be represented by fuzzy numbers which intent to describe the possible values that are inserted in the real-world problems. Then, the resulting multi-objective programming problem with fuzzy parameters would be viewed as the more realistic version of the conventional one. In addition, it is necessary to define a model for the quantification of the imprecise data that interpret the decision maker's judgment. There are many comparison approaches among fuzzy numbers and one of them is the possibility theory which was chosen in this work.

3.1 Fuzzy basic concepts

Mathematical programming problems need a precise definition of both the constraints and the objective function to be optimized. Fuzzy sets help handle uncertainties when multi-objective programming problems are formalized in the following form:

$$\begin{array}{ll} \widetilde{\min} & F(\tilde{\mathbf{a}}; \mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \tilde{\Omega} \end{array} \quad (4)$$

where $F = (f_1, f_2, \ldots, f_m)(m \geq 2)$ is a vector of objectives, $\tilde{\mathbf{a}} \in \mathbb{F}(\mathbb{R}^{n \times m})$ represent the fuzzy parameters in the objective functions and $\tilde{\Omega} \subset \mathbb{F}(\mathbb{R}^n)$ is a subset of feasible solutions. $\mathbb{F}(\mathbb{R})$ defines the set of fuzzy numbers, $\mathbb{F}(\mathbb{R}^n)$ defines the set of n-dimensional vector with fuzzy parameters and $\mathbb{F}(\mathbb{R}^{n \times m})$ defines the set of $n \times m$-dimensional matrix with fuzzy parameters. However, we shall only address the uncertainties of the parameters in the objective functions, in this work.

In order to be able to sort fuzzy numbers in an increasing (decreasing) order, one has to opt for a comparison measure, therefore a possibility measure is used which is defined as:

Definition 5 (Possibility measure) *Let A be a fuzzy subset of U and let $\prod_X$ be a possibility distribution associated with a variable X which takes values in U. The* possibility measure*, $\pi(A)$, of A is defined by*

$$Poss\{X \text{ } is \text{ } A\} \triangleq \pi(A) \triangleq \sup_{u \in U} \min(\mu_A(u), \pi_X(u)) \quad (5)$$

where μ_A is the membership function of A and π_X is the possibility distribution function of X. It can be interpreted as the possibility that the value X belongs to the set A *and it is defined to be numerically equal to the membership function of X.*

Then, it is possible define a way to compare two fuzzy numbers and this index can be formulated as follows

$$Poss\{\tilde{a}_1 \leq \tilde{a}_2\} = \sup_{u,v \in U; u \leq v} \min(\mu_{\tilde{a}_1}(u), \mu_{\tilde{a}_2}(v))$$

where $\mu_{\tilde{a}_1}$ and $\mu_{\tilde{a}_2}$ are membership functions of $\tilde{a}_1$ and $\tilde{a}_2$. Possibility degree $Poss\{\tilde{a}_1 \leq \tilde{a}_2\}$ shows to what extent $\tilde{a}_1$ is possibly less than or equal to $\tilde{a}_2$, as described in [7, 11].

The definition above enables one to define a fuzzily ordered set $\mathbb{F}(\mathbb{R})$ that is an extension of the classical ordered set. A set is said to be *completely ordered* if it satisfies the following conditions:

Definition 6 (Ordered fuzzily set) *A fuzzy subset $A \subset \mathbb{F}(\mathbb{R})$ is fuzzily ordered with respect to the possibility measure if each element in A satisfies the following basic properties:*

1. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_1] = 1$;
2. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_2] \geq \alpha_1$ *and* $Poss[\tilde{\mathbf{a}}_2 \leq \tilde{\mathbf{a}}_3] \geq \alpha_2 \Rightarrow$ $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_3] \geq \min\{\alpha_1, \alpha_2\}$;
3. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_2] \geq \alpha_1$ *and* $Poss[\tilde{\mathbf{a}}_2 \leq \tilde{\mathbf{a}}_1] \geq \alpha_2 \Rightarrow$ $Poss[\tilde{\mathbf{a}}_1 = \tilde{\mathbf{a}}_2] \geq \min\{\alpha_1, \alpha_2\}$;

$\forall \tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{a}}_3 \in A$ *and* $\forall \alpha_1, \alpha_2 \in [0, 1]$.

According to the expressions above, a fuzzy subset $A \subset \mathbb{F}(\mathbb{R})$ is completely ordered. However, a fuzzy subset of $\mathbb{F}(\mathbb{R}^m)$ is only partially ordered. Therefore, the concept of optimal solution for single objective problems, which was defined in [3, 15], does no fit into the multi-objective formulation, unless the problem admits the so-called *ideal solution*, i.e. a single solution that simultaneously minimizes all objective functions as below:

Definition 7 (Ideal solution) *The ideal solution $\tilde{\mathbf{y}}$ of the multi-objective problem is defined as*

$$\tilde{\mathbf{y}}_i = f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^i), \;\; i = 1, \ldots, m$$

where $\mathbf{x}^i = \arg\min_{\mathbf{x} \in \mathbb{R}^n} f_i(\tilde{\mathbf{a}}_i; \mathbf{x})$.

The problem is said to admit an ideal solution whenever the set of arguments $\{\mathbf{x}^i, \; i = 1, \ldots, m\}$, possesses a single element. Because the multi-objective framework is most often

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employed in problems with conflicting objectives, a typical multi-objective problem is unlikely to admit such a solution. However, due the existence of an ideal solution is very rare, such a possibility will not be considered in the present analysis.

Conceptually, an efficient solution is one which is not dominated by any other feasible solution. Hence, the domination concept of a fuzzy multi-objective problem should reflect the decision maker's preferences. In this work, a fuzzy dominance concept is proposed which can be adjusted to the decision maker's preferences. This renders the proposed approach flexible and customizable, and possibly applicable to a wide range of problems. For any point $\mathbf{x}^0 \in \mathbb{R}^n$, consider the following subsets

$$\Omega_{<}(\mathbf{x}^0;\alpha) \triangleq \{\mathbf{x} \in \mathbb{R}^n : Poss[F(\tilde{\mathbf{a}};\mathbf{x}) \leq F(\tilde{\mathbf{a}};\mathbf{x}^0)] \geq \alpha \text{ and } Poss[F(\tilde{\mathbf{a}};\mathbf{x}) = F(\tilde{\mathbf{a}};\mathbf{x}^0)] < 1\}$$

$$\Omega_{\geq}(\mathbf{x}^0;\alpha) \triangleq \{\mathbf{x} \in \mathbb{R}^n : Poss[F(\tilde{\mathbf{a}};\mathbf{x}) \geq F(\tilde{\mathbf{a}};\mathbf{x}^0)] \geq \alpha\}$$

$$\Omega_{\sim}(\mathbf{x}^0;\alpha) \triangleq \{\mathbf{x} \in \mathbb{R}^n : \max\{Poss[F(\tilde{\mathbf{a}};\mathbf{x}) \leq F(\tilde{\mathbf{a}};\mathbf{x}^0)], Poss[F(\tilde{\mathbf{a}};\mathbf{x}) \geq F(\tilde{\mathbf{a}};\mathbf{x}^0)]\} \leq \alpha\}$$

The subset $\Omega_{<}(\mathbf{x}^0;\alpha)$ comprises the points in $\mathbb{R}^n$ that dominate $\mathbf{x}^0$, whereas $\Omega_{\geq}(\mathbf{x}^0;\alpha)$ encompasses the points in $\mathbb{R}^n$ that are dominated by $\mathbf{x}^0$. The set of points that neither dominate nor are dominated by $\mathbf{x}^0$ is denoted by $\Omega_{\sim}(\mathbf{x}^0;\alpha)$. The parameter α is a vector where each one of the terms, α_i with $i = 1,2,\ldots,m$, belong to the interval $[0,1]$. Those sets being defined, one can denote the set of fuzzy Pareto-optimal solutions as below:

Definition 8 (Fuzzy Pareto-optimal solution) $\mathbf{x}^* \in \Omega$ *is said be a fuzzy Pareto-optimal solution if there exists no other* $x \in \Omega$ *such that* $Poss[f_i(\tilde{\mathbf{a}}_i;\mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*)] \geq \alpha_i,\ \forall i$ *and* $Poss[f_j(\tilde{\mathbf{a}}_j;\mathbf{x}) = f_j(\tilde{\mathbf{a}}_j;\mathbf{x}^*)] < 1$ *for at least one* j*, where* $\alpha_i \in [0,1],\ \forall i$.

For difficult optimization problems it is often the case that a local optimal solution is acceptable. A local efficient solution for the proposed problem is defined below:

Definition 9 (Fuzzy local Pareto-optimal solution) $\mathbf{x}^* \in \Omega$ *is said to be a fuzzy local Pareto-optimal solution if there is a real number* $\delta \geq 0$ *such that there exists no other* $x \in \Omega \cap \mathcal{N}(x^*,\delta)$ *such that* $Poss[f_i(\tilde{\mathbf{a}}_i;\mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*)] \geq \alpha_i,\ \forall i$ *and* $Poss[f_j(\tilde{\mathbf{a}}_j;\mathbf{x}) = f_j(\tilde{\mathbf{a}}_j;\mathbf{x}^*)] < 1$ *in at least one* j*, where* $\alpha_i \in [0,1],\ \forall i$.

Note that the definition above implies that a candidate solution to the proposed fuzzy problem is (locally) non-dominated or efficient, if one cannot find (in a certain vicinity) another solution that simultaneously improves all the objective functions. This interpretation matches the classical counterpart of multi-objective optimization.

The convexity hyphotesis determine that the neighbourhood of each one local solution involves whole the feasible region.

Theorem 1 *Let* $f_i : \Omega \subset \mathcal{X} \to \mathbb{F}(\mathcal{Y}),\ i = 1,\ldots,m$ *a convex fuzzy functions about a convex subset* Ω *of a linear space* $\mathcal{X}$*. Then whole locally efficient solution is globaly efficient solution.*

Proof: Let $\mathbf{x}^* \in \Omega$ is a locally efficient solution. By definition of convex subset, we obtain $\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x} \in \Omega,\ \forall \mathbf{x} \in \Omega - \mathcal{N}(\mathbf{x}^*,\epsilon)$, with $\epsilon > 0$ e $\lambda \in [0,1]$. Suppose $\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x} \in \Omega \cap \mathcal{N}(\mathbf{x}^*,\epsilon)$ then by the fuzzy Pareto-optimal solution definition, we obtain $Poss[f_i(\tilde{a}_i;\mathbf{x}^*) \leq f_i(\tilde{a}_i;\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x})] \geq \alpha_i^1,\ i \in \mathcal{I} = \{1,2,\ldots,m\}$, and $Poss[f_j(\tilde{a}_j;\mathbf{x}^*) = f_j(\tilde{a}_j;\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x})] < 1$ for at least one $j \in \mathcal{I}$, where $\alpha_i^1 \in (0,1],\ \forall i \in \mathcal{I}$. By the convex fuzzy function definition, we obtain $f_i(\tilde{a}_i;\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}) \lesssim \lambda f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*) + (1-\lambda)f_i(\tilde{\mathbf{a}}_i;\mathbf{x}),\ \forall i \in \mathcal{I}$, which it can be rewritten by using Possibility Theory as $Poss[f_i(\tilde{a}_i;\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}) \leq \lambda f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*) + (1-\lambda)f_i(\tilde{\mathbf{a}}_i;\mathbf{x})\}] \geq \alpha_i^2$, where $\alpha_i^2 \in (0,1],\ \forall i \in \mathcal{I}$. Thus, by using the ordered fuzzily subset definition, we obtain $Poss[f_i(\tilde{a}_i;\mathbf{x}^*) \leq \lambda f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*) + (1-\lambda)f_i(\tilde{\mathbf{a}}_i;\mathbf{x})\}] \geq \min\{\alpha_i^1,\alpha_i^2\} \Rightarrow Poss[f_i(\tilde{a}_i;\mathbf{x}^*) \leq f_i(\tilde{a}_i;\mathbf{x})] \geq \min\{\alpha_i^1,\alpha_i^2\},\ \forall i \in \mathcal{I}$. By selecting a determined objective function $k \in \mathcal{I}$ and $k \neq i$, by the Theorem 2, we guarantee that $Poss[f_k(\tilde{a}_k;\mathbf{x}^*) = f_k(\tilde{a}_k;\mathbf{x})] < 1$ for at least one $k \in \mathcal{I}$. ■

3.2 Characterization of fuzzy efficient solutions

The characterization of efficient solutions, $efi(\Omega)$, by means of well defined scalar problems is a recurrent approach in fuzzy multi-objective problems. The following theorem relates efficient solutions and scalar problems.

Theorem 2 $\mathbf{x}^* \in efi(\Omega)$ *if and only if* $\mathbf{x}^*$ *solves the* m *scalar problems*

$$\begin{aligned} P_k: \quad & \min_{\mathbf{x}\in\Omega} \quad f_k(\tilde{\mathbf{a}}_k;\mathbf{x}) \\ & s.t. \quad f_l(\tilde{\mathbf{a}}_l;\mathbf{x}) \lesssim f_l(\tilde{\mathbf{a}}_l;\mathbf{x}^*), \\ & \qquad l = 1,2,\ldots,m, \quad \forall\, l \neq k. \end{aligned} \tag{6}$$

Proof: ($\Rightarrow$) If $\mathbf{x}^* \in efi(\Omega)$, then there exist no other $x \in \Omega$ such that $Poss[f_i(\tilde{\mathbf{a}}_i;\mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*)] \geq \alpha_i,\ i = 1,2,\ldots,m$, and $Poss[f_j(\tilde{\mathbf{a}}_j;\mathbf{x}) = f_j(\tilde{\mathbf{a}}_j;\mathbf{x}^*)] < 1$, for any j. In this case $\mathbf{x}^*$ solves (6) for all k.
($\Leftarrow$) Suppose $\mathbf{x}^*$ solves (6), but $\mathbf{x}^* \notin efi(\Omega)$, then there exists another $\mathbf{x} \in \Omega$ such that $Poss[f_i(\tilde{\mathbf{a}}_i;\mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i;\mathbf{x}^*)] \geq \alpha_i,\ \forall i$, and for some j, $Poss[f_j(\tilde{\mathbf{a}}_j;\mathbf{x}) \leq f_j(\tilde{\mathbf{a}}_j;\mathbf{x}^*)] < 1$. Therefore, $\mathbf{x}^*$ does not solve Problem (6). This contradiction concludes the proof. ■

The development of analytical conditions to efficient solutions, based on the characterization of non-dominated solutions to problems $P_k,\ k = 1,2,\ldots,m$, is an important tool in the theoretical analysis. However, such an analysis yields only m non-dominated solutions, one to each scalar problem and is therefore, unable to generate the whole Pareto-optimal set.

Employing a similar analysis to the one presented above, we now establish the relationship between non-dominated solutions of a fuzzy multi-objective problem and solutions to the weighting problem. An alternative characterization based on the linear combination of the objectives can be expressed as

Theorem 3 *Let* $\mathbf{x}^* \in \Omega$ *solve the problem*

$$P_w: \quad \min_{\mathbf{x}\in\Omega} \quad \langle \mathbf{w}, F(\tilde{\mathbf{a}};\mathbf{x})\rangle = \sum_{i=1}^{m} \omega_i f_i(\tilde{\mathbf{a}}_i;\mathbf{x}) \tag{7}$$

for some $\mathbf{w} \in \mathbb{R}^m,\ \mathbf{w} \geq \mathbf{0}$ *and* $\sum_{i=1}^{m} w_i = 1$*. Then* $\mathbf{x}^* \in efi(\Omega)$ *if*

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(i) $\mathbf{x}^*$ *is the unique solution(7), or*

(ii) $w_i > 0,\ i = 1, \ldots, m.$

Proof: *(i)* If $\mathbf{x}^* \in \Omega$ is a unique solution of (7), then $\forall \mathbf{x} \in \Omega$ and by definition, we obtain $Poss\left[\sum_{i=1}^{m} w_i \left(f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*) - f_i(\tilde{\mathbf{a}}_i; \mathbf{x})\right) < 0\right] \geq \min_i\{\alpha_i\}$. Suppose $\mathbf{x}^* \notin efi(\Omega)$, i.e., there exists at least one $\mathbf{x}^0 \in \Omega$ such that $Poss[f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^0) \leq f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*)] \geq \alpha_i,\ i = 1, 2, \ldots, m$ and $Poss[f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^0) = f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^*)] < 1$, for some j. This contradicts the uniqueness hypothesis, because $\mathbf{w} \geq \mathbf{0}$. Thus, $\mathbf{x}^* \in efi(\Omega)$.
(ii) Suppose $\mathbf{x}^* \notin efi(\Omega)$, but $\mathbf{x}^*$ is a solution of (7). Then there exists a $\mathbf{x}^0 \in \Omega$ such that $Poss[f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^0) \leq f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*)] \geq \alpha_i,\ i = 1, 2, \ldots, m$ and $Poss[f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^0) = f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^*)] < 1$, for any j. Hence,

$$Poss\left[\sum_{i=1}^{m} w_i \left(f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*) \leq f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^0)\right) > 0\right] \geq \min_i\{\alpha_i\},\ \forall i.$$

A contradiction and therefore $\mathbf{x}^* \in efi(\Omega)$. ■

4 Results and analysis

The problems we use to evaluate this method are two hypothetic mathematical formulations, which are described in [5], with the fuzzy approach described in Section 3. Nevertheless, they are efficiency in validating the realized study. They were resolved using an modified implementation of **NSGA-II** that solves multi-objective programming problems with constraints or not. This modification was made in the comparison of the objective functions with fuzzy parameters between two feasible solutions which uses the concepts of fuzzy Pareto optimal solutions described in this work.

The vagueness was inserted into the costs of the objective function and fuzzy numbers are interpreted in the form $(a, \underline{a}, \overline{a})_{LR}$ where a is the modal value, $\underline{a}$ is the scattering left and $\overline{a}$ is the scattering right of each fuzzy number.

Example 1 (Schaffer's problem)

$$\begin{aligned} \min \quad & f_1(\tilde{\mathbf{a}}_1; \mathbf{x}) = (x + \tilde{a}_1)^2 \\ \min \quad & f_2(\tilde{\mathbf{a}}_2; \mathbf{x}) = (x - \tilde{a}_2)^2 \\ s.t \quad & -5 \leq \mathbf{x} \leq 5 \end{aligned} \tag{8}$$

where $\tilde{a}_1 = (0, 0, 2)_{LR}$ *and* $\tilde{a}_2 = (2, 1, 1)_{LR}$.

The figures below present the fuzzy solution and fuzzy front.

In Figure 2, each star represents one solution of this multi-objective problem by using the fuzzy Pareto optimal concept defined by Sakawa to $\alpha = 0.8$*, while each square represents one solution by using the fuzzy Pareto optimal concept described in this work to* $\alpha = 0.8$*, too. It can be observed that a range of possible Pareto optimal solutions is formed when the squares are merged. We can also see that many stars are inside some squares, i.e., this solutions have a degree of possibility great or equal to 0.8 and belong to the range of Pareto optimal solutions obtained by the definition that uses possibility theory.*

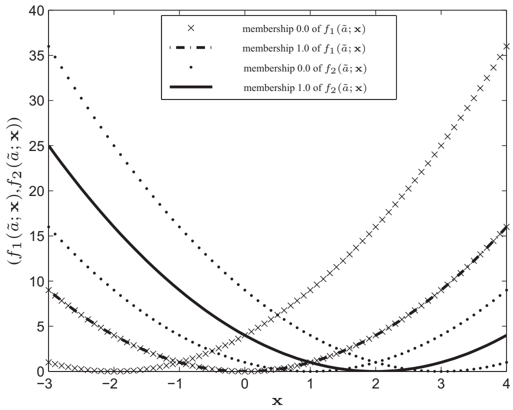

Figure 1: Objective functions

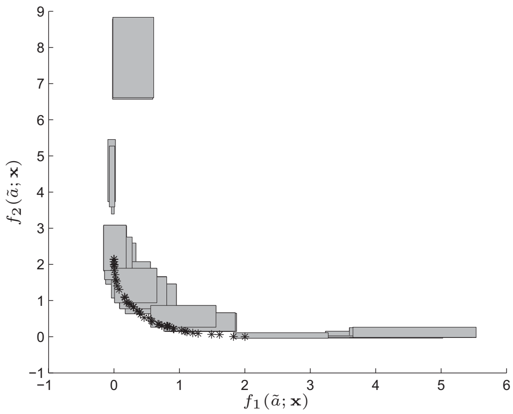

Figure 2: Pareto front

Example 2 (Binh and Kern's problem)

$$\begin{aligned} \min \quad & f_1(\tilde{\mathbf{a}}_1; \mathbf{x}) = \tilde{a}_{11} x_1^2 + \tilde{a}_{12} x_2^2 \\ \min \quad & f_2(\tilde{\mathbf{a}}_2; \mathbf{x}) = (x_1 - \tilde{a}_{21})^2 + (x_2 - \tilde{a}_{22})^2 \\ s.t. \quad & (x_1 - 5)^2 + x_2^2 \leq 25 \\ & (x_1 - 8)^2 + (x_2 + 3)^2 \geq 7.7 \\ & 0 \leq x_1 \leq 5,\ \ 0 \leq x_2 \leq 3 \end{aligned} \tag{9}$$

where $\tilde{a}_{11} = \tilde{a}_{12} = (4, 1, 1)_{LR}$*,* $\tilde{a}_{21} = \tilde{a}_{22} = (5, 1, 1)_{LR}$.

The figures below present the fuzzy solution and fuzzy front.

In Figure 3 are shown the function objectives of Problem (2) where the superior drawing represents the function $f_1(\tilde{\mathbf{a}}_1; \mathbf{x})$ *and the inferior one represents the function* $f_2(\tilde{\mathbf{a}}_2; \mathbf{x})$*. The drawings with solid line represent the value of the objective functions with* $\alpha = 1$ *while the ones with dotted line represent the value of the objective functions with* $\alpha = 0$.

Again, each star represents one solution by using the concept defined by Sakawa to $\alpha = 0.8$*, while each square represents one solution by using the concept described here to* $\alpha = 0.8$*, too. In this case, the stars are in the imaginary bound of the Pareto front range formed by possible Pareto solutions, i.e., the solutions obtained by Sakawa's definition have a satisfaction level closed in* $\alpha = 0.8$.

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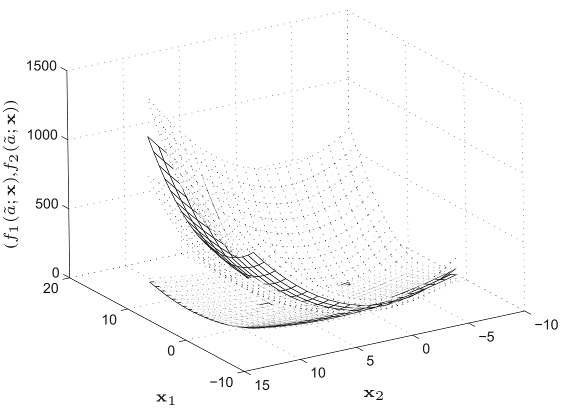

Figure 3: Objective functions

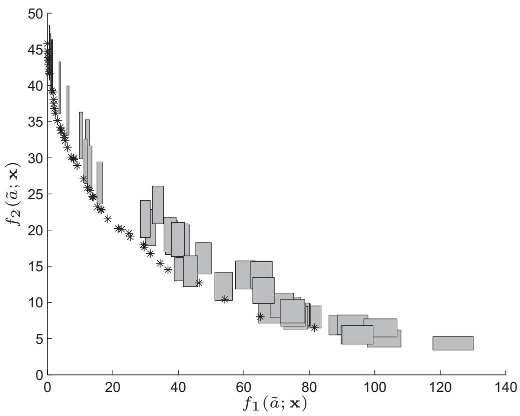

Figure 4: Pareto front

5 Conclusion

The use of defuzzification methods to solve fuzzy mathematical programming problems is very common. These methods transform a fuzzy number into a classical one, but some data are always lost in each case. The Pareto-optimality presented in this work makes use of different defuzzification methods in various stages of the definitions.

Multi-objective Programming problems are very important in a variety of both theoretical and practical areas. As ambiguity and vagueness are natural and ever-present in real-life situations that require precise solutions, it makes perfect sense to attempt to address these problems using Fuzzy Multi-objective Programming problems. In this context, this paper presented a novel theory to determine Pareto-optimality conditions which provide the user with a fuzzy solution. This theory is an expansion of the classical Pareto-optimality theory and demonstrates the necessary conditions for fuzzy Pareto-optimality. Some numerical examples are presented to validate the theory outlined.

The authors aim firstly to extend the line of investigation regarding Fuzzy Quadratic Programming problems in order to try to solve practical real-life problems by facilitating the building of Decision Support Systems. This requires the involvement of fuzzy costs as well as fuzzy coefficients, as a must.

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